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LINEAR OPERATORS THAT PRESERVE BOOLEAN RANK OF BOOLEAN MATRICES

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Abstract. The Boolean rank of a nonzero $m \times n$ Boolean matrix A is the minimum number k such that there exist an $m \times k$ Boolean matrix B and a $k \times n$ Boolean matrix C such that $A = BC$. In the previous research L. B. Beasley and N. J. Pullman obtained that a linear operator preserves Boolean rank if and only if it preserves Boolean ranks 1 and 2. In this paper we extend this characterizations of linear operators that preserve the Boolean ranks of Boolean matrices. That is, we obtain that a linear operator preserves Boolean rank if and only if it preserves Boolean ranks 1 and k for some $1 < k \leqslant m$.

Keywords: Boolean matrix, Boolean rank, Boolean linear operator

MSC 2010: 15A86, 15A04, 15B34

The *binary Boolean algebra* consists of the set $\mathbb{B} = \{0, 1\}$ equipped with two binary operations, addition and multiplication. The operations are defined as usual except that $1 + 1 = 1$.

There are many papers on linear operators on a matrix space that preserve matrix functions over an algebraic structure ([1], [2], [3] and [5]). Boolean matrices also have been the subject of research by many authors ([2]–[5]). Beasley and Pullman ([2]) obtained characterizations of rank-preserving operators of Boolean matrices. Kang and Song ([3]) characterized the linear operators that preserve regular matrices over the Boolean algebra.

In this article we consider the Boolean rank and extend the results of [2] to obtain new characterizations of the linear operators that preserve Boolean rank.

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Let $\mathbf{M}_{m,n}(\mathbb{B})$ be the set of all $m\times n$ matrices with entries in the binary Boolean algebra $\mathbb{B}$. The usual definitions for adding and multiplying matrices apply to Boolean matrices as well. The matrix I_n is the $n\times n$ identity matrix, $O_{m,n}$ is the $m\times n$ zero matrix, and $J_{m,n}$ is the $m\times n$ matrix all of whose entries are 1. We will suppress the superscripts on these matrices when the orders are evident from the context and we write I, O, and J, respectively. Let $E_{i,j}$ be the $m\times n$ matrix whose $(i,j)^{\text{th}}$ entry is 1 and whose other entries are all 0. We call $E_{i,j}$ a *cell*. Further, we let the set of all cells be denoted by $\mathbf{E}$. That is,

$$\mathbf{E}=\{E_{i,j}\in \mathbf{M}_{m,n}(\mathbb{B});\ i=1,\dots,m \text{ and } j=1,\dots,n\}.$$

From now on we will assume that $2\leqslant m\leqslant n$.

The *Boolean rank*, $\beta(A)$, of a nonzero Boolean matrix A in $\mathbf{M}_{m,n}(\mathbb{B})$ is the minimum number k such that there exist matrices $B\in \mathbf{M}_{m,k}(\mathbb{B})$ and $C\in \mathbf{M}_{k,n}(\mathbb{B})$ such that $A=BC$. The Boolean rank of the zero matrix is 0.

It is easy to verify that the Boolean rank of $A\in \mathbf{M}_{m,n}(\mathbb{B})$ is 1 if and only if there exist nonzero (Boolean) vectors $\mathbf{a}\in \mathbf{M}_{m,1}(\mathbb{B})$ and $\mathbf{b}\in \mathbf{M}_{n,1}(\mathbb{B})$ such that $A=\mathbf{ab}^t$. And these vectors $\mathbf{a}$ and $\mathbf{b}$ are uniquely determined by A. It is well known that $\beta(A)$ is the least k such that A is the sum of k matrices of Boolean rank 1 ([2]). It follows that $0\leqslant \beta(A)\leqslant m$ for all nonzero $A\in \mathbf{M}_{m,n}(\mathbb{B})$.

By considering a minimal sum of rank 1 matrices for A and B such as $A=A_1+\ldots+A_k$, and $B=B_1+\ldots+B_l$, we have that $A+B=A_1+\ldots+A_k+B_1+\ldots+B_l$, so that $A+B$ has rank at most $k+l$. This establishes the following lemma.

