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A GENERALIZATION OF THE AUSLANDER TRANSPOSE AND THE GENERALIZED GORENSTEIN DIMENSION

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Abstract. Let R be a left and right Noetherian ring and C a semidualizing R-bimodule. We introduce a transpose $\mathrm{Tr}_{\mathrm{c}}M$ of an R-module M with respect to C which unifies the Auslander transpose and Huang's transpose, see Z. Y. Huang, On a generalization of the Auslander-Bridger transpose, Comm. Algebra 27 (1999), 5791–5812, in the two-sided Noetherian setting, and use $\mathrm{Tr}_{\mathrm{c}}M$ to develop further the generalized Gorenstein dimension with respect to C. Especially, we generalize the Auslander-Bridger formula to the generalized Gorenstein dimension case. These results extend the corresponding ones on the Gorenstein dimension obtained by Auslander in M. Auslander, M. Bridger, Stable Module Theory, Mem. Amer. Math. Soc. vol. 94, Amer. Math. Soc., Providence, RI, 1969.

Keywords: transpose, semidualizing module, generalized Gorenstein dimension, depth, Auslander-Bridger formula

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1. Introduction and preliminaries

Throughout this paper, unless otherwise specified, R is a left and right Noetherian ring with identity, all modules under consideration will be assumed finitely generated.

As usual, ${}_RM$ or M_R denotes, respectively, a left or right R-module. $\mathrm{Add}_R\,M$ or $\mathrm{add}_R\,M$ stands for the category consisting of all R-modules isomorphic to direct summands of direct or, respectively, finite direct sums of copies of ${}_RM$. Similarly, we have the notations $\mathrm{Add}\,M_R$ and $\mathrm{add}\,M_R$. When (R, m, k) is a commutative Noetherian local ring, for an R-module M, we write $\mathrm{depth}\,M$ for the length of maximal M-regular sequences in m, i.e., the depth of M, and use freely the formula

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depth $M = \min\{i\colon \operatorname{Ext}^i_R(k, M) \neq 0\}$. General background material can be found in [1], [4], [5], [14].

We first recall some known notions and facts needed in the sequel.

An R-bimodule C is called semidualizing if

(1) ${}_RC$ and C_R are finitely generated;
(2) the natural maps $R^{\mathrm{op}} \to \operatorname{End}({}_RC)$ and $R \to \operatorname{End}(C_R)$ are isomorphisms;
(3) $\operatorname{Ext}^i_R(C, C) = 0 = \operatorname{Ext}^i_{R^{\mathrm{op}}}(C, C)$ for all $i \geqslant 1$.

In what follows, C always denotes a semidualizing R-bimodule.

A left R-module M is said to have generalized Gorenstein dimension zero with respect to C if

(1) $M \cong \operatorname{Hom}_{R^{\mathrm{op}}}(\operatorname{Hom}_R(M, C), C)$;
(2) $\operatorname{Ext}^i_R(M, C) = 0 = \operatorname{Ext}^i_{R^{\mathrm{op}}}(\operatorname{Hom}_R(M, C), C)$ for all $i \geqslant 1$.

We denote by $G_c(R)$ or $G_c(R^{\mathrm{op}})$ the class of all left or right R-modules, respectively, having generalized Gorenstein dimension zero with respect to C.

Let $\mathcal{W}$ be a class of R-modules and M an R-module. For a non-negative integer n, M is said to have $\mathcal{W}$-dimension at most n, denoted by $\mathcal{W}\text{-}\dim M \leqslant n$, if there is an exact sequence $0 \to W_n \to \ldots \to W_1 \to W_0 \to M \to 0$ with $W_i \in \mathcal{W}$ for all $0 \leqslant i \leqslant n$.

A left R-module M (non-finitely generated) is called C-Gorenstein projective if

(1) $\operatorname{Ext}^i_R(M, C \otimes_R P) = 0$ for all projective left R-modules P and $i \geqslant 1$;
(2) there is an exact sequence $0 \to M \to C \otimes_R P_0 \to C \otimes_R P_1 \to \ldots$ with P_i projective left R-modules for all $i \geqslant 0$ such that this sequence stays exact when we apply to it the functor $\operatorname{Hom}_R(-, C \otimes_R Q)$ for any projective left R-module Q.

Semidualizing modules (i.e., PG-modules of rank one) were first introduced by Foxby [8] over commutative Noetherian rings, and have been recently defined and studied by Holm and White over arbitrary associative rings [10]. A semidualizing module was also called a generalized tilting module in the sense of Wakamatsu [16] or a faithfully balanced selforthogonal module in [11].

Auslander and Bridger [1] introduced the Gorenstein dimension (abbr. G-dimension) for finitely generated modules, and proved that, over commutative Noetherian local rings R, a finitely generated R-module M with finite G-dimension satisfies the Auslander-Bridger formula: $\text{G-}\dim M + \operatorname{depth} M = \operatorname{depth} R$. Then Auslander and Reiten [2] generalized the notion of G-dimension to that of generalized Gorenstein dimension with respect to a semidualizing module C.

Assume that $P_1 \xrightarrow{f} P_0 \to M \to 0$ is a projective resolution of a left R-module M. Dualizing this sequence by $\operatorname{Hom}_R(-, R)$, the Auslander transpose of M, $\operatorname{Tr} M$, is defined as $\operatorname{coker}(\operatorname{Hom}_R(f, R))$. It is well known that $\operatorname{Tr} M$ depends on the choice of the projective resolution of M, but it is unique up to projective equivalence. The Auslander transpose plays a very important role in the representation theory and Gorenstein dimension theory. Over an artinian algebra Λ, using the minimal projective resolution of M, replacing the functor $\mathrm{Hom}_{\Lambda}(-,\Lambda)$ by $\mathrm{Hom}_{\Lambda}(-,C)$, where C is a semidualizing Λ-bimodule, Huang [11] introduced and studied $\mathrm{Tr}_{\mathrm{c}}M$, a transpose of M with respect to C. It is the aim of this paper to extend this notion to two-sided Noetherian rings, which unifies the Auslander transpose and that of Huang, and use $\mathrm{Tr}_{\mathrm{c}}M$ to develop further the generalized Gorenstein dimension. Especially, we obtain a generalization of the Auslander-Bridger formula on the generalized Gorenstein dimension. These results generalize the corresponding ones on the Gorenstein dimension obtained by Auslander in [1].

