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STATE-SPACE REALIZATION OF NONLINEAR CONTROL SYSTEMS: UNIFICATION AND EXTENSION VIA PSEUDO-LINEAR ALGEBRA

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In this paper the tools of pseudo-linear algebra are applied to the realization problem, allowing to unify the study of the continuous- and discrete-time nonlinear control systems under a single algebraic framework. The realization of nonlinear input-output equation, defined in terms of the pseudo-linear operator, in the classical state-space form is addressed by the polynomial approach in which the system is described by two polynomials from the non-commutative ring of skew polynomials. This allows to simplify the existing step-by-step algorithm-based solution. The paper presents explicit formulas to compute the differentials of the state coordinates directly from the polynomial description of the nonlinear system. The method is straight-forward and better suited for implementation in different computer algebra packages such as *Mathematica* or *Maple*.

Keywords: nonlinear control systems, input-output models, realization, pseudo-linear algebra

Classification: 93E12, 62A10

1. INTRODUCTION

This paper has to be understood as continuation of research in [11], the polynomial tools we employ are the same as those in [11]. The former paper applied the pseudo-linear algebra to unify the results on system reduction for continuous- and discrete-time nonlinear control systems whereas this paper focuses on the closely related problem of state-space realization. Namely, the realization procedure ends up with the controllable (accessible) realization iff the input-output (i/o) equation is reduced to the simplest form, being transfer equivalent to the original equation. The explicit polynomial formulas for finding the differentials of the state coordinates have been represented earlier separately for continuous-time [20], shift operator based discrete-time [14] and the difference operator based discrete-time case [2], respectively. In this paper we derive general tools which allow to formalize and handle different cases under a single unified framework. The results of those papers follow as special cases from the results of this paper. In computation of the differentials of the state variables left[1] polynomial division operation is applied

[1]Note that polynomials are from the non-commutative ring.

repeatedly. Our method uses left quotients of polynomials and thus it is different from Euclidean division algorithm, producing the sequence of remainders.

The realization problem has been addressed in many papers, but the polynomial method has several advantages if compared with the earlier algorithm-related methods, based either on the sequence of the subspaces of differential one-forms [6], on the sequence of distributions of vector fields as in [18], on the iterative Lie brackets of the vector fields as in [7]. The most powerful argument is computation speed, but also the program code is shorter and more compact. What is also important, these formulas practically coincide in the special case of linear time-invariant (LTI) systems with the respective formulas of the LTI systems theory [17], except that in the linear case the polynomials may be understood as operators that are applied to the variables u and y whereas in the nonlinear case they are applied to their differentials. The latter aspect brings along the integrability restriction which is a well-known phenomena since it is known for a long time that nonlinear i/o equations, unlike linear i/o equations, are not always realizable in the state space form.

Note that necessary and sufficient realizability conditions for nonlinear i/o equations, defined in terms of the pseudo-linear operator, have been given in [12]. Whereas the paper [12] suggests that the state coordinates may be found by integrating the exact basis vectors of certain vector space of differential one-forms, it did not address the computation of this vector space. The main result of this paper is to provide explicit formulas for computation of the basis one-forms of the above vector space. This is achieved by using the polynomial framework built upon the formalism of differential one-forms, like in [11].

The paper is organized as follows. Section 2 recalls the basic notions of the algebraic framework used in this paper. The next section presents the solution of the realization problem. In Section 4 polynomial formulas within the context of pseudo-linear algebra are illustrated on several examples, followed by a brief description of implementation of the theoretical results from this paper and from [11] in *Mathematica* within the symbolic software package NLControl. Concluding remarks are drawn in the last section.

2. ALGEBRAIC FRAMEWORK

In this paper, the realization problem is stated and solved in a unified manner. In particular, this means that both the i/o and state equations are described in terms of the pseudo-linear operator, and the formulas to find the state coordinates are also given in terms of these operators. For the special cases of continuous- and discrete-time systems, these operators take the form of differential, difference or shift operators. Below we shortly recall the algebraic setup from [11] that we use in this paper, see also [3].

Let $\mathcal{K}$ be a field and $\sigma : \mathcal{K} \to \mathcal{K}$ an automorphism of $\mathcal{K}$. A map $\delta : \mathcal{K} \to \mathcal{K}$ which satisfies

$$\begin{aligned} \delta(a+b) &= \delta(a) + \delta(b), \\ \delta(ab) &= \sigma(a)\delta(b) + \delta(a)b \end{aligned}$$

for $a, b \in \mathcal{K}$, is called a pseudo- or σ-derivation. A σ-differential field is a triple $(\mathcal{K}, \sigma, \delta)$, where $\mathcal{K}$ is a field, σ is an automorphism of $\mathcal{K}$ and δ is a σ-derivation. Hereinafter $(\mathcal{K}, \sigma, \delta)$ will be denoted by $\mathcal{K}$.

Let V be a vector space over the field $\mathcal{K}$. An operator $\theta : V \to V$ is called pseudo-linear if

$$\begin{aligned} \theta(v+w) &= \theta(v)+\theta(w), \\ \theta(aw) &= \sigma(a)\theta(w)+\delta(a)w \end{aligned} \tag{1}$$

for any $a \in \mathcal{K}$, $v, w \in V$. Note that any field $\mathcal{K}$ is a vector space itself. Hence, (1) holds for any $a, v, w \in \mathcal{K}$. Any pseudo-derivation $\delta : \mathcal{K} \to \mathcal{K}$ is a pseudo-linear operator by letting $\theta = \delta$. Also for a shift operator, when $\delta = 0$, (1) is clearly satisfied by letting $\theta = \sigma$. Thus, pseudo-linear operators allow to handle differential, shift and difference structures from a unified standpoint. The basic types of operators that can be addressed within the pseudo-linear algebra are listed in Table 1.

