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CANONICAL CHARACTERS ON SIMPLE GRAPHS

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Abstract. A multiplicative functional on a graded connected Hopf algebra is called the character. Every character decomposes uniquely as a product of an even character and an odd character. We apply the character theory of combinatorial Hopf algebras to the Hopf algebra of simple graphs. We derive explicit formulas for the canonical characters on simple graphs in terms of coefficients of the chromatic symmetric function of a graph and of canonical characters on quasi-symmetric functions. These formulas and properties of characters are used to derive some interesting numerical identities relating multinomial and central binomial coefficients.

Keywords: Hopf algebra, simple graph, quasi-symmetric function, character

MSC 2010: 05C25, 16T30, 05E05

1. Introduction

Many combinatorial objects with the natural meaning of compositions and decompositions produce the structure of a graded Hopf algebra. The classical example is the Hopf algebra of simple graphs, introduced in [3]. The combinatorial Hopf algebra $(\mathcal{H},\zeta)$ is a graded connected Hopf algebra $\mathcal{H}$ equipped with a multiplicative functional ζ, called the character. The character theory on Hopf algebras is developed in [1]. The Hopf algebra of quasi-symmetric functions $Q\mathrm{Sym}$, equipped with the universal character ζ_Q, is the terminal object in the category of combinatorial Hopf algebras. This universal property produces the natural morphism from combinatorial objects to quasi-symmetric functions. Many enumerative combinatorial invariants are obtained in this way, among them Stanley's chromatic symmetric function of simple graphs [4].

Every character on a graded connected Hopf algebra decomposes uniquely as a product of an even character and an odd character. Explicit formulas for the

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even and odd parts of the universal character ζ_Q on the Hopf algebra of quasi-symmetric functions $Q\mathrm{Sym}$ are obtained in [2]. These formulas are used to derive some interesting identities relating bivariate Catalan numbers and central binomial coefficients.

In this paper we use the chromatic symmetric function of a graph and the canonical characters on quasi-symmetric functions to derive explicit formulas for the canonical characters on simple graphs. These formulas are used to obtain some numerical identities relating multinomial and central binomial coefficients.

2. Canonical characters on quasi-symmetric functions

Definition 2.1. A combinatorial Hopf algebra $(\mathcal{H}, \zeta)$ is a graded connected Hopf algebra $\mathcal{H} = \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$ over a field $\mathbb{K}$ equipped with a multiplicative linear functional $\zeta\colon \mathcal{H} \to \mathbb{K}$, called a character.

A morphism of combinatorial Hopf algebras $\varphi\colon (\mathcal{H}_1, \zeta_1) \to (\mathcal{H}_2, \zeta_2)$ is a morphism of graded Hopf algebras $\varphi\colon \mathcal{H}_1 \to \mathcal{H}_2$ such that $\zeta_2 \circ \varphi = \zeta_1$.

The set of characters of an arbitrary Hopf algebra forms a group under the convolution product

$$\varphi\psi = m_{\mathbb{K}} \circ (\varphi \otimes \psi) \circ \Delta_{\mathcal{H}},$$

with the unit $\varepsilon_{\mathcal{H}}$ and the inverse $\varphi^{-1} = \varphi \circ S_{\mathcal{H}}$, where $\varepsilon_{\mathcal{H}}$ and $S_{\mathcal{H}}$ are the counit and the antipode of the Hopf algebra $\mathcal{H}$.

Any graded Hopf algebra $\mathcal{H}$ carries the canonical involution

$$h \mapsto \bar{h} = (-1)^n h,$$

for homogeneous elements $h \in \mathcal{H}_n$. This induces the involution $\varphi \mapsto \bar{\varphi}$ on the character group, where the conjugate character $\bar{\varphi}$ is defined on homogeneous elements by

$$\bar{\varphi}(h) = \varphi(\bar{h}).$$

A character φ is said to be even if $\bar{\varphi} = \varphi$ and it is said to be odd if $\bar{\varphi} = \varphi^{-1}$. Any character φ decomposes uniquely as a product of characters

$$\varphi = \varphi_+ \varphi_-,$$

with φ_+ even and φ_- odd, [1, Theorem 1.5.].

Definition 2.2. The canonical characters of a combinatorial Hopf algebra $(\mathcal{H}, \zeta)$ are the even and odd parts ζ_+ and ζ_- of the character ζ.