In Section 2, for a left R-module M, the notion of a transpose of M with respect to C, denoted by $\mathrm{Tr}_{\mathrm{c}}M$, is defined over the left and right Noetherian ring R. It is shown that $\mathrm{Tr}_{\mathrm{c}}M$, although depending on the choice of the projective resolution of M, is unique up to $\operatorname{add} C$-equivalence (Definition 2.1), and that $M \in G_c(R)$ if and only if $\mathrm{Ext}^i_R(M,C) = 0 = \mathrm{Ext}^i_{R^{\mathrm{op}}}(\mathrm{Tr}_{\mathrm{c}}M,C)$ for all $i \geqslant 1$ (Proposition 2.6) if and only if $\mathrm{Tr}_{\mathrm{c}}M \in G_c(R^{\mathrm{op}})$ (Proposition 2.7). C-Gorenstein projective modules (non-finitely generated) were first introduced in [9] over commutative Noetherian rings and then extended to commutative non-Noetherian setting in [17]. Here it is proved that a finitely generated left R-module M is C-Gorenstein projective if and only if $M \in G_c(R)$ (Proposition 2.8), a result already shown for commutative rings in [17] with a different proof.

Section 3 is devoted to investigating the generalized Gorenstein dimension under changes of commutative Noetherian rings. For an R-module M and a non-negative integer n, it is proved that $G_c(R)\text{-}\dim M \leqslant n$ if and only if $G_{c_P}(R_P)\text{-}\dim M_P \leqslant n$ for all prime (maximal) ideals P (Corollary 3.4), and that $G_{c/xc}(R/xR)\text{-}\dim(M/xM) \leqslant G_c(R)\text{-}\dim M$, where x is regular on both R and M; furthermore, if M has finite $G_c(R)$-dimension and $x \in J(R)$ (the Jacobson radical of R), then $G_{c/xc}(R/xR)\text{-}\dim(M/xM) = G_c(R)\text{-}\dim M$ (Proposition 3.7).

In Section 4, as an application of the results obtained in the last two sections, we show the following theorem which extends the Auslander-Bridger formula [1, Theorem 4.13 (b)] and Strooker's result [15, Theorem 6.1] to the generalized Gorenstein dimension setting.

Theorem. *Let (R,m) be a commutative Noetherian local ring and M a nonzero R-module with finite $G_c(R)$-dimension. Then* $G_c(R)\text{-}\dim M + \operatorname{depth} M = \operatorname{depth} C$.

For a left R-module M and a homomorphism f of left R-modules, we put $M^* = \operatorname{Hom}_R(M, C)$, $f^* = \operatorname{Hom}_R(f, C)$ and use $\sigma_M\colon M \to M^{**}$, defined by $\sigma_M(x)(g) = g(x)$ for any $x \in M$ and $g \in M^*$, to denote the canonical evaluation homomorphism.

Definition 2.1. Two left R-modules M and N are said to be add ${}_RC$-equivalent, denoted by $M \approx_c N$, if there exist $A, B \in \operatorname{add}{}_RC$ such that $M \oplus A \cong N \oplus B$. For right R-modules we have a similar definition.

It is clear that $\approx_c$ is an equivalence relation on the category of all finitely generated R-modules and the $\operatorname{add}_R C$-equivalence is just the projective equivalence [1] when $C = R$.

Proposition 2.2. *Let* $P_1 \xrightarrow{\mu} P_0 \xrightarrow{f} M \to 0\,(\pi_1)$ *and* $Q_1 \xrightarrow{\nu} Q_0 \xrightarrow{g} M \to 0\,(\pi_2)$ *be projective resolutions of a left* R*-module* M*. Then* $\operatorname{coker}\mu^* \approx_c \operatorname{coker}\nu^*$.

P r o o f. Extending the projective resolutions (π_1) and (π_2) to the left and lifting id_M to φ_0, we get the commutative diagram with exact rows

$$\begin{tikzcd}
P_2 \arrow[r, "\mu'"] & P_1 \arrow[r, "\mu"] & P_0 \arrow[r, "f"] \arrow[d, "\varphi_0"] & M \arrow[r] \arrow[d, equal] & 0 \\
Q_2 \arrow[r, "\nu'"] & Q_1 \arrow[r, "\nu"] & Q_0 \arrow[r, "g"] & M \arrow[r] & 0.
\end{tikzcd}$$

Let $h\colon P_0 \oplus Q_0 \to M$, $h((p_0, q_0)) = f(p_0) + g(q_0)$ for any $p_0 \in P_0$, $q_0 \in Q_0$. Then it is clear that h is surjective. So we have the projective resolutions of M

$$E \xrightarrow{\alpha} P_0 \oplus Q_0 \xrightarrow{h} M \to 0\,(\pi_3).$$

Let $\lambda_0\colon P_0 \oplus Q_0 \to Q_0$, $\lambda_0((p_0, q_0)) = \varphi_0(p_0) + q_0$ for any $p_0 \in P_0$, $q_0 \in Q_0$. λ_0 is clearly surjective, and $g\lambda_0 = h$. Lifting λ_0 to $\delta\colon E \to Q_1$, we have the exact commutative diagram

$$\begin{tikzcd}
E \arrow[r, "\alpha"] \arrow[d, "\delta"] & P_0 \oplus Q_0 \arrow[r, "h"] \arrow[d, "\lambda_0"] & M \arrow[r] \arrow[d, equal] & 0 \\
Q_1 \arrow[r, "\nu"] & Q_0 \arrow[r, "g"] & M \arrow[r] & 0.
\end{tikzcd}$$

Let $\omega\colon E \oplus P_2 \oplus Q_2 \to P_0 \oplus Q_0$, $\omega((e, p_2, q_2)) = \alpha(e)$ for any $e \in E$, $p_2 \in P_2$, $q_2 \in Q_2$. Since $\operatorname{im}\omega = \operatorname{im}\alpha = \ker h$ by (π_3), we have the exact sequence

$$E \oplus P_2 \oplus Q_2 \xrightarrow{\omega} P_0 \oplus Q_0 \xrightarrow{h} M \to 0\,(\pi_4).$$