Tab. 1. Basic types of operators.

Operator	σ	δ	θ	$f^{\langle 1\rangle}(t)$
differential	$\mathrm{id}_{\mathcal{K}}$	$\frac{\mathrm{d}}{\mathrm{d}t}$	δ	$\frac{\mathrm{d}f(t)}{\mathrm{d}t}$
shift	σ	0	σ	$f(t+1)$
difference	σ	Δ	δ	$\frac{1}{\mu}(f(t+1)-f(t))$

Hereinafter we use the abridged notation $\theta(y(t)) = y^{\langle 1\rangle}$. It can be a derivation $y^{\langle 1\rangle} = \dot{y}$ that corresponds to the continuous-time case, a shift $y^{\langle 1\rangle} = \sigma(y)$, or a difference $y^{\langle 1\rangle} = \frac{1}{\mu}(\sigma(y) - y)$ with $\mu \in \mathbb{R}$ that correspond to two alternative discrete-time cases. Moreover, we use notation $\theta^k(y(t)) = y^{\langle k\rangle}$ for the k-fold application of the pseudo-linear operator.

Consider a nonlinear control system, described by the i/o equation

$$y^{\langle n\rangle} = \phi\left(y, \ldots, y^{\langle n-1\rangle}, u, \ldots, u^{\langle s\rangle}\right), \tag{2}$$

where $u, y \in \mathbb{R}$ are the input and the output of the system, respectively, ϕ is a real analytic function, and n, s are non-negative integers such that $s < n$. Assume that system (2) is generically submersive, i.e.

$$\operatorname{rank} \frac{\partial \sigma^n(y)}{\partial(y,u)} \not\equiv 0. \tag{3}$$

Note that assumption (3) is not restrictive since it is necessary condition for system accessibility. Besides, it reduces to the well-known condition in case of the discrete-time nonlinear systems when $y^{\langle 1\rangle} = \sigma(y)$ [8], and is trivially satisfied in case of the continuous-time systems $y^{\langle 1\rangle} = \dot{y}$ when $\sigma(y) = y$.

Let $\mathcal{K}$ denote from now on the field of meromorphic functions in the independent system variables $\mathcal{C} = \left\{y, y^{\langle 1\rangle}, \ldots, y^{\langle n-1\rangle}, u^{\langle k\rangle}, k \geq 0\right\}$ and let δ be a pseudo-derivation defined on $\mathcal{K}$. The field $\mathcal{K}$ may be endowed with a σ-differential structure $(\mathcal{K}, \sigma, \delta)$, determined by the system equations (2), see [11]. Define a pseudo-linear operator $\theta : \mathcal{K} \to \mathcal{K}$ as follows

$$\theta(\zeta) = \begin{cases} \delta(\zeta), & \text{if } \delta \neq 0 \\ \sigma(\zeta), & \text{if } \delta = 0. \end{cases}$$

Moreover, it should be mentioned that application of the operator θ to $y^{\langle n-1\rangle}$ results in $y^{\langle n\rangle}$ which, according to (2), has to be replaced by $\phi(\cdot)$, whenever it occurs in some expression.

Under assumption (3), there exists, up to an isomorphism, a unique difference overfield $\mathcal{K}^* \supseteq \mathcal{K}$, called the inversive closure of $\mathcal{K}$, with σ being an automorphism of $\mathcal{K}^*$, see [5]. An explicit construction of inversive closure is given in [1] and [9] for the cases when θ is the difference or shift operator, respectively. In the continuous-time case when $\sigma = \mathrm{id}_{\mathcal{K}}$, $\mathcal{K}^* = \mathcal{K}$.

In general, the new independent variables of the (isomorphic) field extension may be chosen in two different ways, either as $\sigma^{-k}(y)$, $k \geq 1$, or as $\sigma^{-k}(u)$, $k \geq 1$. Here the σ^{-k} means the k-time application of the backward-shift operator σ^{-1}. The other variables, that is, $\sigma^{-k}(u)$, or $\sigma^{-k}(y)$, respectively, may be calculated from the i/o equation (2), applying to it σ^{-1} the required number of times. Over the field $\mathcal{K}^*$ one can define the vector space $\mathcal{E} := \mathrm{span}_{\mathcal{K}^*}\mathrm{d}\mathcal{C}$ of differential one-forms, where either

$$\mathrm{d}\mathcal{C} = \left\{\mathrm{d}y, \mathrm{d}y^{\langle 1\rangle}, \ldots, \mathrm{d}y^{\langle n-1\rangle}, \mathrm{d}y^{\langle -k\rangle}, k \geq 1, \mathrm{d}u^{\langle l\rangle}, l \geq 0\right\}$$

or

$$\mathrm{d}\mathcal{C} = \left\{\mathrm{d}y, \mathrm{d}y^{\langle 1\rangle}, \ldots, \mathrm{d}y^{\langle n-1\rangle}, \mathrm{d}u^{\langle l\rangle}, l \geq 0, \mathrm{d}u^{\langle -k\rangle}, k \geq 1\right\},$$

respectively. The space $\mathcal{E}$ may be also endowed with the pseudo-linear operator $\theta : \mathcal{E} \to \mathcal{E}$ as follows

$$\theta(\alpha\mathrm{d}\zeta) = \sigma(\alpha)\mathrm{d}(\theta(\zeta)) + \delta(\alpha)\mathrm{d}\zeta.$$

Note that the operator θ commutes with the operator d, $\theta(\mathrm{d}\varphi) = \mathrm{d}(\theta(\varphi))$.