Lemma 1. *For matrices A and B in $\mathbf{M}_{m,n}(\mathbb{B})$, we have*

$$\beta(A+B)\leqslant \beta(A)+\beta(B).$$

If A and B are matrices in $\mathbf{M}_{m,n}(\mathbb{B})$, we say that B *dominates* A (written $A\leqslant B$ or $B\geqslant A$) if $b_{i,j}=0$ implies $a_{i,j}=0$ for all i and j. Equivalently, $A\leqslant B$ if and only if $A+B=B$. This provides a reflexive and transitive relation on $\mathbf{M}_{m,n}(\mathbb{B})$.

We let $|A|$ denote the number of nonzero entries in the matrix A.

A mapping $T\colon \mathbf{M}_{m,n}(\mathbb{B})\to \mathbf{M}_{m,n}(\mathbb{B})$ is called a *Boolean linear operator* if T preserves sums and the zero matrix.

For a Boolean linear operator $T\colon \mathbf{M}_{m,n}(\mathbb{B})\to \mathbf{M}_{m,n}(\mathbb{B})$, we say that T

(1) *preserves Boolean rank* k if $\beta(T(X))=k$ whenever $\beta(X)=k$ for all $X\in \mathbf{M}_{m,n}(\mathbb{B})$;
(2) *preserves Boolean rank* if it preserves Boolean rank k for every k ($\leqslant m$).

A Boolean linear operator $T\colon \mathbf{M}_{m,n}(\mathbb{B}) \to \mathbf{M}_{m,n}(\mathbb{B})$ is called a (P,Q)-*operator* if there are permutation matrices P and Q such that $T(X) = PXQ$ for all $X \in \mathbf{M}_{m,n}(\mathbb{B})$, or $m = n$ and $T(X) = PX^tQ$ for all $X \in \mathbf{M}_{m,n}(\mathbb{B})$, where X^t is the transpose of X.

In this note we prove the following theorem:

Theorem 1. *For a Boolean linear operator $T\colon \mathbf{M}_{m,n}(\mathbb{B}) \to \mathbf{M}_{m,n}(\mathbb{B})$, the following are equivalent:*
(1) *T preserves Boolean rank;*
(2) *T preserves Boolean ranks 1 and k for some $1 < k \leqslant m$;*
(3) *T is a (P,Q)-operator.*

Hereafter, $T\colon \mathbf{M}_{m,n}(\mathbb{B}) \to \mathbf{M}_{m,n}(\mathbb{B})$ will denote a Boolean linear operator. Further, the adjective "Boolean" will be omitted and we say "rank" for "Boolean rank", "linear operator" for "Boolean linear operator", etc.

Lemma 2. *Let E be a cell, $E \in \mathbf{M}_{m,n}(\mathbb{B})$, and Z a matrix such that $E \leqslant Z$. Let $T\colon \mathbf{M}_{m,n}(\mathbb{B}) \to \mathbf{M}_{m,n}(\mathbb{B})$ be a linear operator. If $|T(Z)| \leqslant |Z|$ and $|T(E)| \geqslant 2$ then there exists a cell $F \neq E$ such that $T(Z \setminus F) = T(Z)$.*

Proof. Suppose Z is a matrix such that $|T(Z)| \leqslant |Z|$. Further, suppose that a cell E_1 satisfies $E_1 \leqslant Z$ and $|T(E_1)| > 1$. If $T(E_1) \neq T(Z)$, there is a cell $E_2 \leqslant Z$ such that $|T(E_1 + E_2)| > |T(E_1)|$. Continuing in this manner, we find cells $E_1, E_2, \ldots, E_i$ such that $E_1 + E_2 + \ldots + E_i \leqslant Z$ and $|T(E_1 + E_2 + \ldots + E_i)| > |T(E_1 + E_2 + \ldots + E_{i-1})|$ for $i \leqslant j$ for some j. Since $|Z|$ and $|T(Z)|$ are finite, there exists some $j < |T(Z)|$ such that $T(E_1 + E_2 + \ldots + E_j) = T(Z)$. Let k be the first such j, so that $|T(E_1 + E_2 + \ldots + E_k)| > |T(E_1 + E_2 + \ldots + E_{k-1})|$ and $T(E_1 + E_2 + \ldots + E_k) = T(Z)$. We must now have that $k < |Z|$. It now follows that there is a cell $F \leqslant Z$ such that $T(Z \setminus F) = T(Z)$. □