Suppose that $\varphi\colon (\mathcal{H}_1, \zeta_1) \to (\mathcal{H}_2, \zeta_2)$ is a morphism of combinatorial Hopf algebras. Then φ preserves the canonical characters [2, Lemma 2.2.].

$$(\zeta_2)_+ \circ \varphi = (\zeta_1)_+, \ (\zeta_2)_- \circ \varphi = (\zeta_1)_-.$$

2.1. Quasi-symmetric functions. Let $R[[x_1, x_2, \ldots]]$ be the algebra of formal power series in countably commuting variables over the commutative ring R. It is a graded algebra by letting all variables to have the same rank $\mathrm{rk}(x_n) = 1, n \in \mathbb{N}$. The coefficients of monomials $x_{i_1}^{a_1} x_{i_2}^{a_2} \ldots x_{i_k}^{a_k}$ are indexed by all pairs of strictly increasing sequences $i_1 < i_2 < \ldots < i_k$ indexing variables and sequences $(a_1, a_2, \ldots, a_k)$ of exponents. A power series $f \in R[[x_1, x_2, \ldots]]$ of a bounded degree is a quasi-symmetric function if all monomials with the same exponents have equal coefficients. The algebra $Q\mathrm{Sym} = \bigoplus_{n \geqslant 0} Q\mathrm{Sym}_n$ of quasi-symmetric functions is a graded subalgebra of the algebra $R[[x_1, x_2, \ldots]]$. It is linearly spanned by monomial quasi-symmetric functions

$$M_\alpha = \sum_{i_1 < i_2 < \ldots < i_k} x_{i_1}^{a_1} x_{i_2}^{a_2} \ldots x_{i_k}^{a_k},$$

defined for any positive integer n and a composition $\alpha = (a_1, a_2, \ldots a_k) \models n$. The multiplication on the monomial basis is given by the quasi-shuffle product q on the set of compositions

$$M_\alpha M_\beta = \sum_{\gamma = q(\alpha, \beta)} M_\gamma.$$

The algebra $Q\mathrm{Sym}$ is a graded Hopf algebra with the comultiplication given by

$$\Delta(M_\alpha) = \sum_{\alpha = \beta\gamma} M_\beta \otimes M_\gamma,$$

where $\beta\gamma$ is the concatenation of compositions β and γ.

Let $\zeta\colon\ R[[x_1, x_2, \ldots]] \to R$ be a ring morphism defined on variables by

$$\zeta(x_i) = \begin{cases} 1, & i = 1, \\ 0, & i \neq 1. \end{cases}$$

The universal character ζ_Q on $Q\mathrm{Sym}$ is the restriction $\zeta_Q = \zeta|_{Q\mathrm{Sym}}$, determined on the monomial basis by

$$\zeta_Q(M_\alpha) = \begin{cases} 1, & \alpha = (n) \text{ or } (\ ), \\ 0, & \text{otherwise}. \end{cases}$$

For an arbitrary combinatorial Hopf algebra $(\mathcal{H}, \zeta)$, there is a unique morphism of combinatorial Hopf algebras $\Psi\colon (\mathcal{H}, \zeta) \to (Q\mathrm{Sym}, \zeta_Q)$, [1, Theorem 4.1.]. It is defined on homogeneous elements $h \in \mathcal{H}_n$ by

$$\Psi(h) = \sum_{\alpha \models n} \zeta_\alpha(h) M_\alpha, \tag{2.1}$$

where, for a composition $\alpha = (a_1, \ldots, a_k) \models n$, ζ_α is the convolution product

$$\zeta_{a_1} \ldots \zeta_{a_k} \colon \mathcal{H} \xrightarrow{\Delta^{(k-1)}} \mathcal{H}^{\otimes k} \xrightarrow{\mathrm{proj}} \mathcal{H}_{a_1} \otimes \ldots \otimes \mathcal{H}_{a_k} \xrightarrow{\zeta^{\otimes k}} \mathbb{K}.$$

The morphism Ψ we call the canonical morphism of the combinatorial Hopf algebra $(\mathcal{H}, \zeta)$.

2.2. Canonical characters of QSym. The canonical characters $(\zeta_Q)_+$ and $(\zeta_Q)_-$ of the Hopf algebra of quasi-symmetric functions QSym are explicitly calculated in [2, Theorem 3.2]. For a composition $\alpha = (a_1, \ldots, a_k) \models n$, let $k_e(\alpha)$ and $k_o(\alpha)$ denote the numbers of the even and odd components of α. Let $C(0, m) = \binom{2m}{m}$ and $\frac{1}{2}C(1, m) = (m+1)^{-1}\binom{2m}{m}$ be the central binomial coefficients and the Catalan numbers. Then

$$(\zeta_Q)_-(M_\alpha) = \begin{cases} \dfrac{(-1)^{k_e(\alpha)}}{2^{2\lfloor k_o(\alpha)/2 \rfloor}} C(0, \lfloor k_o(\alpha)/2 \rfloor), & a_k \text{ odd}, \\ 0, & a_k \text{ even}; \end{cases} \tag{2.2}$$

$$(\zeta_Q)_+(M_\alpha) = \begin{cases} \dfrac{(-1)^{k_e(\alpha)+1}}{2^{k_o(\alpha)}} C(1, k_o(\alpha)/2 - 1), & a_1, a_k \text{ odd}, n \text{ even}, \\ 1, & \alpha = (n), n \text{ even}, \\ 0, & \text{otherwise}. \end{cases} \tag{2.3}$$