A *left* polynomial can be uniquely written in the form $a = \sum_{i=0}^{n} \alpha_i z^{n-i}$, $\alpha_i \in \mathcal{K}^*$. If $\alpha_0 \not\equiv 0$, then n is called the degree of a, denoted by $\deg(a)$. The pseudo-linear operator θ induces a (left) skew polynomial ring of polynomials in z (if z is interpreted as θ) over $\mathcal{K}^*$ with the commutation rule given by

$$z \cdot \alpha = \sigma(\alpha)z + \delta(\alpha) \tag{4}$$

for any $\alpha \in \mathcal{K}^*$. A ring is called an *integral domain*, if it does not contain any zero divisors. The ring $\mathcal{K}^*[z; \sigma, \delta]$ is an integral domain [15].

The nonlinear system (2) may be represented in terms of two skew polynomials in the ring $\mathcal{K}^*[z; \sigma, \delta]$, since by differentiating (2) we obtain

$$\mathrm{d}y^{\langle n\rangle} - \sum_{i=0}^{n-1} \frac{\partial\phi}{\partial y^{\langle i\rangle}}\mathrm{d}y^{\langle i\rangle} - \sum_{j=0}^{s} \frac{\partial\phi}{\partial u^{\langle j\rangle}}\mathrm{d}u^{\langle j\rangle} = 0 \tag{5}$$

which may be rewritten as

$$p\mathrm{d}y + q\mathrm{d}u = 0, \tag{6}$$

where $p = z^n - \sum_{i=0}^{n-1} p_i z^i$, $q = -\sum_{j=0}^{s} q_j z^j$ and $p_i = \frac{\partial\phi}{\partial y^{\langle i\rangle}} \in \mathcal{K}^*$, $q_j = \frac{\partial\phi}{\partial u^{\langle j\rangle}} \in \mathcal{K}^*$, i.e. are polynomials over the σ-differential field $\mathcal{K}^*$.

3. REALIZATION

The realization problem can be stated as follows. Given an i/o equation of the form (2), find, if possible, the state coordinates $x = \psi\left(y, \dots, y^{\langle n-1\rangle}, u, \dots, u^{\langle s\rangle}\right) \in \mathbb{R}^n$ such that in these coordinates the system takes the classical state-space form

$$\begin{aligned} x^{\langle 1\rangle} &= f(x,u) \\ y &= h(x) \end{aligned} \tag{7}$$

and sequences $\{u(t), y(t), t \geq 0\}$, generated by descriptions (2) and (7), coincide. The i/o equation (2) is said to be realizable if it admits a realization of the form (7). Note that since we are looking for minimal, i.e. accessible and observable realization, irreducibility plays an important role. A function $\varphi \not\equiv$ constant in $\mathcal{K}^*$, such that $\varphi(0, \dots, 0) = 0$, is said to be an autonomous variable for control system (2) if there exist an integer $\nu \geq 1$ and a non-constant analytic function F so that $F\left(\varphi, \varphi^{\langle 1\rangle}, \dots, \varphi^{\langle \nu\rangle}\right) = 0$. Control system (2) is said to be irreducible if it does not admit any non-constant autonomous variable in $\mathcal{K}^*$. Otherwise system (2) is called reducible, see [11] for details.

An nth-order realization of equation (2) is accessible if and only if system (2) is irreducible, see [11] for technical details. Besides, according to [19], system (7) is said to be single-experiment observable if the observability matrix has generically full rank

$$\operatorname{rank}_{\mathcal{K}^*} \frac{\partial\left(h(x), \dots, h^{\langle n-1\rangle}\left(x, u, \dots, u^{\langle n-2\rangle}\right)\right)}{\partial x} = n.$$

Define the non-increasing sequence $\{\mathcal{H}_k\}_{k=1}^{\infty}$ of subspaces from $\mathcal{E}$ as follows

$$\begin{aligned} \mathcal{H}_1 &= \operatorname{span}_{\mathcal{K}^*}\left\{\mathrm{d}y, \dots, \mathrm{d}y^{\langle n-1\rangle}, \mathrm{d}u, \dots, \mathrm{d}u^{\langle s\rangle}\right\}, \\ \mathcal{H}_{k+1} &= \left\{\omega \in \mathcal{H}_k \mid \omega^{\langle 1\rangle} \in \mathcal{H}_k\right\}, \quad k \geq 1, \end{aligned} \tag{8}$$

playing the key role in the study of realization problem, see [12]. There exists an integer k^* such that $\mathcal{H}_1 \supset \mathcal{H}_2 \supset \cdots \supset \mathcal{H}_{k^*} \supset \mathcal{H}_{k^*+1} = \mathcal{H}_{k^*+2} = \cdots =: \mathcal{H}_\infty$. Existence of k^* comes from the fact that each $\mathcal{H}_k$ is finite-dimensional $\mathcal{K}^*$-vector space, so that at each step either its dimension decreases or $\mathcal{H}_{k+1} = \mathcal{H}_k$. We assume that the i/o equation (2) is in the irreducible form, i.e. $\mathcal{H}_\infty = \{0\}$, see [11] for details.