Lemma 3. *If $T\colon \mathbf{M}_{m,n}(\mathbb{B}) \to \mathbf{M}_{m,n}(\mathbb{B})$ preserves rank 1 then for any $A \in \mathbf{M}_{m,n}(\mathbb{B})$,*

$$\beta(T(A)) \leqslant \beta(A).$$

Proof. The rank of a matrix A being k is equivalent to k being the minimum number of rank 1 matrices whose sum is A. Thus, the image of any rank k matrix is the sum of the images of k rank 1 matrices (all of which have rank 1) and, hence, the image of A has rank at most k. □

Let $\mathbf{N}_k$ be the set of all rank 1 matrices in $\mathbf{M}_{m,n}(\mathbb{B})$ which are dominated by a rank k matrix. Suppose that w is the largest weight of any matrix in $\mathbf{N}_k$. Let $\mathbf{N}_k^+$ be the set of all elements of $\mathbf{N}_k$ that are of weight w. Since $X \in \mathbf{N}_k^+$ implies $PXQ \in \mathbf{N}_k^+$ for any permutation matrices, P and Q of appropriate orders, the following is easily seen.

Lemma 4. *Let E be a cell in $\mathbf{M}_{m,n}(\mathbb{B})$. Then there is an element of $\mathbf{N}_k^+$ dominating E.*

Elementary arguments easily establish the following:

Lemma 5. *If $p \leqslant m$ and $q \leqslant n$, and $\begin{bmatrix} J_{p,q} & O \\ O & O \end{bmatrix} \in \mathbf{N}_k^+$, then $(m-p)+(n-q) = k-1$. As a consequence, $m-p \leqslant q-1$ and $n-q \leqslant p-1$.*

An operator $T\colon \mathbf{M}_{m,n}(\mathbb{B}) \to \mathbf{M}_{m,n}(\mathbb{B})$ is *singular* if $T(X) = O$ for some nonzero $X \in \mathbf{M}_{m,n}(\mathbb{B})$; otherwise T is *nonsingular*. Notice that if T is a (P,Q)-operator, then T is nonsingular.

Lemma 6. *If $T\colon \mathbf{M}_{m,n}(\mathbb{B}) \to \mathbf{M}_{m,n}(\mathbb{B})$ preserves ranks 1 and k for some $1 < k \leqslant m$ then T maps cells to cells.*

Proof. Clearly, we may assume that $2 \leqslant m \leqslant n$.

Since T preserves rank 1, T is nonsingular. Suppose that the image of some cell dominates two or more cells. Say, $E = E_1$ is such a cell so that $|T(E_1)| > 1$. By Lemma 4 there is $Z \in \mathbf{N}_k^+$ that dominates E_1. That is, $E_1 \leqslant Z$ and $|T(E_1)| > 1$. Since Z is of rank one and T preserves both rank one and rank k, $T(Z) \in \mathbf{N}_k$, thus, $|T(Z)| \leqslant |Z|$. By Lemma 2 there is a cell $F \leqslant Z$ such that $T(Z \setminus F) = T(Z)$. Without loss of generality, we may assume that $Z = \begin{bmatrix} J_{p,q} & O \\ O & O \end{bmatrix}$ and that $F = E_{p,q}$.

If $q = n$ then we must have $p = m-k+1$ by Lemma 5. For $A = \begin{bmatrix} O & O \\ I_{k-1} & O \end{bmatrix}$, $A+Z$ is of rank k.

For $m \neq k$ so that $p \neq 1$, let $B = (A+Z) \setminus (E_{p,q} + E_{m,k-1})$. Then $\beta(B) = k$, while $\beta(B+E_{p,q}) = k-1$. Thus, $\beta(T(B+E_{p,q})) \leqslant k-1$ by Lemma 3, and $\beta(T(B)) = k$, since T preserves rank k. But $T(B) = T(B+E_{p,q})$, a contradiction.

For $m = k$ so that $p = 1$, we must have $m = n$, for otherwise, $U = \begin{bmatrix} J_{2,n-1} & O \\ O & O \end{bmatrix} \in \mathbf{N}_k$ and $|U| = 2(n-1) > n = |Z|$, contradicting that $Z \in \mathbf{N}_k^+$ since $n > 2$. We now have that $m = k = n$. Let $B = (A+Z) \setminus E_{1,n}$. Then $\beta(B) = k-1$ while $\beta(B+E_{1,n}) = k$. Thus, $\beta(T(B+E_{p,q})) \leqslant k$, and $\beta(T(B)) = k$ by Lemma 3, since T preserves rank k. But $T(B) = T(B+E_{p,q})$, a contradiction.