Define $\lambda_1\colon\ E\oplus P_2\oplus Q_2 \to Q_1$, $\lambda_1((e,p_2,q_2)) = \delta(e)+\nu'(q_2)$ for any $e\in E$, $p_2\in P_2$, $q_2\in Q_2$. Since $\nu\lambda_1((e,p_2,q_2)) = \nu(\delta(e)+\nu'(q_2)) = \nu(\delta(e)) = \lambda_0\alpha(e) = \lambda_0\omega(e)$, we have the exact commutative diagram

$$\begin{tikzcd}
E\oplus P_2\oplus Q_2 \arrow[r, "\omega"] \arrow[d, "\lambda_1"] & P_0\oplus Q_0 \arrow[r, "h"] \arrow[d, "\lambda_0"] & M \arrow[r] \arrow[d, equal] & 0 \\
Q_1 \arrow[r, "\nu"] & Q_0 \arrow[r, "g"] & M \arrow[r] & 0.
\end{tikzcd}$$

We claim that λ_1 is surjective. Indeed, for any $q_1\in Q_1$, $\nu(q_1)\in Q_0$ we have $h((0,\nu(q_1))) = g\lambda_0((0,\nu(q_1))) = g(\nu(q_1)) = 0$, so $(0,\nu(q_1))\in\ker h = \operatorname{im}\alpha$. Take $e\in E$ with $\alpha(e) = (0,\nu(q_1))$. Then $\nu\delta(e) = \lambda_0\alpha(e) = \lambda_0(0,\nu(q_1)) = \nu(q_1)$. So $q_1-\delta(e)\in\ker\nu = \operatorname{im}\nu'$. Thus $q_1 = \delta(e)+\nu'(q_2) = \lambda_1((e,0,q_2))$ for some $q_2\in Q_2$, as desired.

Let $\varpi = \omega|_{\ker\lambda_1}\colon\ \ker\lambda_1\to\ker\lambda_0$. Next we prove that ϖ is epic. Let $(p_0,q_0)\in\ker\lambda_0$ with $p_0\in P_0$, $q_0\in Q_0$. Then $h((p_0,q_0)) = g\lambda_0((p_0,q_0)) = 0$. So $(p_0,q_0)\in\ker h = \operatorname{im}\alpha$. Take $e\in E$ such that $(p_0,q_0) = \alpha(e) = \omega((e,0,0))$. On the other hand, since $\nu\delta(e) = \nu\lambda_1((e,0,0)) = \lambda_0\omega((e,0,0)) = \lambda_0((p_0,q_0)) = 0$ we have $\delta(e)\in\ker\nu = \operatorname{im}\nu'$. So $\delta(e) = \nu'(q_2)$ for some $q_2\in Q_2$. Thus $(e,0,-q_2)\in\ker\lambda_1$, and $\varpi((e,0,-q_2)) = \alpha(e) = (p_0,q_0)$. Therefore ϖ is epic.

Now set $W_0 = P_0\oplus Q_0$, $W_1 = E\oplus P_2\oplus Q_2$, $K_i = \ker\lambda_i$ for $i=0,1$, and note that the sequences $0\to K_i\to W_i\to Q_i\to 0$ are split exact since Q_i are projective for $i=0,1$. Thus K_0 and K_1 are projective, and then ϖ is split epic. Dualizing the last commutative diagram we get the exact commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M^* \arrow[r] \arrow[d, equal] & Q_0^* \arrow[r, "\nu^*"] \arrow[d, "\lambda_0^*"] & Q_1^* \arrow[r] \arrow[d, "\lambda_1^*"] & \operatorname{coker}\nu^* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M^* \arrow[r] & W_0^* \arrow[r, "\omega^*"] \arrow[d] & W_1^* \arrow[r] \arrow[d] & \operatorname{coker}\omega^* \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & K_0^* \arrow[r, "\varpi^*"] \arrow[d] & K_1^* \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & 0, \\
 & & 0 & 0 & 0 &
\end{tikzcd}$$

where $K = \operatorname{coker}\varpi^*$. The sequence $0\to\operatorname{coker}\nu^*\to\operatorname{coker}\omega^*\to K\to 0$ is exact by the Snake Lemma, and since the sequences $0\to Q_1^*\to W_1^*\to K_1^*\to 0$ and $0\to K_0^*\to K_1^*\to K\to 0$ are split as well. So $\operatorname{coker}\omega^*\cong\operatorname{coker}\nu^*\oplus K$ with $K\in\operatorname{add}C_R$ for $K_1^*\in\operatorname{add}C_R$. By a dual argument, we have that $\operatorname{coker}\omega^*\cong\operatorname{coker}\mu^*\oplus K^{'}$ for some $K^{'}\in\operatorname{add}C_R$. Therefore $\operatorname{coker}\mu^*\approx_c\operatorname{coker}\nu^*$. $\square$

Using minimal projective resolutions of modules, the transpose $\mathrm{Tr_c}M$ of an R-module M with respect to a semidualizing bimodule C is defined in [11] over an artinian algebra. Now we generalize this notion and the Auslander transpose to two-sided Notherian rings.

Definition 2.3. Let $P_1 \xrightarrow{f} P_0 \to M \to 0$ be a projective resolution of a left R-module M. Then we have the exact sequence $0 \to M^* \to P_0^* \xrightarrow{f^*} P_1^* \to \operatorname{coker} f^* \to 0$. We call $\operatorname{coker} f^*$ a transpose of M with respect to C, and denote it by $\mathrm{Tr_c}M$. Similarly, we have the concept for right R-modules.

Remark 2.4. (1) For a left R-module M, it is clear that $\mathrm{Tr_c}M$ depends on the choice of the projective resolution of M, but it is unique up to $\operatorname{add} C_R$-equivalence by Proposition 2.2. So each $\mathrm{Ext}^i_{R^{\mathrm{op}}}(\mathrm{Tr_c}M, C)$ is identical up to isomorphisms for any $i \geqslant 1$ since $\mathrm{Ext}^i_{R^{\mathrm{op}}}(C, C) = 0$ for all $i \geqslant 1$. In the following, we will use $\mathrm{Tr_c}M$ to indicate a right R-module and will be careful to specify, when necessary, that a particular resolution is used. In many instances, the distinction is irrelevant.