The even and odd parts of the inverse ζ_Q^{-1} of the universal character, calculated in [2, Theorem 8.4.], are given by

$$(\zeta_Q^{-1})_-(M_\alpha) = \begin{cases} \dfrac{(-1)^{k(\alpha)}}{2^{2\lfloor k_o(\alpha)/2 \rfloor}} C(0, \lfloor k_o(\alpha)/2 \rfloor), & a_1 \text{ odd}, \\ 0, & a_1 \text{ even}; \end{cases} \tag{2.4}$$

$$(\zeta_Q^{-1})_+(M_\alpha) = \begin{cases} \dfrac{(-1)^{k(\alpha)}}{2^{k_o(\alpha)}} C(0, k_o(\alpha)/2), & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases} \tag{2.5}$$

3. Canonical characters on simple graphs

Recall the definition of the Hopf algebra of simple graphs $\mathcal{G}$. The graph $\Gamma = (V(\Gamma), E(\Gamma))$ with the set of vertices $V(\Gamma)$ and with the set of edges $E(\Gamma)$ is called simple if there are neither multiple edges nor loops. Let $\mathcal{G}$ be the $\mathbb{K}$-vector space spanned by all equivalence classes of finite simple graphs. It is graded by the rank function

$$\mathcal{G} = \bigoplus_{n \geqslant 0} \mathcal{G}_n,$$

where $\mathcal{G}_n$ is the $\mathbb{K}$-vector space spanned by all equivalence classes of simple graphs on n vertices. The space $\mathcal{G}$ is a Hopf algebra with the multiplication given by disjoint union of graphs $\Gamma_1 \cdot \Gamma_2 = \Gamma_1 \sqcup \Gamma_2$ and the comultiplication given by

$$\Delta(\Gamma) = \sum_{I \subset V(\Gamma)} \Gamma|_I \otimes \Gamma|_{I^c},$$

where $\Gamma|_I$ is the restriction of the graph Γ to the set $I \subset V(\Gamma)$.

The Hopf algebra $\mathcal{G}$ is graded, commutative and cocommutative. Define a character ζ on $\mathcal{G}$ by

$$\zeta(\Gamma) = \begin{cases} 1, & \Gamma \text{ is discrete}, \\ 0, & \text{otherwise}. \end{cases}$$

The canonical morphism $\Psi\colon (\mathcal{G}, \zeta) \to (Q\mathrm{Sym}, \zeta_Q)$ of combinatorial Hopf algebra $(\mathcal{G}, \zeta)$ is defined on the n-th homogeneous element $\Gamma \in \mathcal{G}_n$ by

$$\Psi(\Gamma) = \sum_{\alpha \models n} \zeta_\alpha(\Gamma) M_\alpha.$$

The image of a graph Γ under the morphism Ψ is Stanley's chromatic function of the graph Γ.

Theorem 3.1. *The canonical characters ζ_+ and ζ_- of a combinatorial Hopf algebra $(\mathcal{H}, \zeta)$ are given on homogeneous elements $h \in \mathcal{H}_n$ by*

$$\zeta_\pm(h) = \sum_{\alpha \models n} \zeta_\alpha(h)(\zeta_Q)_\pm(M_\alpha). \tag{3.1}$$

The even and odd parts $(\zeta^{-1})_+$ and $(\zeta^{-1})_-$ of the inverse character ζ^{-1} are given on homogeneous elements $h \in \mathcal{H}_n$ by

$$(\zeta^{-1})_\pm(h) = \sum_{\alpha \models n} \zeta_\alpha(h)(\zeta_Q^{-1})_\pm(M_\alpha). \tag{3.2}$$

