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BOUNDS OF THE MATRIX EIGENVALUES AND ITS EXPONENTIAL BY LYAPUNOV EQUATION

Guang-Da Hu and Taketomo Mitsui

We are concerned with bounds of the matrix eigenvalues and its exponential. Combining the Lyapunov equation with the weighted logarithmic matrix norm technique, four sequences are presented to locate eigenvalues of a matrix. Based on the relations between the real parts of the eigenvalues and the weighted logarithmic matrix norms, we derive both lower and upper bounds of the matrix exponential, which complement and improve the existing results in the literature. Some numerical examples are also given.

Keywords: Lyapunov equation, weighted logarithmic matrix norm, location of eigenvalues, bounds of the matrix exponential

Classification: 15A18, 15A60, 34D20

1. MOTIVATION AND INTRODUCTION

In control problems involving the long-term stability of the closed-loop systems, one is often interested in showing whether all the eigenvalues of a matrix lie in the left half-plane or inside the unit circle [6, 12, 14] of the complex plane $\mathcal{C}$. Furthermore one also wants to locate the eigenvalues of a matrix in a bounded set that is easily characterized. The knowledge that all eigenvalues of a matrix A are located in a disc in the complex plane centered at the origin and having radius $\|A\|$ is well known in linear algebra textbooks, and naturally quite crude. Sharper locating regions are given by the Geršgorin disc theorem and its extensions [5, 12]. Along the line of the theorem, a lot of results have been obtained, e. g., [5, 12]. Recently the bounds of modulus of eigenvalues are derived by the weighted matrix norms which are constructed by Stein equation [10].

Hereafter the symbol ı stands for the imaginary unit, $\Re\lambda_j(A)$ and $\Im\lambda_j(A)$ are the real part and the imaginary part of the jth eigenvalue of matrix A, respectively. The following lemma shows that the logarithmic matrix norm $\mu[\cdot]$ is useful to locate eigenvalues of a matrix.

Lemma 1.1. [3] For any matrix $A \in \mathcal{C}^{n\times n}$, the bounds

$$-\mu[-A] \le \Re\lambda_j(A) \le \mu[A] \tag{1}$$

and

$$-\mu[\mathrm{i}A] \le \Im\lambda_j(A) \le \mu[-\mathrm{i}A] \tag{2}$$

hold.

Remark 1.2. Inequality (1) can be found in [3]. By noting that the imaginary part of an eigenvalue of the matrix A is equal to the real part of the eigenvalue of the matrix $-\imath A$, we can derive inequality (2) from inequality (1).

The logarithmic matrix norms have been studied in [2, 3, 11, 13, 15] to develop its properties and is known to be applied to estimate bounds of the matrix exponential [1, 2, 3, 8, 11]. Since the matrix exponential $\exp(At)$ is called the state transition matrix in the control theory, its bounds are useful to analyze and design control systems. The following lemma relates an upper bound of the matrix exponential with its logarithmic norm whose definition will be given in the following section.

Lemma 1.3. (Desoer and Vidyasagar [3]) For any matrix $A \in \mathcal{C}^{n\times n}$, the estimation

$$\|\exp(At)\| \le \exp(\mu[A]t) \tag{3}$$

holds.

The Lyapunov equation is an important tool in stability analysis of dynamical systems and control theory [1, 6, 12, 14]. The weighted logarithmic norms, which has been recently developed in [7, 8, 9] from the Lyapunov equation, can be constructed to be less than either 1-, 2- or ∞-logarithmic matrix norm in magnitude. Based on the weighted logarithmic norm, an upper bound of the matrix exponential is given in [8]. To the best of the authors' knowledge, no other results exist in the literature for lower bounds of the matrix exponential. This motivates to develop lower bounds of the matrix exponential, which can give more insight of the matrix exponential.