(2) Let k be a positive integer. By [12, Definition 2], a left R-module M is C-k-torsionfree if and only if $\mathrm{Ext}^i_{R^{\mathrm{op}}}(\mathrm{Tr_c}M, C) = 0$ for all $1 \leqslant i \leqslant k$.

Lemma 2.5 [13, Lemma 2.1]. *Let M be a left R-module. Then we have the following two exact sequences:*

(1) $0 \to \mathrm{Ext}^1_{R^{\mathrm{op}}}(\mathrm{Tr_c}M, C) \to M \xrightarrow{\sigma_M} M^{**} \to \mathrm{Ext}^2_{R^{\mathrm{op}}}(\mathrm{Tr_c}M, C) \to 0.$

(2) $0 \to \mathrm{Ext}^1_R(M, C) \to \mathrm{Tr_c}M \xrightarrow{\sigma_{\mathrm{Tr_c}M}} (\mathrm{Tr_c}M)^{**} \to \mathrm{Ext}^2_R(M, C) \to 0.$

By Lemma 2.5, we immediately obtain the following proposition which extends [1, Proposition 3.8].

Proposition 2.6. *Let M be a left R-module. Then $M \in G_c(R)$ if and only if it satisfies $\mathrm{Ext}^i_R(M, C) = 0 = \mathrm{Ext}^i_{R^{\mathrm{op}}}(\mathrm{Tr_c}M, C)$ for all $i \geqslant 1$.*

Proposition 2.7. *The following implications hold for a left R-module M:*

(1) *If $M \in G_c(R)$ then $M^* \in G_c(R^{\mathrm{op}})$.*

(2) *$M \in G_c(R)$ if and only if $\mathrm{Tr_c}M \in G_c(R^{\mathrm{op}})$.*

Proof. (1) is clear by the definition.

(2) Let $P_1 \to P_0 \to M \to 0$ be a projective resolution of M. Since $P_i \cong P_i^{**}$ for $i = 0, 1$, we have the following two exact sequences:

$$0 \to M^* \to P_0^* \to P_1^* \to \mathrm{Tr_c}M \to 0,$$
$$0 \to (\mathrm{Tr_c}M)^* \to P_1^{**} \to P_0^{**} \to M \to 0.$$

Then for all $i \geqslant 1$ we have

$$\begin{aligned}\operatorname{Ext}^i_{R^{\rm op}}(M^*, C) &\cong \operatorname{Ext}^{i+2}_{R^{\rm op}}(\operatorname{Tr_c} M, C),\\ \operatorname{Ext}^i_R((\operatorname{Tr_c} M)^*, C) &\cong \operatorname{Ext}^{i+2}_R(M, C).\end{aligned}$$

So if $M \in G_c(R)$ then $\operatorname{Tr_c} M \cong (\operatorname{Tr_c} M)^{**}$ and $\operatorname{Ext}^i_{R^{\rm op}}(\operatorname{Tr_c} M, C) = 0$ for $i = 1, 2$ by Lemma 2.5, and then $\operatorname{Ext}^i_{R^{\rm op}}(\operatorname{Tr_c} M, C) = 0 = \operatorname{Ext}^i_R((\operatorname{Tr_c} M)^*, C)$ for all $i \geqslant 1$ by the isomorphisms above. Thus $\operatorname{Tr_c} M \in G_c(R^{\rm op})$. The proof of the other direction is similar, so we omit it. $\square$

Enochs and Jenda defined in [7] Gorenstein projective modules whether the modules are finitely generated or not. Also, they defined the Gorenstein projective dimension for arbitrary (non-finitely generated) modules. It is well known that for finitely generated modules over a commutative Noetherian ring, the Gorenstein projective dimension agrees with the Gorenstein dimension. We have here a similar result for C-Gorenstein projective dimension and the generalized Gorenstein dimension with respect to C which is shown in the commutative setting in [17] with a different proof.

Proposition 2.8. *Let M be a left R-module. Then $M \in G_c(R)$ if and only if M is C-Gorenstein projective.*

Proof. We first note that $C \otimes_R \operatorname{Proj} = \operatorname{Add}_R C$, where Proj is the class of all (non-finitely generated) projective left R-modules. It is clear that $C \otimes_R \operatorname{Proj} \subseteq \operatorname{Add}_R C$. Conversely, suppose $M \oplus N = C^{(I)}$ for some left R-module N and some index set I. Then

$$\begin{aligned}&C \otimes_R \operatorname{Hom}_R(C, M) \oplus C \otimes_R \operatorname{Hom}_R(C, N)\\ &\qquad \cong C \otimes_R \operatorname{Hom}_R(C, C^{(I)})\\ &\qquad \cong C \otimes_R (\operatorname{Hom}_R(C, C))^{(I)} \text{ (for } {}_RC \text{ is finitely generated)}\\ &\qquad \cong C \otimes_R R^{(I)} \text{ (for } \operatorname{Hom}_R(C, C) \cong R)\\ &\qquad \cong C^{(I)}\end{aligned}$$

and $\operatorname{Hom}_R(C, M) \oplus \operatorname{Hom}_R(C, N) \cong \operatorname{Hom}_R(C, C^{(I)}) \cong R^{(I)}$. Thus $M \cong C \otimes_R \operatorname{Hom}_R(C, M)$ with $\operatorname{Hom}_R(C, M)$ a projective left R-module. So $M \in C \otimes_R \operatorname{Proj}$.