P r o o f. Let $\Psi\colon (\mathcal{G},\zeta)\to(Q\mathrm{Sym},\zeta_Q)$ be the canonical morphism of the combinatorial Hopf algebra $(\mathcal{H},\zeta)$, given by (2.1). The identity (3.1) follows at once from the identity

$$\zeta_\pm = (\zeta_Q)_\pm \circ \Psi.$$

Let $S_\mathcal{H}$ and S_Q be the antipodes of the Hopf algebras $\mathcal{H}$ and $Q\mathrm{Sym}$. The inverse characters ζ^{-1} and ζ_Q^{-1} are given by $\zeta^{-1}=\zeta\circ S_\mathcal{H}$ and $\zeta_Q^{-1}=\zeta_Q\circ S_Q$. The morphism Ψ, as a morphism of Hopf algebras, commutes with the antipodes, $S_Q\circ\Psi=\Psi\circ S_\mathcal{H}$. Hence,

$$\zeta_Q^{-1}\circ\Psi=\zeta_Q\circ S_Q\circ\Psi=\zeta_Q\circ\Psi\circ S_\mathcal{H}=\zeta\circ S_\mathcal{H}=\zeta^{-1}.$$

It means that $\Psi\colon(\mathcal{H},\zeta^{-1})\to(Q\mathrm{Sym},\zeta_Q^{-1})$ is a morphism of combinatorial Hopf algebras. Therefore,

$$(\zeta^{-1})_\pm=(\zeta_Q^{-1})_\pm\circ\Psi.$$

This proves the identity (3.2). □

Corollary 3.1. *For the combinatorial Hopf algebra of simple graphs $(\mathcal{G},\zeta)$ we have the following identities:*

$$\zeta(\Gamma)=\sum_{J\subset V(\Gamma)}\sum_{\alpha\models|J|,\beta\models|J^c|}\zeta_\alpha(\Gamma|_J)\zeta_\beta(\Gamma|_{J^c})(\zeta_Q)_+(M_\alpha)(\zeta_Q)_-(M_\beta), \tag{3.3}$$

$$\zeta^{-1}(\Gamma)=\sum_{J\subset V(\Gamma)}\sum_{\alpha\models|J|,\beta\models|J^c|}\zeta_\alpha(\Gamma|_J)\zeta_\beta(\Gamma|_{J^c})(\zeta_Q^{-1})_+(M_\alpha)(\zeta_Q^{-1})_-(M_\beta). \tag{3.4}$$

P r o o f. The given identities are obtained by inserting the formulas (3.1) and (3.2) for the Hopf algebra $\mathcal{G}$ into the identities

$$\zeta=\zeta_+\zeta_- \text{ and } \zeta^{-1}=(\zeta^{-1})_+(\zeta^{-1})_-.$$

□

3.1. The application. The formulas (3.1), (3.2), (3.3) and (3.4) produce various numerical identities depending on the combinatorics of graphs. We illustrate this in the following propositions. Denote by $\binom{n}{\alpha}=\binom{n}{a_1a_2\ldots a_k}=\frac{n!}{a_1!a_2!\ldots a_k!}$ the multinomial coefficient corresponding to the composition $\alpha=(a_1,a_2,\ldots,a_k)\models n$.

Proposition 3.1. *For any even positive integer n,*

$$1 + \sum_{\alpha \models n \; a_1, a_{k(\alpha)} \text{ odd}} \binom{n}{\alpha} \frac{(-1)^{k_e(\alpha)+1}}{2^{k_o(\alpha)}} C(1, k_o(\alpha)/2 - 1) = 0,$$

$$\sum_{\alpha \models n} \binom{n}{\alpha} \frac{(-1)^{k(\alpha)}}{2^{k_o(\alpha)}} C(0, k_o(\alpha)/2) = 0.$$

P r o o f. Denote by D_n the discrete graph on n vertices. Since the characters ζ_+ and ζ_+^{-1} are even and multiplicative, we have that $\zeta_+(D_n) = \zeta_+^{-1}(D_n) = 0$ for any positive integer n. The above identities are obtained from (2.3) and (2.5) by setting $\Gamma = D_n$ in (3.1) and (3.2). □

Proposition 3.2. *For any positive integer m,*

$$\sum_{l=1}^{m} \frac{1}{l} \binom{2l-2}{l-1} \binom{2m-2l}{m-l} = \frac{1}{2} \binom{2m}{m},$$

$$\sum_{l=0}^{m} \binom{2l}{l} \binom{2m-2l}{m-l} = 2^{2m}.$$

P r o o f. Let K_n be the complete graph on n vertices. Then

$$\zeta_\alpha(K_n) = \begin{cases} n!, & \alpha = (1)^n, \\ 0, & \alpha \neq (1)^n. \end{cases}$$

The above identities are obtained from (2.2), (2.3), (2.4) and (2.5) by setting $\Gamma = K_{2m}$ in (3.3) and (3.4). □

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