Throughout the paper, $\overline{z}$ and $|z|$ denote the complex conjugate and the modulus of a complex number z, respectively. The symbol I denotes the unit matrix, while A^{T} and A^* stand for the transposition and the conjugated transposition of a matrix A, respectively. Furthermore, $\lambda_j(A)$ denotes the jth eigenvalue of A, and $\lambda_{\min}(H)$ and $\lambda_{\max}(H)$ stand for the minimal and the maximal eigenvalues of a Hermitian matrix H, respectively. The usual inner product on $\mathcal{C}^n$ is denoted by $(\cdot,\cdot)$ and $\|\cdot\|_2$ denotes its subordinate norm. Finally, we introduce a vector-valued function, closely related to the Kronecker one [6, 12], associated with a matrix. For a matrix $F \in \mathcal{C}^{n\times n}$ write $F = [F_{*1}, F_{*2}, \ldots, F_{*n}]$, where $F_{*j} \in \mathcal{C}^n, j = 1, 2, \ldots, n$. Then the vector F^{T} is said to be the *vec*-function of F and written $vec(F)$. It is the vector formed by "stacking" the columns of F into one long vector.

The outline of this paper is as follows. In Section 2, the weighted logarithmic matrix norm derived from Lyapunov equation is reviewed. Section 3 presents four sequences to locate eigenvalues of a matrix by combining the Lyapunov equation with the weighted logarithmic matrix norm technique. In Section 4, both lower and upper bounds of the matrix exponential are derived which improve and complement the results that exist in the literature.

2. WEIGHTED LOGARITHMIC MATRIX NORM DERIVED FROM THE LYAPUNOV EQUATION

Here we will give definitions and lemmas of the logarithmic matrix norm with or without weight and review their relationship with Lyapunov equation. The obtained results will be employed in Section 3.

The logarithmic norm of a matrix A, which is often called the measure of the matrix, is defined through

$$\mu_p[A] = \lim_{\Delta \to 0_+} \frac{\|I + \Delta A\|_p - 1}{\Delta} \tag{4}$$

with the matrix norm $\|\cdot\|_p$ induced by a certain vector norm in $\mathcal{C}^n$. For the usual 1-, 2- and ∞-matrix norms, the following identities are well-known [2, 3]:

$$\mu_1[A] = \max_j \left[\Re a_{jj} + \sum_{i=1, i \neq j}^{n} |a_{ij}| \right], \quad \mu_2[A] = \lambda_{\max} \left(\frac{A + A^*}{2} \right)$$

and

$$\mu_\infty[A] = \max_i \left[\Re a_{ii} + \sum_{j=1, j \neq i}^{n} |a_{ij}| \right].$$

Remark 2.1. The above results tell that the eigenvalue computation of a positive definite matrix is not involved for $\mu_1[A]$ and $\mu_\infty[A]$. When the matrix A is Hermitian, obviously $\mu_1[A] = \mu_\infty[A]$. The $\mu_1[A]$ will be used in the following section.

Definition 2.2. (Horn and Johnson [5] and Lancaster and Tismenetsky [12]) A Hermitian matrix H is said to be positive (respectively, negative) definite if all its eigenvalues are positive (respectively, negative).

Definition 2.3. (Horn and Johnson [5], Lancaster and Tismenetsky [12], Rugh [14]) A matrix $A \in \mathcal{C}^{n \times n}$ is said to be stable if all its eigenvalues have negative real parts.

Definition 2.4. (Horn and Johnson [5] and Lancaster and Tismenetsky [12]) Assume that a Hermitian matrix H is positive definite. The inner product on $\mathcal{C}^n$ induced by $(x, y)_H = y^* H x$ is said to be the inner product with weight H (or simply with H) to distinguish from the standard (or Euclidean) inner product $(x, y)_2 = y^* x$.

Lemma 2.5. (Dekker and Verwer [2]) For any inner product on $\mathcal{C}^n$ and its subordinate norm $\|\cdot\|$, we have

$$\mu[A] = \max_{x \neq 0} \frac{\Re\{(Ax, x)\}}{\|x\|^2}.$$

Definition 2.6. (Hu and Hu [7], Hu and Liu [8])

Assume that a Hermitian matrix H is positive definite. For any vector x and any matrix A, the vector norm with weight H, the matrix norm with weight H, and the logarithmic norm with weight H defined, respectively, by

$$\|x\|_H = \sqrt{x^* H x}, \quad \|A\|_H = \max_{x \in \mathcal{C}^n, x \neq 0} \frac{\|Ax\|_H}{\|x\|_H} \quad \text{and} \quad \mu_{(H)}[A] = \max_{x \neq 0} \frac{\Re\{(Ax, x)_H\}}{{\|x\|^2}_H}.$$

The last identity can be guaranteed by Lemma 2.5.