Let $M \in G_c(R)$. Since M is finitely generated and $\operatorname{Ext}^i_R(M, C) = 0$ for all $i \geqslant 1$, $\operatorname{Ext}^i_R(M, N) = 0$ for all $N \in \operatorname{Add}_R C$ and all $i \geqslant 1$. On the other hand, since M is C-k-torsionfree for all $k \geqslant 1$ by Proposition 2.6 and Remark 2.4 (2), there exists an exact sequence $0 \to M \to C^{n_0} \to C^{n_1} \to \ldots$ which is $\operatorname{Hom}_R(-, C)$ exact by [12, Theorem 1], and then $\operatorname{Hom}_R(-, \operatorname{Add}_R C)$ exact, where n_j are positive integers for all $j \geqslant 0$. So M is C-Gorenstein projective. Conversely, if M is C-Gorenstein projective, then $\operatorname{Ext}^i_R(M,C)=0$ for all $i \geqslant 1$ and there exists an exact sequence $0 \to M \to C^{(A_0)} \to C^{(A_1)} \to \dots$ which is $\operatorname{Hom}_R(-, \operatorname{Add}_R C)$ exact, where A_j are index sets for all $j \geqslant 0$. In fact, by an argument similar to the proof of [5, Theorem 4.2.6], we can construct a $\operatorname{Hom}_R(-,C)$ exact exact sequence $0 \to M \to C^{n_0} \to C^{n_1} \to \dots$ with n_j a positive integer for each $j \geqslant 0$. So M is C-k-torsionfree for all $k \geqslant 1$ by [12, Theorem 1] again, and then $\operatorname{Ext}^i_{R^{\rm op}}(\operatorname{Tr_c} M, C)=0$ for all $i \geqslant 0$ by Remark 2.4 (2). Thus $M \in G_c(R)$ by Proposition 2.6. □

3. Generalized Gorenstein dimension under changes of rings

In the following two sections let R be commutative Noetherian and C a given semidualizing R-module. We begin with the study of semidualizing modules.

Proposition 3.1.

(1) *For any $P \in \operatorname{Spec} R$, C_P is a semidualizing R_P-module.*

(2) *Let $x \in R$ be R-regular (i.e., x is a nonzero-divisor on R). Then C/xC is a semidualizing R/xR-module. In general, if $x_1, x_2, \dots, x_n$ is an R-regular sequence then $C/(x_1, x_2, \dots, x_n)C$ is a semidualizing $R/(x_1, x_2, \dots, x_n)R$-module.*

Proof. (1) is immediate by [6, Proposition 5.8].

(2) By the definition of semidualizing modules and [3, Proposition 10, p. 267], we have $\operatorname{Ass} R = \operatorname{Ass}(\operatorname{Hom}_R(C,C)) = \operatorname{Ass} C \cap \operatorname{Supp} C = \operatorname{Ass} C$. So R and C have the same zero-divisors by [14, Corollary 2, p.50]. If x is R-regular then it is also C-regular. So there exists an exact sequence

$$(\sharp) \qquad 0 \to C \xrightarrow{x} C \to C/xC \to 0.$$

Since $\operatorname{Ext}^1_R(C,C)=0$, applying the functor $\operatorname{Hom}_R(C,-)$ to the sequence above we obtain the following exact sequence

$$0 \to \operatorname{Hom}_R(C,C) \xrightarrow{x} \operatorname{Hom}_R(C,C) \to \operatorname{Hom}_R(C,C/xC) \to 0.$$

Then we have $\operatorname{Hom}_R(C,C/xC) \cong R/xR$ since $\operatorname{Hom}_R(C,C) \cong R$. So by the adjoint isomorphism we have $\operatorname{Hom}_{R/xR}(C/xC, C/xC) \cong \operatorname{Hom}_R(C, C/xC) \cong R/xR$.

On the other hand, we have $\operatorname{Ext}^i_R(C, C/xC) = 0$ for all $i \geqslant 1$ by ($\sharp$) since $\operatorname{Ext}^i_R(C,C)=0$ for all $i \geqslant 1$. So we have

$$\operatorname{Ext}^i_{R/xR}(C/xC, C/xC) \cong \operatorname{Ext}^i_R(C, C/xC) = 0$$

for all $i \geqslant 1$ by [1, Lemma 4.7]. Therefore C/xC is a semidualizing R/xR-module. The last conclusion is immediate by induction. □

Corollary 3.2. *Let $x_i \in R$ for all $i = 1, 2, \ldots, n$. Then $x_1, x_2, \ldots, x_n$ is an R-regular sequence if and only if $x_1, x_2, \ldots, x_n$ is a C-regular sequence. In particular, if R is local then* $\operatorname{depth} C = \operatorname{depth} R$.

In the following, we put $\overline{R} = R/xR$, $\overline{M} = M/xM$ for any $x \in R$ and any R-module M.

Proposition 3.3. *Let M be an R-module. Then the following assertions hold:*

(1) $(\mathrm{Tr}_{\mathrm{c}} M)_P \simeq_{c_P} \mathrm{Tr}_{\mathrm{c_P}} M_P$ *for any $P \in \operatorname{Spec} R$.*
(2) $(\mathrm{Tr}_{\mathrm{c}} M) \otimes_R \overline{C} \simeq_{\bar{c}} \mathrm{Tr}_{\bar{c}} \overline{M}$ *for any R-regular element x.*

Proof. In fact, we can prove a more general result: if $f\colon R \to S$ is a homomorphism of commutative Noetherian rings such that $C \otimes_R S$ is a semidualizing S-module, then $(\mathrm{Tr}_{\mathrm{c}} M) \otimes_R S$ and $\mathrm{Tr}_{c \otimes_R s}(M \otimes_R S)$ are $\mathrm{add}(C \otimes_R S)$-equivalent for all R-modules M.

Let $P_1 \to P_0 \to M \to 0$ be a projective resolution of M. Then $P_1 \otimes_R S \to P_0 \otimes_R S \to M \otimes_R S \to 0$ is an S-projective resolution of $M \otimes_R S$. For $i = 0, 1$, we have

$$\begin{aligned}\mathrm{Hom}_S(P_i \otimes_R S, C \otimes_R S) &\cong \mathrm{Hom}_R(P_i, C \otimes_R S) \text{ (by the adjoint isomorphism)}\\ &\cong \mathrm{Hom}_R(P_i, C) \otimes_R S \text{ (by the tensor evaluation isomorphism).}\end{aligned}$$

So we get the following commutative diagram with exact rows:

$$\begin{tikzcd}
\mathrm{Hom}_S(P_0 \otimes_R S, C \otimes_R S) \arrow[r] \arrow[d] & \mathrm{Hom}_S(P_1 \otimes_R S, C \otimes_R S) \arrow[r] \arrow[d] & \mathrm{Tr}_{\mathrm{c} \otimes_R s}(M \otimes_R S) \arrow[r] \arrow[d] & 0 \\
P_0^* \otimes_R S \arrow[r, "\nu"] & P_1^* \otimes_R S \arrow[r, "g"] & \mathrm{Tr}_{\mathrm{c}} M \otimes_R S \arrow[r] & 0.
\end{tikzcd}$$

Therefore $(\mathrm{Tr}_{\mathrm{c}} M) \otimes_R S \approx_{c \otimes_R s} \mathrm{Tr}_{c \otimes_R s}(M \otimes_R S)$ by Remark 2.4. □

We use temporarily $\mathrm{Ext}_R^{\geqslant 1}(M, N) = 0$ to indicate $\mathrm{Ext}_R^i(M, N) = 0$ for all $i \geqslant 1$ for two R-modules M and N. The following corollary is a generalization of [1, Corollary 4.15].