We say that $\omega \in \mathcal{E}$ is an *exact* one-form, if there exists $\xi \in \mathcal{K}^*$ such that $\mathrm{d}\xi = \omega$. A one-form ω for which $\mathrm{d}\omega = 0$ is said to be *closed*. A subspace is said to be completely integrable or closed, if it has locally a basis which consists only of exact one-forms. Integrability of the subspace of one-forms may be checked by the Frobenius theorem below, where the symbol $\mathrm{d}\omega$ denotes the exterior derivative of one-form ω and $\wedge$ means the exterior or wedge product.

Theorem 3.1. (Choquet et al. [4]) Let $\mathcal{V} = \operatorname{span}_{\mathcal{K}^*}\{\omega_1, \dots, \omega_r\}$ be a subspace of $\mathcal{E}$. $\mathcal{V}$ is closed if and only if $\mathrm{d}\omega_i \wedge \omega_1 \wedge \cdots \wedge \omega_r = 0$ for all $i = 1, \dots, r$.

We recall now the necessary and sufficient realizability conditions.

Theorem 3.2. (Kotta et al. [12]) The nonlinear i/o equation (2) has an observable state-space realization if and only if the subspace $\mathcal{H}_{s+2}$, defined by (8), is completely integrable.

Though [12] provides necessary and sufficient realizability conditions for i/o equation (2), and the sufficiency part of the proof suggests that the integrable basis of $\mathcal{H}_{s+2}$ defines the differentials of the state coordinates $\mathrm{d}x_i$, $i = 1, \ldots, n$, it does not address the computation of the subspace $\mathcal{H}_{s+2}$. Whereas [12] and this paper both use the formalism of differential forms, we build upon the latter the polynomial framework as in [11]. Our main result formulated below in Theorem 3.3 provides explicit polynomial formulas for computing the basis vectors of the subspace $\mathcal{H}_{s+2}$ in Theorem 3.2.

Since σ is an automorphism of $\mathcal{K}^*$, the *left division* operation is well-defined in $\mathcal{K}^*[z;\sigma,\delta]$. Given two polynomials $p, q \in \mathcal{K}^*[z;\sigma,\delta]$, $q \neq 0$ with $\deg(p) > \deg(q)$ there exist a unique *left quotient* polynomial γ and unique *left remainder* polynomial r such that $p = q\gamma + r$ and $\deg(r) < \deg(q)$. Below we need certain sequences of left quotients, which are computed by starting with the skew polynomials $p_0 := p$ and $q_0 := q$ in (6) and then the element p_l (q_l) for $l = 1, \ldots, n$ is found as the left quotient of p_{l-1} (q_{l-1}) and the polynomial z:

$$\begin{aligned} p_{l-1} &= z \cdot p_l + r_l, \quad \deg r_l = 0, \\ q_{l-1} &= z \cdot q_l + \rho_l, \quad \deg \rho_l = 0. \end{aligned} \tag{9}$$

We introduce certain one-forms, in terms of which we will formulate our main result in Theorem 3.3:

$$\omega_l = \begin{bmatrix} p_l & q_l \end{bmatrix} \begin{bmatrix} \mathrm{d}y \\ \mathrm{d}u \end{bmatrix}, \qquad l = 1, \ldots, n. \tag{10}$$

Theorem 3.3. For the input-output model (2), the subspaces $\mathcal{H}_k$ may be calculated as

$$\mathcal{H}_k = \operatorname{span}_{\mathcal{K}^*} \left\{ \omega_1, \ldots, \omega_n, \mathrm{d}u, \ldots, \mathrm{d}u^{\langle s-k+1 \rangle} \right\} \tag{11}$$

for $k = 1, \ldots, s+1$ and

$$\mathcal{H}_{s+2} = \operatorname{span}_{\mathcal{K}^*} \{ \omega_1, \ldots, \omega_n \}. \tag{12}$$

Proof. The proof is based on the principle of mathematical induction. First, we show that formula (11) holds for $k = 1$. To show this, prove that $\mathcal{H}_1$ in (8) may be represented as (11) for $k = 1$. In order to simplify the proof note that the recursive formulas (9) may be rewritten for $l = 1, \ldots, n$ explicitly as

$$\begin{aligned} p &= z^l \cdot p_l + R_l, \quad \deg R_l < l, \\ q &= z^l \cdot q_l + P_l, \quad \deg P_l < l, \end{aligned} \tag{13}$$

with $R_l = \sum_{i=1}^{l} z^{i-1} r_i$ and $P_l = \sum_{j=1}^{l} z^{j-1} \rho_j$.