The following lemma shows that the logarithmic matrix norm with weight H can locate the matrix eigenvalues. It is an alternative representation of Lemma 1.1 with respect to the vector norm with H.

Lemma 2.7. (Dekker and Verwer [2], Desoer and Vidyasagar [3]) For any matrix $A \in \mathcal{C}^{n\times n}$ and a positive definite matrix H, the inequalities

$$-\mu_H[-A] \le \Re\lambda_j(A) \le \mu_H[A] \tag{5}$$

and

$$-\mu_H[\imath A] \le \Im\lambda_j(A) \le \mu_H[-\imath A] \tag{6}$$

hold.

Now we introduce a relationship which combines the logarithmic matrix norm with weight and Lyapunov equation.

Theorem 2.8. (Hu and Liu [8]) For any complex matrix A and any real σ satisfying $\sigma > \max_i\{\Re\lambda_i(A)\}$, there exists a positive definite Hermitian matrix which fulfils Lyapunov equation

$$(A-\sigma I)^*H + H(A-\sigma I) = -2I \tag{7}$$

and gives the weighted logarithmic matrix norm of A satisfying

$$\mu_H[A] = \sigma - \frac{1}{\lambda_{\max}(H)}. \tag{8}$$

Proof of the above theorem requires the following well-known result.

Lemma 2.9. (Horn and Johnson [6], Lancaster and Tismenetsky [12], Rugh [14]) Let $A, W \in \mathcal{C}^{n\times n}$ and W be positive definite. The matrix A is stable if and only if there is a positive definite matrix H which is the unique solution of the following Lyapunov equation

$$A^*H + HA = -2W. \tag{9}$$

In the real matrix case, Theorem 2.8 was proved in [8]. We describe a detailed proof of the theorem although there is only a little difference from that in [8].

Proof of Theorem 2.8. Let

$$B = A - \sigma I. \tag{10}$$

Since the matrix $A - \sigma I$ is stable, due to Lemma 2.9, there is a positive definite matrix H satisfying the following Lyapunov equation

$$B^*H + HB = -2I.$$

By Definition 2.6, we have

$$\mu_H[B] = \max_{x\neq 0} \frac{\Re\{(Bx,x)_H\}}{\|x\|_H^2} \tag{11}$$

where
$$\|x\|_H = \sqrt{x^* H x}. \tag{12}$$

Noting the identities

$$(x, Bx)_H = x^* B^* H x, \quad (Bx, x)_H = x^* H B x, \quad (x, Bx)_H = \overline{(Bx, x)}_H,$$

we can calculate

$$\Re\{(Bx, x)_H\} = \frac{(Bx, x)_H + (x, Bx)_H}{2} = \frac{x^*(B^* H + HB)x}{2}. \tag{13}$$

Substituting (13) into (11), we obtain

$$\mu_H[B] = \max_{x \neq 0} \frac{x^*(B^* H + HB)x}{2\|x\|_H^2} = \max_{x \neq 0} \frac{-x^* x}{x^* H x} = -\frac{1}{\lambda_{\max}(H)}. \tag{14}$$

According to the definition (10) and properties of the logarithmic norm of a matrix [3], we have

$$\mu_H[B] = \mu_H[A - \sigma I] = \mu_H[A] - \sigma. \tag{15}$$

By (14) and (15) we can conclude

$$\mu_H[A] = \sigma - \frac{1}{\lambda_{\max}(H)}.$$

The proof is completed. □

3. LOCATION OF EIGENVALUES

The following theorem gives a location of the maximum as well as the minimum of real part of all eigenvalues. The proof of the theorem is similar to that in [9] or [10].

Theorem 3.1. For any complex matrix A, the following iterative locating procedure holds.