Corollary 3.4. *Let M be an R-module and n a non-negative integer. Then $G_c(R)$-$\dim M \leqslant n$ if and only if $G_{c_P}(R_P)$-$\dim M_P \leqslant n$ for all prime (maximal) ideals P. Therefore*

$$G_c(R)-\dim M = \sup\{G_{c_P}(R_P)-\dim M_P \colon P \in \operatorname{Spec} R\}.$$

P r o o f. By Propositions 2.6 and 3.3 we have that

$$\begin{aligned}
&M \in G_c(R)\\
&\Leftrightarrow \operatorname{Ext}_R^{\geqslant 1}(M,C) = 0 = \operatorname{Ext}_R^{\geqslant 1}(\mathrm{Tr}_c M, C)\\
&\Leftrightarrow (\operatorname{Ext}_R^{\geqslant 1}(M,C))_P = 0 = (\operatorname{Ext}_R^{\geqslant 1}(\mathrm{Tr}_c M, C))_P \text{ for all prime (maximal) ideals } P\\
&\Leftrightarrow \operatorname{Ext}_{R_P}^{\geqslant 1}(M_P,C_P) = 0 = \operatorname{Ext}_{R_P}^{\geqslant 1}((\mathrm{Tr}_c M)_P, C_P) \text{ for all prime (maximal) ideals } P\\
&\Leftrightarrow M_P \in G_{c_P}(R_P) \text{ for all prime (maximal) ideals } P.
\end{aligned}$$

So the "only if" part is clear since localization preserves exactness. Conversely, assume that $G_{c_P}(R_P)$-dim$M_P \leqslant n$ for all prime (maximal) ideals P. Let

$$0 \to K \to G_{n-1} \to \ldots \to G_0 \to M \to 0$$

be an exact sequence with $G_i \in G_c(R)$ for all $0 \leqslant i \leqslant n-1$. Then localizing at P we get $K_P \in G_{c_P}(R_P)$ by [17, Proposition 3.12]. So $K \in G_c(R)$ by the previous proof, and then $G_c(R)$-dim$M \leqslant n$ by [17, Proposition 3.12] again. □

Corollary 3.5. *Let $x \in R$ be R-regular and $M \in G_c(R)$. Then $\overline{M} \in G_{\bar{c}}(\overline{R})$.*

P r o o f. Since x is R-regular, x is C-regular by Corollary 3.2. So x is also M-regular since there exists an exact sequence $0 \to M \to C^n$ for some positive integer n by the proof of Proposition 2.8. Because x is C-regular, we have the exact sequence $0 \to C \xrightarrow{x} C \to \overline{C} \to 0$ which induces the exact sequence $\operatorname{Ext}_R^i(M,C) \to \operatorname{Ext}_R^i(M,\overline{C}) \to \operatorname{Ext}_R^{i+1}(M,C)$ for all $i \geqslant 1$. So $\operatorname{Ext}_R^i(M,\overline{C}) = 0$ for all $i \geqslant 1$ since $\operatorname{Ext}_R^i(M,C) = 0$ for all $i \geqslant 1$ by $M \in G_c(R)$. Since x is both R and M-regular, $\operatorname{Tor}_i^R(M,\overline{R}) = 0$ for all $i \geqslant 1$ by [1, Lemma 4.7]. Thus, by [4, Proposition 4.1.3, p. 118], we have that $\operatorname{Ext}_{\overline{R}}^i(\overline{M},\overline{C}) \cong \operatorname{Ext}_R^i(M,\overline{C}) = 0$ for all $i \geqslant 1$. On the other hand, since $\mathrm{Tr}_c M \in G_c(R)$ by Proposition 2.7, we have $\operatorname{Ext}_{\overline{R}}^i((\mathrm{Tr}_c M) \otimes_R \overline{R}, \overline{C}) = 0$ for all $i \geqslant 1$ by the previous proof. So $\operatorname{Ext}_{\overline{R}}^i((\mathrm{Tr}_{\bar{c}} \overline{M}, \overline{C}) = 0$ for all $i \geqslant 1$ by Proposition 3.3. Thus $\overline{M} \in G_{\bar{c}}(\overline{R})$ by Proposition 2.6. □

Remark 3.6. By an inductive argument, we immediately obtain that if $x_1, x_2, \ldots, x_n$ is an R-regular sequence and $M \in G_c(R)$ then

(1) $x_1, x_2, \ldots, x_n$ is an M-regular sequence,

(2) $M/(x_1, x_2, \ldots, x_n)M \in G_{c/(x_1,x_2,\ldots,x_n)c}(R/(x_1, x_2, \ldots, x_n)R)$.

Proposition 3.7. *Let M be an R-module and let x be regular on both R and M. Then the following assertions hold:*

(1) $G_{\overline{c}}(\overline{R})\text{-}\dim \overline{M} \leqslant G_c(R)\text{-}\dim M$.

(2) *If M has finite $G_c(R)$-dimension and $x \in J(R)$ (the Jacobson radical of R) then* $G_{\overline{c}}(\overline{R})\text{-}\dim \overline{M} = G_c(R)\text{-}\dim M$.