Suppose $l = n$. According to (10), $\omega_n = p_n \mathrm{d}y + q_n \mathrm{d}u$. Due to the structure of the i/o equation, $\deg(p) = n$, and p is monic. Then, it follows from (13) that p_n is a left quotient of p and z^n, i.e. $p_n = 1$. Notice that $s < n$ meaning that the quotient of q and z^n is equal to zero. Consequently, $\omega_n = \mathrm{d}y$. Next, take $l = n-1$ and compute ω_{n-1}. Now, it follows from (13) that p_{n-1} is a polynomial of the first order. Thus, $\omega_{n-1} = \mathrm{d}y^{\langle 1 \rangle} + \alpha\omega_n + \beta \mathrm{d}u$ with $\alpha, \beta \in \mathcal{K}^*$, where ω_n and $\mathrm{d}u$ are independent elements in $\mathcal{H}_1$, so ω_{n-1} may be replaced by the more simple one-form $\mathrm{d}y^{\langle 1 \rangle}$. Continuing in the similar manner, it is possible to show that the remaining basis one-forms ω_l, for

$l = n-2, \dots, 1$, in (12) may be replaced by $\mathrm{d}y^{\langle 2\rangle}, \dots, \mathrm{d}y^{\langle n-1\rangle}$, respectively. As a result, the statement is true for $k = 1$.

Assume now that formula (11) holds for r and prove it to be valid for $r+1$. We have to prove that

$$\mathcal{H}_{r+1} = \operatorname{span}_{\mathcal{K}^*}\left\{\omega_1, \dots, \omega_n, \mathrm{d}u, \dots, \mathrm{d}u^{\langle s-r\rangle}\right\}, \tag{14}$$

calculated according to formula (11), satisfies condition (8).

First, note that the one-forms $\omega_1, \dots, \omega_n, \mathrm{d}u, \dots, \mathrm{d}u^{\langle s-r\rangle} \in \mathcal{H}_r$, since (11) holds for r. We have to prove that also the derivatives of the basis one-forms in (14) belong to $\mathcal{H}_r$. By (10), we have for $l = 1, \dots, n$

$$\omega_l^{\langle 1\rangle} = \begin{bmatrix} z \cdot p_l & z \cdot q_l \end{bmatrix} \begin{bmatrix} \mathrm{d}y \\ \mathrm{d}u \end{bmatrix}.$$

Using relations (9), we get

$$\omega_l^{\langle 1\rangle} = \begin{bmatrix} p_{l-1} - r_l & q_{l-1} - \rho_l \end{bmatrix} \begin{bmatrix} \mathrm{d}y \\ \mathrm{d}u \end{bmatrix}$$

or after reordering the terms

$$\omega_l^{\langle 1\rangle} = \begin{bmatrix} p_{l-1} & q_{l-1} \end{bmatrix} \begin{bmatrix} \mathrm{d}y \\ \mathrm{d}u \end{bmatrix} - \begin{bmatrix} r_l & \rho_l \end{bmatrix} \begin{bmatrix} \mathrm{d}y \\ \mathrm{d}u \end{bmatrix}. \tag{15}$$

Thus, the one-form $\omega_l^{\langle 1\rangle}$ is represented as a sum of two terms. For the first term we consider two separate cases. In case $l = 1$, the first term yields $p_0\mathrm{d}y + q_0\mathrm{d}u = p\mathrm{d}y + q\mathrm{d}u = 0$ due to polynomial system description (6). In case $l = 2, \dots, n$, the first term of (15) is equal to ω_{l-1} by (10) and, therefore, in $\mathcal{H}_r$. The second term of (15) is a linear combination of $\mathrm{d}y, \mathrm{d}u \in \mathcal{H}_r$, since the elements of r_l and ρ_l are functions from $\mathcal{K}^*$. Consequently, $\omega_l^{\langle 1\rangle} \in \mathcal{H}_r$ for $l = 1, \dots, n$. Finally, we observe that the derivatives of the rest of the basis one-forms in (14) are $\mathrm{d}u^{\langle 1\rangle}, \dots, \mathrm{d}u^{\langle s-r+1\rangle}$, which are also in $\mathcal{H}_r$. It should be mentioned that the subspace $\mathcal{H}_{s+2}$ does not contain the elements $\mathrm{d}u^{\langle j\rangle}$, $j = 1, \dots, s-k+1$. Thus, we have shown that $\mathcal{H}_k$, computed according to (11) for $k = 1, \dots, s+1$ and (12) for $k = s+2$, agrees with definition (8). □

Note that from the computational point of view the polynomial formulas (9) are faster, straight-forward and therefore better suited for implementation in symbolic software than the algorithm, based on definition (8).

Remark 3.4. Note that, according to Theorem 3.2, in order to find the minimal state-space realization, one has to check the integrability of the subspace $\mathcal{H}_{s+2}$, computed for irreducible i/o equation. However, Theorem 3.3 allows to find not only this subspace, but all the previous subspaces as well. Though the latter subspaces are not necessary to solve the realization problem, they may be important in the solutions of the related problems, for example, in the problem of lowering the input derivatives in the generalized state equations [13].

Remark 3.5. Note that though in case of the realizable i/o equation, $\mathcal{H}_{s+2}$, defined by (12), is completely integrable, the one-forms ω_l for $l = 1, \dots, n$ are not necessarily always exact. In such a case, one has to find for $\mathcal{H}_{s+2}$ a new (locally) exact basis, using linear transformations over the field $\mathcal{K}^*$.

In the algorithm below we summarize the realization procedure:

Step 1. Given the i/o equation (2), find the polynomial description of the system by rewriting (5) in the form (6).