(i) Put $\alpha_0 = \mu_1[A]$. We can recursively define a monotone decreasing sequence $\{\alpha_k\}$ and a sequence of positive definite matrices $\{H_k\}$ through

$$\begin{cases} (A - \alpha_0 I)^* H_1 + H_1(A - \alpha_0 I) = -2I, \quad \alpha_1 = \mu_1[A] - \frac{1}{\mu_1[H_1]}, \\ \quad\vdots \\ (A - \alpha_k I)^* H_{k+1} + H_{k+1}(A - \alpha_k I) = -2I, \quad \alpha_{k+1} = \alpha_k - \frac{1}{\mu_1[H_{k+1}]}, \\ \qquad\qquad \text{for} \quad k > 1, \end{cases} \tag{16}$$

which means that H_{k+1} is constructed from H_k. Furthermore,

$$\alpha_k \geq \max_i\{\Re\lambda_i(A)\}, \quad \lim_{k\to\infty} \alpha_k = \max_i\{\Re\lambda_i(A)\} \tag{17}$$

and
$$\mu_{H_k}[A] \leq \alpha_k \tag{18}$$

hold.

(ii) Put $\beta_0 = -\mu_1[-A]$. We can recursively define a monotone increasing sequence $\{\beta_k\}$ and a sequence of positive definite matrices $\{L_k\}$ through

$$\begin{cases} (\beta_0 I - A)^* L_1 + L_1(\beta_0 I - A) = -2I, \quad \beta_1 = -\mu_1[-A] + \frac{1}{\mu_1[L_1]}, \\ \quad \vdots \\ (\beta_k I - A)^* L_{k+1} + L_{k+1}(\beta_k I - A) = -2I, \quad \beta_{k+1} = \beta_k + \frac{1}{\mu_1[L_{k+1}]}, \\ \qquad\qquad \text{for} \quad k > 1, \end{cases} \tag{19}$$

which means that L_{k+1} is constructed from L_k. Furthermore,

$$\beta_k \le \min_i\{\Re\lambda_i(A)\}, \quad \lim_{k\to\infty} \beta_k = \min_i\{\Re\lambda_i(A)\} \tag{20}$$

and

$$-\mu_{L_k}[-A] \ge \beta_k \tag{21}$$

hold.

(iii) The above two sequences of positive definite matrices $\{H_k\}$ and $\{L_k\}$ keep the partial order relations

$$H_{k+1} > H_k \tag{22}$$

and

$$L_{k+1} > L_k \tag{23}$$

for any $k \ge 1$ when the strict inequalities $\alpha_{k+1} < \alpha_k$ and $\beta_{k+1} > \beta_k$ hold.

Proof. First we will prove the assertion (i). The first bound of Lemma 1.1 yields $\Re\lambda_i(A) \le \mu_1[A]$. If $\max_i\{\Re\lambda_i(A)\} = \mu_1[A]$, the proof completes. Otherwise, we have $\Re\lambda_i(A) < \mu_1[A]$, which implies the matrix $A - \mu_1[A]I$ is stable. Let $\sigma = \mu_1[A]$ in Theorem 2.1, we obtain

$$\mu_{H_1}[A] = \mu_1[A] - \frac{1}{\lambda_{\max}(H_1)}, \tag{24}$$

where the matrix H_1 satisfies the Lyapunov equation (7) for $\sigma = \mu_1[A]$, i.e.,

$$(A - \mu_1[A]I)^* H_1 + H_1(A - \mu_1[A]I) = -2I.$$

From the properties of the logarithmic matrix norm, (1) in Lemma 1.1,

$$\lambda_{\max}(H_1) \le \mu_1[H_1].$$

By (24), we have

$$\mu_{H_1}[A] = \mu_1[A] - \frac{1}{\lambda_{\max}(H_1)} \le \mu_1[A] - \frac{1}{\mu_1[H_1]} = \alpha_1. \tag{25}$$

From (5) in Lemma 2.2 and (25), we obtain

$$\max_i\{\Re\lambda_i(A)\} \le \mu_{H_1}[A] \le \alpha_1. \tag{26}$$

When the equal sign in (26) holds, i. e., $\max_i\{\Re\lambda_i(A)\} = \alpha_1$, the proof is terminated. Otherwise we can obtain the matrix H_2 by

$$\begin{cases} (A - \alpha_1 I)^* H_2 + H_2(A - \alpha_1 I) = -2I, \\ \mu_{H_2}[A] = \alpha_1 - \frac{1}{\lambda_{\max}(H_2)} \le \alpha_1 - \frac{1}{\mu_1[H_2]} = \alpha_2. \end{cases} \tag{27}$$