Proof. (1) If $G_c(R)\text{-}\dim M = \infty$ then the inequality obviously holds, so we assume that M has finite $G_c(R)$-dimension. When $G_c(R)\text{-}\dim M = 0$ we have $G_{\overline{c}}(\overline{R})\text{-}\dim \overline{M} = 0$ by Corollary 3.5. Now let $G_c(R)\text{-}\dim M = n \geqslant 1$, and assume that the statement is true for all modules of smaller dimension. Then there exists an exact sequence $0 \to K \to G \to M \to 0$ with $G \in G_c(R)$ and $G_c(R)\text{-}\dim K \leqslant n-1$. Since x is regular on both R and M, we have $\operatorname{Tor}_i^R(M, \overline{R}) = 0$ for all $i \geqslant 1$ by [1, Lemma 4.7]. So we have the exact sequence $0 \to \overline{K} \to \overline{G} \to \overline{M} \to 0$. On the other hand, we have $G_{\overline{c}}(\overline{R})\text{-}\dim \overline{K} \leqslant n-1$ and $\overline{G} \in G_{\overline{c}}(\overline{R})$ by the induction hypothesis. Thus $G_{\overline{c}}(\overline{R})\text{-}\dim \overline{M} \leqslant n$.

(2) It is sufficient to prove $G_c(R)\text{-}\dim M \leqslant G_{\overline{c}}(\overline{R})\text{-}\dim \overline{M}$ by (1). This inequality obviously holds if $\overline{M}$ has infinite $G_{\overline{c}}(\overline{R})$-dimension. So we assume $G_{\overline{c}}(\overline{R})\text{-}\dim \overline{M} = n < \infty$. Then $\operatorname{Ext}^i_{\overline{R}}(\overline{M}, \overline{C}) = 0$ for all $i > n$ by [17, Proposition 3.13]. Since $\operatorname{Tor}_i^R(M, \overline{R}) = 0$ for all $i \geqslant 1$ by the proof of (1), $\operatorname{Ext}^i_R(M, \overline{C}) \cong \operatorname{Ext}^i_{\overline{R}}(\overline{M}, \overline{C}) = 0$ for all $i > n$ by [4, Proposition 4.1.3, p. 118]. On the other hand, since x is also C-regular, there exists an exact sequence $0 \to C \xrightarrow{x} C \to \overline{C} \to 0$ which yields the exact sequence $\operatorname{Ext}^i_R(M, C) \xrightarrow{x} \operatorname{Ext}^i_R(M, C) \to \operatorname{Ext}^i_R(M, \overline{C}) = 0$ for all $i > n$. Therefore $\operatorname{Ext}^i_R(M, C) = 0$ for all $i > n$ by Nakayama's Lemma. So $G_c(R)\text{-}\dim M \leqslant n$ by [17, Proposition 3.13] again since $G_c(R)\text{-}\dim M < \infty$. This completes the proof. □

4. A generalization of the Auslander-Bridger formula

The purpose of this section is to prove the theorem from Introduction using the results obtained in the previous two sections.

Lemma 4.1. *Let (R, m) be a local ring. If* $\operatorname{depth} R = 0$, *then all R-modules with finite $G_c(R)$-dimension belong to $G_c(R)$.*

Proof. Let n be a positive integer and M a nonzero R-module with $G_c(R)\text{-}\dim M \leqslant n$. We proceed by induction on n. First we assume $G_c(R)\text{-}\dim M \leqslant 1$. Then $\operatorname{Ext}^i_R(M, C) = 0$ for all $i \geqslant 2$ by [17, Proposition 3.13]. It is enough to prove $\operatorname{Ext}^1_R(M, C) = 0$ by [17, Proposition 3.13] again since $G_c(R)\text{-}\dim M < \infty$. By assumption, we have the exact sequence $0 \to G_1 \to G_0 \to M \to 0$ with $G_i \in G_c(R)$ for $i = 0, 1$. Since $\operatorname{Ext}^1_R(G_0, C) = 0$, applying the functor $\operatorname{Hom}_R(-, C)$ twice to this sequence we have the exact sequence $0 \to (\mathrm{Ext}^1_R(M,C))^* \to (G_1)^{**} \to (G_0)^{**}$. So $(\mathrm{Ext}^1_R(M,C))^* = 0$ since $(G_i)^{**} \cong G_i$ for $i = 0,1$. Thus, by [3, Proposition 10, p. 267], we have

$$\emptyset = \mathrm{Ass}((\mathrm{Ext}^1_R(M,C))^*) = \mathrm{Ass}\, C \cap \mathrm{Supp}(\mathrm{Ext}^1_R(M,C)).$$

Since $\operatorname{depth} C = \operatorname{depth} R = 0$ by Corollary 3.2, $\mathrm{Ass}\, C = \{m\}$. So m does not belong to $\mathrm{Supp}(\mathrm{Ext}^1_R(M,C))$, and then $\mathrm{Ext}^1_R(M,C) = 0$.

Now suppose that $G_c(R)$-$\dim M \leqslant n-1$ implies $M \in G_c(R)$. If $G_c(R)$-$\dim M \leqslant n$, then we have the exact sequence $0 \to K \to G \to M \to 0$ with $G_c(R)$-$\dim K \leqslant n-1$ and $G \in G_c(R)$. So $K \in G_c(R)$ by the induction hypothesis, and then $G_c(R)$-$\dim M \leqslant 1$. Therefore $M \in G_c(R)$ by the case $n = 1$ already proved. $\square$

Lemma 4.2. *Let (R,m) be a local ring and M a nonzero R-module with finite $G_c(R)$-dimension. Then the following assertions are equivalent:*

(1) $M \in G_c(R)$.
(2) $\operatorname{depth} M = \operatorname{depth} C$.
(3) $\mathrm{Ext}^i_R(M,C) = 0$ *for all* $i > 0$.
(4) $\mathrm{Ext}^i_R(M,N) = 0$ *for all* $i > 0$ *and all R-modules N with finite* $\mathrm{add}\, C$*-dimension.*

P r o o f. (1) $\Rightarrow$ (2). It is enough to show that $\operatorname{depth} M \leqslant \operatorname{depth} C$ by Remark 3.6 and Corollary 3.2. If $\operatorname{depth} C = 0$ then $\mathrm{Ass}\, C = \{m\}$. Since $M \neq 0$ and $M \cong M^{**}$ by $M \in G_c(R)$, we have $\mathrm{Ass}\, M = \mathrm{Ass}\, C \cap \mathrm{Supp}\, M^* = \{m\}$. So $\operatorname{depth} M = 0$.