Step 2. Given $p_0 := p$ and $q_0 := q$, obtained at **Step 1**, calculate, according to (9), two sequences $\{p_l\}_{l=1}^n$, $\{q_l\}_{l=1}^n$ of left quotients of polynomials p and q, respectively.

Step 3. Construct the vector space $\mathcal{H}_{s+2} = \mathrm{span}_{\mathcal{K}^*}\{\omega_1, \dots, \omega_n\}$, where the one-forms $\omega_l := p_l \mathrm{d}y + q_l \mathrm{d}u$, for $l = 1, \dots, n$, and simplify the basis elements of $\mathcal{H}_{s+2}$ whenever possible.

Step 4. Check the integrability of the vector space $\mathcal{H}_{s+2}$. If $\mathcal{H}_{s+2}$ is integrable, go to **Step 5**. Otherwise, inform that the i/o equation is not realizable and go to **Step 7**.

Step 5. Check whether the basis one-forms of $\mathcal{H}_{s+2}$ are exact or not. If this is true, integrate the one-forms $\omega_1, \dots, \omega_n$ to get $x_1, \dots, x_n$. Otherwise, use before a linear transformation to find a new integrable basis.

Step 6. Compute the state equations, applying to $x_1, \dots, x_n$ the pseudo-linear operator.

Step 7. End of the algorithm.

4. EXAMPLES AND SYMBOLIC SOFTWARE

Example 4.1. ([22]) Consider the i/o equation

$$y^{\langle 2\rangle} + \alpha_1 y^{\langle 1\rangle} + \alpha_0 y(1 + \varepsilon_1 y^2) = \beta_0(1 + \varepsilon_2 y)u, \tag{16}$$

where $\alpha_0, \alpha_1, \beta_0, \varepsilon_1, \varepsilon_2 \in \mathbb{R}$. In [22] the system was studied separately for continuous- and discrete-time cases, the latter being based on the difference operator description. Here, however, we address the model within the framework of pseudo-linear algebra which accommodates both special cases in a single model.

Equation (16) can be described as in (6) by two polynomials $p = z^2 + \alpha_1 z + \alpha_0 + 3\alpha_0\varepsilon_1 y^2 - \beta_0\varepsilon_2 u$ and $q = -\beta_0(1 + \varepsilon_2 y)$. From (16), $n = 2$ and $s = 0$. Given $p_0 = p$ and $q_0 = q$, compute iteratively, according to (9), the polynomials p_l and q_l for $l = 1, 2$ dividing respectively p_{l-1} and q_{l-1} by z from the left:

$$\begin{aligned} p_0 &= z^2 + \alpha_1 z + \alpha_0 + 3\alpha_0\varepsilon_1 y^2 - \beta_0\varepsilon_2 u, & q_0 &= -\beta_0(1 + \varepsilon_2 y),\\ p_1 &= z + \alpha_1, & q_1 &= 0,\\ p_2 &= 1, & q_2 &= 0. \end{aligned}$$

Since $s = 0$, according to Remark 3.4 and using (10), the basis elements of the last subspace $\mathcal{H}_{s+2} = \mathcal{H}_2 = \operatorname{span}_{\mathcal{K}^*}\{\omega_1, \omega_2\}$ can be represented in the following form

$$\omega_1 = p_1 \mathrm{d}y + q_1 \mathrm{d}u = (z + \alpha_1)\mathrm{d}y,$$
$$\omega_2 = p_2 \mathrm{d}y + q_2 \mathrm{d}u = \mathrm{d}y.$$

Finally, we get $\mathcal{H}_2 = \operatorname{span}_{\mathcal{K}^*}\left\{\mathrm{d}y, \mathrm{d}y^{\langle 1\rangle} + \alpha_1 \mathrm{d}y\right\}$. Simplifying the basis one-forms, the subspace may be rewritten as $\mathcal{H}_2 = \operatorname{span}_{\mathcal{K}^*}\left\{\mathrm{d}y, \mathrm{d}y^{\langle 1\rangle}\right\}$. The basis elements are exact, so one may choose $\mathrm{d}x_1 = \mathrm{d}y, \mathrm{d}x_2 = \mathrm{d}y^{\langle 1\rangle}$ and the state equations are

$$\begin{aligned} x_1^{\langle 1\rangle} &= x_2 \\ x_2^{\langle 1\rangle} &= -\alpha_1 x_2 - \alpha_0\left(1 + \varepsilon_1 x_1^2\right) x_1 + \beta_0(1 + \varepsilon_2 x_1)u \\ y &= x_1. \end{aligned} \tag{17}$$

For the special cases of continuous- and discrete-time models, (17) takes the forms

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x_1, x_2, u) \\ y &= x_1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x_1^{\Delta} &= x_2 \\ x_2^{\Delta} &= f(x_1, x_2, u) \\ y &= x_1 \end{aligned}$$

respectively, with $f(x_1, x_2, u) = -\alpha_1 x_2 - \alpha_0(1 + \varepsilon_1 x_1^2)x_1 + \beta_0(1 + \varepsilon_2 x_1)u$, like in [22].

It should be mentioned that since equation (16) depends only on u, but not on $u^{\langle k\rangle}, k \geq 1$, $\mathcal{H}_2 = \operatorname{span}_{\mathcal{K}^*}\left\{\mathrm{d}y, \mathrm{d}y^{\langle 1\rangle}\right\}$ by (8), see [6] for details. In fact, we may skip the intermediate computations and directly write out the state space realization of i/o equations (16); however, we decided to show them to illustrate the theory presented above.