From (5) in Lemma 2.7 and (27), we attain

$$\max_i\{\Re\lambda_i(A)\} \le \mu_{H_2}[A] \le \alpha_2. \tag{28}$$

We can repeat the above process for $k \ge 2$ to establish

$$(A-\alpha_k I)^* H_{k+1} + H_{k+1}(A-\alpha_k I) = -2I$$

and

$$\mu_{H_{k+1}}[A] = \alpha_k - \frac{1}{\lambda_{\max}(H_{k+1})} \le \alpha_k - \frac{1}{\mu_1[H_{k+1}]} = \alpha_{k+1},$$

which implies

$$\alpha_{k+1} = \alpha_k - \frac{1}{\mu_1[H_{k+1}]} \ge \mu_{H_{k+1}}[A]. \tag{29}$$

By (29), we obtain $\{\alpha_{k+1}\}$ satisfying

$$\alpha_{k+1} < \alpha_k, \tag{30}$$

which, together with (5) in Lemma 2.7, derives

$$\max_i\{\Re\lambda_i(A)\} \le \mu_{H_k}[A] \le \alpha_k. \tag{31}$$

When the equal sign holds in the above, the proof is terminated. The obtained monotone decreasing sequence $\{\alpha_k\}$ has a limit since it is bounded below. We introduce equation

$$G_k x_k = c, \tag{32}$$

where

$$G_k = I \otimes (A-\alpha_k I)^* + (A-\alpha_k I)^{\mathrm{T}} \otimes I,$$
$$x_k = vec(H_{k+1}), \quad c = -2\, vec(I)$$

and $\otimes$ stands for the Kronecker product. By a similar way to that in [9] or [10], we can prove

$$\lim_{k\to\infty} \alpha_k = \max_i\{\Re\lambda_i(A)\}.$$

Thus the proof of (i) is completed.

Now we will proceed to (ii). From (1) in Lemma 1.1, it is known that $\Re\lambda_i(A) \ge -\mu_1[-A]$. If $\min_i\{\Re\lambda_i(A)\} = -\mu_1[-A]$, the proof is terminated. Otherwise, the inequality $\Re\lambda_i(A) > -\mu_1[-A]$ means the matrix $-\mu_1[-A]I - A$ is stable. Let $\sigma = \mu_1[-A]$. Then Theorem 2.8 guarantees the existence of the matrix L_1 which fulfils

$$\mu_{L_1}[-A] = \mu_1[-A] - \frac{1}{\lambda_{\max}(L_1)} \tag{33}$$

as well as the Lyapunov equation (7) with respect to $\sigma = \mu_1[-A]$, i.e.,

$$(-\mu_1[-A]I - A)^* L_1 + L_1(-\mu_1[-A]I - A) = -2I.$$

The properties of the logarithmic matrix norm and the bound (1) in Lemma 1.1 imply

$$\lambda_{\max}(L_1) \le \mu_1[L_1].$$

By (33), we have

$$\mu_{L_1}[-A] = \mu_1[-A] - \frac{1}{\lambda_{\max}(L_1)} \leq \mu_1[-A] - \frac{1}{\mu_1[L_1]}, \tag{34}$$

which, together with (5) in Lemma 2.7, yields

$$\min_i\{\Re\lambda_i(A)\} \geq -\mu_{L_1}[-A] \geq -\mu_1[-A] + \frac{1}{\mu_1[L_1]} = \beta_1. \tag{35}$$

The rest proof of (ii) is almost identical with that of (i). By noting the fact that the sequence $\{\beta_k\}$ is monotone increasing and bounded above, the proof of (ii) is completed.