Now suppose that $\operatorname{depth} R \geqslant 1$ and that the implication holds for all rings of smaller depth. Since $\operatorname{depth} M \geqslant \operatorname{depth} R \geqslant 1$, we can find $x \in R$ such that x is both R and M-regular, and then C-regular. Thus $\operatorname{depth} \overline{M} = \operatorname{depth} M - 1$, $\overline{M} \in G_{\bar{c}}(\overline{R})$ by Corollary 3.5, and $\operatorname{depth} \overline{R} = \operatorname{depth} \overline{C} = \operatorname{depth} C - 1 = \operatorname{depth} R - 1$. So we have $\operatorname{depth} \overline{M} = \operatorname{depth} \overline{C}$ by the induction hypothesis, and then $\operatorname{depth} M = \operatorname{depth} C$.

(2) $\Rightarrow$ (3). We prove (3) by induction on $\operatorname{depth} R$. If $\operatorname{depth} R = 0$ then $M \in G_c(R)$ by Lemma 4.1 since M has finite $G_c(R)$-dimension. So $\mathrm{Ext}^i_R(M,C) = 0$ for all $i > 0$ by [17, Proposition 3.13]. Assume that $\operatorname{depth} R \geqslant 1$ and the implication holds for all rings of smaller depth. As above we can choose $x \in R$ such that x is both R and M-regular and C-regular since $\operatorname{depth} M = \operatorname{depth} C = \operatorname{depth} R \geqslant 1$ by assumption. Then $\operatorname{depth} \overline{M} = \operatorname{depth} M - 1 = \operatorname{depth} R - 1 = \operatorname{depth} \overline{R} = \operatorname{depth} \overline{C}$, and $G_{\bar{c}}(\overline{R})$-$\dim \overline{M} < \infty$ by Proposition 3.7. So $\mathrm{Ext}^i_{\overline{R}}(\overline{M},\overline{C}) = 0$ for all $i > 0$ by the induction hypothesis. Since x is regular on both R and M, $\mathrm{Tor}^R_i(M,\overline{R}) = 0$ for all $i > 0$ by [1, Lemma 4.7]. Therefore $\mathrm{Ext}^i_R(M,\overline{C}) \cong \mathrm{Ext}^i_{\overline{R}}(\overline{M},\overline{C}) = 0$ for all $i > 0$ by [4, Proposition 4.1.3, p. 118]. On the other hand, since x is C-regular, there exists an exact sequence $0 \to C \xrightarrow{x} C \to \overline{C} \to 0$ which yields the exact

sequence $\mathrm{Ext}^i_R(M,C) \xrightarrow{x} \mathrm{Ext}^i_R(M,C) \to \mathrm{Ext}^i_R(M,\overline{C}) = 0$ for all $i > 0$. Therefore $\mathrm{Ext}^i_R(M,C) = 0$ for all $i > 0$ by Nakayama's Lemma.

(3) $\Rightarrow$ (4) is clear by the usual dimension shifting argument and (4) $\Rightarrow$ (1) is immediate by [17, Proposition 3.13] since $G_c(R)$-$\dim M < \infty$. $\square$

Lemma 4.3. *Let (R,m,k) be a local ring with k the residue field and M an R-module. If $\operatorname{depth} C = d$ and $G_c(R)$-$\dim M = 1$ then $\operatorname{depth} M = d-1$.*

Proof. We prove this by induction on d. Since $\operatorname{depth} R = \operatorname{depth} C$, it is trivial when $d = 0$ by Lemma 4.1. Let $d \geqslant 1$ and let the equality hold for all rings of smaller depth. Since $G_c(R)$-$\dim M = 1$, there is an exact sequence $0 \to G_1 \to G_0 \to M \to 0$ with $G_i \in G_c(R)$ for $i = 0,1$. Then we have the exact sequence $\mathrm{Ext}^i_R(k,G_0) \to \mathrm{Ext}^i_R(k,M) \to \mathrm{Ext}^{i+1}_R(k,G_1)$ for $i \geqslant 1$. Because $\operatorname{depth} G_0 = \operatorname{depth} G_1 = d$ by Lemma 4.2, we have $\mathrm{Ext}^i_R(k,M) = 0$ for $i \leqslant d-2$. So $\operatorname{depth} M \geqslant d-1$. If $\operatorname{depth} M \geqslant d$ then we can choose an element $x \in m$ which is both R and M-regular, and also C-regular. Thus $G_{\bar{c}}(\overline{R})$-$\dim \overline{M} = 1$ by Proposition 3.7, and $\operatorname{depth} \overline{C} = \operatorname{depth} \overline{R} = d-1$, $\operatorname{depth} \overline{M} = \operatorname{depth} M - 1 \geqslant d-1$. On the other hand, we have $\operatorname{depth} \overline{M} = \operatorname{depth} \overline{C} - 1 = d-2$ by the induction hypothesis. This is a contradiction. So $\operatorname{depth} M = d-1$. $\square$

Now we are in position to prove the theorem from Introduction.

Theorem 4.4. *Let (R,m) be a commutative Noetherian local ring and M a nonzero R-module with finite $G_c(R)$-dimension. Then $G_c(R)$-$\dim M + \operatorname{depth} M = \operatorname{depth} C$.*

Proof. Let $G_c(R)$-$\dim M = n$. We proceed by induction on n. If $n = 0$, the result is contained in Lemma 4.2. If $n \geqslant 1$, then $\operatorname{depth} C = \operatorname{depth} R = d \geqslant 1$ by Lemma 4.1, and the case of $n = 1$ is immediate by Lemma 4.3. We now suppose $n \geqslant 2$. Then there is an exact sequence $0 \to K \to G \to M \to 0$ with $G \in G_c(R)$ and $G_c(R)$-$\dim K = n-1$. So $\operatorname{depth} G = \operatorname{depth} C = d$ by Lemma 4.2, and $\operatorname{depth} K = d-(n-1)$ by the induction hypothesis. Thus $\operatorname{depth} K < \operatorname{depth} G$ since $n \geqslant 2$. So we have the isomorphisms $\mathrm{Ext}^i_R(k,M) \cong \mathrm{Ext}^{i+1}_R(k,K)$ for all $i \leqslant d-2$, and hence $\operatorname{depth} M = d-n$. $\square$

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