Example 4.2. Consider the i/o equation

$$y^{\langle 2\rangle} = y^{\langle 1\rangle} u^{\langle 1\rangle} + uy$$

that may be described as in (6) by two polynomials $p = z^2 - u^{\langle 1\rangle} z - u$ and $q = -y^{\langle 1\rangle} z - y$. Note that $n = 2$ and $s = 1$. Given $p_0 := p$ and $q_0 := q$, compute, according to (9), two sequences of the left quotients as follows

$$p_1 = z - \sigma^{-1}\left(u^{\langle 1\rangle}\right), \qquad q_1 = -\sigma^{-1}\left(y^{\langle 1\rangle}\right),$$
$$p_2 = 1, \qquad q_2 = 0.$$

By (10), the one-forms of the subspace $\mathcal{H}_{s+2} = \mathcal{H}_3 = \operatorname{span}_{\mathcal{K}^*}\{\omega_1, \omega_2\}$ are

$$\omega_1 = p_1 \mathrm{d}y + q_1 \mathrm{d}u = \left(z - \sigma^{-1}\left(u^{\langle 1\rangle}\right)\right)\mathrm{d}y - \sigma^{-1}\left(y^{\langle 1\rangle}\right)\mathrm{d}u,$$
$$\omega_2 = p_2 \mathrm{d}y + q_2 \mathrm{d}u = \mathrm{d}y.$$

Since $\mathrm{d}y$ is the basis vector of the subspace $\mathcal{H}_3$, ω_1 may be simplified, resulting in $\mathcal{H}_3 = \operatorname{span}_{\mathcal{K}^*}\left\{\mathrm{d}y, \mathrm{d}y^{\langle 1\rangle} - \sigma^{-1}(y^{\langle 1\rangle})\mathrm{d}u\right\}$. Note that integrability of $\mathcal{H}_3$ depends on

$\sigma^{-1}\left(y^{\langle 1\rangle}\right)$. Next, we separately consider three typical cases. In the continuous-time case, when $\sigma = \sigma^{-1} = \mathrm{id}_{\mathcal{K}}$, the subspace

$$\mathcal{H}_3 = \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}y, \mathrm{d}\dot{y} - \dot{y}\mathrm{d}u\}$$

is, by Theorem 3.1, integrable. The choice $x_1 = y$, $x_2 = e^{-u}\dot{y}$ yields the classical state equations

$$\begin{aligned} \dot{x}_1 &= e^u x_2 \\ \dot{x}_2 &= e^{-u} u x_1 \\ y &= x_1. \end{aligned}$$

In the discrete-time case, when $\theta = \sigma$ and $\sigma^{-1}(\sigma(y)) = y$, the subspace

$$\mathcal{H}_3 = \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}y, \mathrm{d}\sigma(y) - y\mathrm{d}u\}$$

is again, by Theorem 3.1, integrable, yielding the state coordinates $x_1 = y$, $x_2 = \sigma(y) - uy$ and the state equations

$$\begin{aligned} \sigma(x_1) &= ux_1 + x_2 \\ \sigma(x_2) &= ux_1 \\ y &= x_1. \end{aligned}$$

In the discrete-time case, when $\theta = \Delta$, the subspace

$$\mathcal{H}_3 = \mathrm{span}_{\mathcal{K}^*}\left\{\mathrm{d}y, \mathrm{d}y^{\Delta} - \sigma^{-1}(y^{\Delta})\mathrm{d}u\right\} = \mathrm{span}_{\mathcal{K}^*}\left\{\mathrm{d}y, \frac{1}{\mu}\mathrm{d}\sigma(y) + \frac{1}{\mu}\left(\sigma^{-1}(y) - y\right)\mathrm{d}u\right\}$$

is, according to Theorem 3.1, not integrable, since $\mathrm{d}\omega_2 \wedge \omega_1 = \frac{1}{\mu}\mathrm{d}[\sigma^{-1}(y)] \wedge \mathrm{d}u \wedge \mathrm{d}y \neq 0$. Recall that either $\sigma^{-1}(y)$ or $\sigma^{-1}(u)$ may be chosen as the independent variable of field extension $\mathcal{K}^*$. In the latter case $\sigma^{-1}(y) = \frac{\sigma(y) - 2y - yu + \sigma^{-1}(u)y}{(\mu^2+1)\sigma^{-1}(u) - u - 1}$, yielding again that $\mathrm{d}\omega_2 \wedge \omega_1 \neq 0$.

Example 4.3. Consider the "ball and beam" system, with input being the angle and output being the ball position. The input-output equation of the system is

$$y^{\langle 2\rangle} = \frac{mR^2}{J + mR^2}\left(y\left(u^{\langle 1\rangle}\right)^2 - g\sin(u)\right), \tag{18}$$

where the constant parameters J, R, m represent, respectively, the inertia, radius and mass of the ball, and g is the gravitational constant. Usually, system (18) is considered separately for continuous- and discrete-time cases, see for example [10] and [16], respectively. Here, however, we consider the pseudo-linear operator based system description which accommodates both continuous- and discrete-time models.