Thus, for $q = n$, we have that the image of a cell is a cell.

If $p = m$ a similar argument shows that T maps cells to cells.

Now, assume that $p < m$ and $q < n$. Here, $k \geqslant 3$ since $p+q = m+n-k+1$. Since $Z \in \mathbf{N}_k^+$, we must have by Lemma 5 that $(m-p)+(n-q) = k-1$ and $m-p \leqslant q-1$ and $n-q \leqslant p-1$.

Let Q_l be an $l \times l$ $(0,1)$-matrix such that for $Q_l = (q_{u,v})$, $q_{u,v} = 1$ if and only if $u+v \leqslant l+1$.

So,

$$Q_l = \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Let

$$B = \begin{bmatrix} O_{p,q} & \begin{bmatrix} J_{m-k,n-q} \\ Q_{n-q} \\ \mathbf{0}^t_{n-q} \end{bmatrix} \\ [\, J_{m-p,n-k} \quad Q_{m-p} \quad \mathbf{0}_{m-p} \,] & O_{m-p,n-q} \end{bmatrix}.$$

So,

$$B+Z = \begin{bmatrix} J_{m-k,n-k} & J_{m-k,k} \\ J_{k,n-k} & Q_k \end{bmatrix}.$$

Clearly $\beta(B+Z) = k$, and hence, $\beta(T(B+Z)) = k$.

Further, $\beta((B+Z) \setminus E_{p,q}) = k-1$ since the p^{th} row and the $(p+1)^{\text{st}}$ row of $((B+Z) \setminus E_{p,q})$ are the same. Thus, $\beta(T((B+Z) \setminus E_{p,q})) \leqslant k-1$ by Lemma 3. But $T((B+Z) \setminus E_{p,q}) = T(B+Z)$ since $T(E_{p,q}) \leqslant T(Z)$. Thus $\beta(T(B+Z)) \leqslant k-1$.

This is a contradiction since $T(B+Z)$ cannot have rank both k and something strictly less than k. ☐

Lemma 7. *If $T\colon \mathbf{M}_{m,n}(\mathbb{B}) \to \mathbf{M}_{m,n}(\mathbb{B})$ preserves ranks 1 and k for some $1 < k \leqslant m$ then T is a bijection on $\mathbf{E}$ and hence invertible on $\mathbf{M}_{m,n}(\mathbb{B})$.*

P r o o f. We only need to show that T is injective on $\mathbf{E}$. By Lemma 6, the image of a cell is a cell. Suppose that T is not injective on the set of cells, then, without loss of generality, we may assume that $T(E_{1,1}) = T(E_{i,j})$ and $i \leqslant 2$. Let $Z = \begin{bmatrix} J_{m-k+2,n} \\ O_{k-2,n} \end{bmatrix}$ and $A = \begin{bmatrix} O & O \\ O & I_{k-2} \end{bmatrix}$, $X = Z + A$, and $Y = (Z \setminus E_{1,1}) + A$. Then $\beta(X) = k-1$ while $\beta(Y) = k$ and $T(X) = T(Y)$, an impossibility since T preserves rank k and by Lemma 3 $\beta(T(X)) \leqslant k-1 < l = \beta(T(Y))$. Thus, T is bijective on the set of cells. ☐

The following theorem completes our necessary preliminaries.

Theorem 2 [2, Theorem 3.1]**.** *Let* $T\colon \mathbf{M}_{m,n}(\mathbb{B}) \to \mathbf{M}_{m,n}(\mathbb{B})$ *be a linear operator. Then* T *preserves rank* 1 *and is invertible if and only if* T *is a* (P,Q)*-operator.*

The proof of Theorem 1. If $T\colon \mathbf{M}_{m,n}(\mathbb{B}) \to \mathbf{M}_{m,n}(\mathbb{B})$ preserves ranks 1 and k for some $1 < k \leqslant m$ then T is a bijection on $\mathbf{E}$ and hence invertible on $\mathbf{M}_{m,n}(\mathbb{B})$ by Lemma 7. Thus T is a (P,Q)-operator by Theorem 2. This establishes that (3) implies (4).

The other implications are obvious. □

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