Now we prove (iii). (16) for k and $k+1$ reads

$$\begin{array}{rcl} (A - \alpha_{k-1}I)^* H_k + H_k(A - \alpha_{k-1}I) & = & -2I, \\ (A - \alpha_k I)^* H_{k+1} + H_{k+1}(A - \alpha_k I) & = & -2I. \end{array}$$

The difference of both sides of the above identities yields

$$\begin{array}{rcl} A^*(H_{k+1} - H_k) + (H_{k+1} - H_k)A & = & 2\alpha_k H_{k+1} - 2\alpha_{k-1}H_k \\ & = & 2\alpha_k(H_{k+1} - H_k) + 2\alpha_k H_k - 2\alpha_{k-1}H_k, \end{array}$$

which can be rearranged to

$$(A - \alpha_k I)^*(H_{k+1} - H_k) + (H_{k+1} - H_k)(A - \alpha_k I) = -2(\alpha_{k-1} - \alpha_k)H_k. \tag{36}$$

Since $\alpha_k < \alpha_{k-1}$, the inequality

$$-2(\alpha_{k-1} - \alpha_k)H_k < 0 \tag{37}$$

holds. Since the matrix $A - \alpha_k I$ is stable, Lemma 2.9 and (37), (36) implies

$$H_{k+1} - H_k > 0.$$

This completes the proof. □

As for the location of the maximal and the minimal imaginary parts of all eigenvalues, the following theorem can be established in parallel with the previous one.

Theorem 3.2. For any complex matrix A, the following iterative locating procedure holds.

(i) Put $\gamma_0 = \mu_1[-iA]$. We can recursively define a monotone decreasing sequence $\{\gamma_k\}$ and a sequence of positive definite matrices $\{M_k\}$ through

$$\begin{cases} (-\imath A - \gamma_0 I)^* M_1 + M_1(-\imath A - \gamma_0 I) = -2I, \quad \gamma_1 = \mu_1[-\imath A] - \frac{1}{\mu_1[M_1]}, \\ \quad\vdots \\ (-\imath A - \gamma_k I)^* M_{k+1} + M_{k+1}(-\imath A - \gamma_k I) = -2I, \quad \gamma_{k+1} = \gamma_k - \frac{1}{\mu_1[M_{k+1}]}, \\ \quad \text{for} \quad k > 1, \end{cases} \tag{38}$$

which means that M_{k+1} is constructed from M_k. Furthermore,

$$\gamma_k \geq \max_i\{\Im\lambda_i(A)\} \quad \text{and} \quad \lim_{k\to\infty} \gamma_k = \max_i\{\Im\lambda_i(A)\}. \tag{39}$$

(ii) Put $\delta_0 = -\mu_1[\imath A]$. We can recursively define a monotone increasing sequence $\{\delta_k\}$ and a sequence of positive definite matrices $\{N_k\}$ through

$$\begin{cases} (\imath A + \delta_0 I)^* N_1 + N_1(\imath A + \delta_0 I) = -2I, \quad \delta_1 = -\mu_1[iA] + \frac{1}{\mu_1[N_1]}, \\ \quad \vdots \\ (\imath A + \delta_k I)^* N_{k+1} + N_{k+1}(\imath A + \delta_k I) = -2I, \qquad \delta_{k+1} = \delta_k + \frac{1}{\mu_1[N_{k+1}]}, \\ \quad \text{for} \quad k > 1, \end{cases} \tag{40}$$

which means that N_{k+1} is constructed from N_k. Furthermore,

$$\delta_k \leq \min_i\{\Im\lambda_i(A)\} \quad \text{and} \quad \lim_{k\to\infty} \delta_k = \min_i\{\Im\lambda_i(A)\}. \tag{41}$$

(iii) The above two sequences of positive definite matrices $\{M_k\}$ and $\{N_k\}$ keep the partial order relations

$$M_{k+1} > M_k \tag{42}$$

and

$$N_{k+1} > N_k. \tag{43}$$

for any $k \geq 1$ when the strict inequalities $\gamma_{k+1} < \gamma_k$ and $\delta_{k+1} > \delta_k$ hold.

P r o o f is similar to that of Theorem 3.1. Here the only difference is to employ (6) in place of (5) in Lemma 2.7. □

Theorems 3.1 and 3.2 can be interpreted as follows.