System (18) can be described as in (6) by two polynomials $p = z^2 - \frac{mR^2}{J+mR^2}\left(u^{\langle 1\rangle}\right)^2$ and $q = -\frac{2mR^2}{J+mR^2} y u^{\langle 1\rangle} z + \frac{gmR^2}{J+mR^2}\cos(u)$. Note that $n = 2$ and $s = 1$. Given $p_0 := p$ and

$q_0 := q$, compute, according to (9), two sequences of the left quotients as follows

$$p_1 = z, \qquad q_1 = -\frac{2mR^2}{J+mR^2}\sigma^{-1}\left(yu^{\langle 1\rangle}\right),$$
$$p_2 = 1, \qquad q_2 = 0.$$

By (10), the one-forms of the subspace $\mathcal{H}_{s+2} = \mathcal{H}_3 = \mathrm{span}_{\mathcal{K}^*}\{\omega_1, \omega_2\}$ are

$$\omega_1 = p_1\mathrm{d}y + q_1\mathrm{d}u = \mathrm{d}y^{\langle 1\rangle} - \frac{2mR^2}{J+mR^2}\sigma^{-1}\left(yu^{\langle 1\rangle}\right)\mathrm{d}u,$$
$$\omega_2 = p_2\mathrm{d}y + q_2\mathrm{d}u = \mathrm{d}y.$$

Next, we consider separately three typical cases. In the continuous-time case, when $\sigma = \sigma^{-1} = \mathrm{id}_{\mathcal{K}}$, the subspace

$$\mathcal{H}_3 = \mathrm{span}_{\mathcal{K}^*}\left\{\mathrm{d}y, \mathrm{d}\dot{y} - \tfrac{2mR^2}{J+mR^2}y\dot{u}\mathrm{d}u\right\}$$

is, by Theorem 3.1, not integrable.

In the discrete-time case, when $\theta = \sigma$, the subspace

$$\mathcal{H}_3 = \mathrm{span}_{\mathcal{K}^*}\left\{\mathrm{d}y, \mathrm{d}\sigma(y) - \tfrac{2mR^2}{J+mR^2}\sigma^{-1}(y)u\mathrm{d}u\right\}$$

is again not integrable.

In the discrete-time case, when $\theta = \Delta$, the subspace

$$\mathcal{H}_3 = \mathrm{span}_{\mathcal{K}^*}\left\{\mathrm{d}y, \mathrm{d}y^{\Delta} - \tfrac{2mR^2}{J+mR^2}\sigma^{-1}(yu^{\Delta})\mathrm{d}u\right\}$$

is not integrable.

Thus, we may conclude that it is not possible to find the classical state-space realization of system (18) for the cases listed above.

The above examples are all quite simple, but in case of more complicated i/o equations the computations can be very labour-consuming. In order to facilitate the calculations, we have created a set of *Mathematica* functions, implementing the reduction algorithm from [11] and the realization algorithm from this paper. These functions are part of our previously developed *Mathematica* package NLControl, devoted to modelling, analysis and synthesis problems of nonlinear control systems [21]. The user has to specify the σ-differential field $\mathcal{K}$, i.e. the operators σ and δ.

The main advantage of the pseudo-linear approach is that it allows to optimize the code. The multiplication of the skew polynomials in NLControl package is based on formula (4), therefore the multiplication program is universal – it can handle polynomials with coefficients either in difference or differential field. Moreover, one can easily define the skew polynomial ring based on the new type of the pseudo-linear operator, for instance either q-shift or q-difference operator [11], not originally included into NLControl, and the multiplication of such polynomials can be performed immediately. The latter also means that one can define the control system in terms of the new type of pseudo-linear operator and both the reduction and realization functions are immediately applicable to this system.

Note that the NLControl package is made partly available through the web site [24]. The main benefit of the web site is that user does not need *Mathematica* to be installed into local computer, only internet connection and a browser are necessary to run the functions. The implemented functions from NLControl are grouped according to the time domain, e.g. continuous- and discrete-time, in order to make their use more convenient for users. However, the functionality behind the interface is based on the pseudo-linear algebra.

5. CONCLUSIONS

In this paper the minimal realization problem has been studied for nonlinear single-input single-output equation defined in terms of the pseudo-linear operator. The pseudo-linear algebra allowed to unify the realization theory of continuous- and discrete-time systems. Three main cases (continuous-time, shift- and difference operators based discrete-time) are merged into a single formalism. Moreover, we employed the tools from the theory of the non-commutative ring of skew polynomials. The latter requires system to be represented via two polynomials. The explicit formulas to compute the basis one-forms of certain vector space directly from the polynomial system description are presented. If this vector space is not integrable, the i/o equation is not realizable in the state-space form. However, when the vector space is integrable, integration of its exact basis vectors results in the desired state coordinates. Combining the results of this paper with those presented in [11], the complete procedure for deriving the minimal state equations starting from the possibly reducible i/o equation is worked out. In addition, we have implemented the results of this paper and those from [11] in *Mathematica* package NLControl [21]. Thus we may conclude that the program code of the introduced algorithm is shorter and more compact compared to those of the previous methods.

The possible direction for the future extension of this work is to construct the polynomial realization method for the multi-input multi-output equations. Moreover, detailed comparison of our results with those in [23] should be addressed since minimality in these two papers is defined in a different manner. Whereas we call the realization minimal when it is both accessible and observable, the paper [23] defines minimality by the minimal dimension of the state-space. The comparison is, however, not a simple task, since the mathematical tools employed are different.

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