Remark 3.3. For any matrix $A \in \mathcal{C}^{n\times n}$, we can introduce the four numerical sequences $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$ and $\{\delta_k\}$ satisfying

$$\beta_k \leq \Re\lambda_j(A) \leq \alpha_k, \quad \lim_{k\to\infty} \alpha_k = \max_i\{\Re\lambda_i(A)\} \quad \text{and} \quad \lim_{k\to\infty} \beta_k = \min_i\{\Re\lambda_i(A)\}, \tag{44}$$

and

$$\delta_k \leq \Im\lambda_j(A) \leq \gamma_k, \quad \lim_{k\to\infty} \gamma_k = \max_i\{\Im\lambda_i(A)\} \quad \text{and} \quad \lim_{k\to\infty} \delta_k = \min_i\{\Im\lambda_i(A)\}. \tag{45}$$

In fact the four sequences are those which are defined by (16), (19) in Theorem 3.1, (38) and (40) in Theorem 3.2, respectively. For each k, a rectangular region whose four vertices have the coordinate $(\alpha_k, \gamma_k), (\beta_k, \gamma_k), (\beta_k, \delta_k)$ and (α_k, δ_k) in the complex plane encloses all the eigenvalues of A. Moreover, since all the four sequences are monotone, the sequence of rectangular regions is monotonically shrinking. Therefore, the rectangular regions can give a sharper location than Lemma 1.1.

Remark 3.4. However, we have to point out that the four sequences in the previous remark are not necessary practical for finding the extremal real and imaginary parts of the eigenvalues of a matrix because they are many times more expensive than the standard method for finding eigenvalues (e. g., QR method), see Chapter 7 of [4]. By Gaussian elimination method, the operation cost of solving (32) is $O(n^6)$ since G_k is a $n^2 \times n^2$ matrix. Because G_k is sparse, the operation cost of solving (32) may be reduced by iterative methods, see [4].

4. BOUNDS OF THE MATRIX EXPONENTIAL

In this section, bounds of the matrix exponential will be derived by Theorem 3.1. The two inequalities (18) and (21) are utilized for lower and upper bounds of the matrix exponential, respectively. The proofs of the following results are similar to those in [8] and [9].

Theorem 4.1. For any matrix $A \in \mathcal{C}^{n\times n}$, the following inequalities hold.
(i) As for the weighted norms,

$$\|\exp(At)\|_{L_k} \geq \exp(-\mu_{L_k}[-A]t) \geq \exp(\beta_k t) \tag{46}$$

and

$$\|\exp(At)\|_{H_k} \leq \exp(\mu_{H_k}[A]t) \leq \exp(\alpha_k t), \tag{47}$$

where the scalars α_k and β_k, the matrices H_k and L_k are given in Theorem 3.1.
(ii) As for the 2-norm,

$$\theta_k \exp(\beta_k t) \leq \|\exp(At)\|_2 \leq \eta_k \exp(\alpha_k t), \tag{48}$$

where

$$\theta_k = \sqrt{\frac{\lambda_{\min}(L_k)}{\lambda_{\max}(L_k)}} \quad \text{and} \quad \eta_k = \sqrt{\frac{\lambda_{\max}(H_k)}{\lambda_{\min}(H_k)}}, \tag{49}$$

respectively, and the scalars α_k and β_k, the matrices H_k and L_k are given in Theorem 3.1.

Proof. (i) The bound (47) is a direct consequence from Lemma 1.3 and (18) in Theorem 3.1. Now we will prove (46). Letting $-\mu_{L_k}[-A] = d$, we have

$$\Re\lambda_i(-A) \leq \mu_{L_k}[-A] = -d$$

by Lemma 2.7. According to Lemma 1.3 and the above inequality, we have

$$\|\exp(-At)\|_{L_k} \leq \exp(\mu_{L_k}[-A]t) = \exp(-dt),$$

which implies

$$1 \leq \|\exp(-At)\|_{L_k}\|\exp(At)\|_{L_k} \leq \exp(-dt)\|\exp(At)\|_{L_k}.$$

This, together with (21), leads

$$\|\exp(At)\|_{L_k} \geq \exp(dt) = \exp(-\mu_{L_k}[-A]t) \geq \exp(\beta_k t).$$

(ii) First we prove

$$\|\exp(At)\|_2 \leq \eta \exp(\alpha_k t).$$

Denoting $H_k = D$, $Q = \sqrt{D}$ and $\tilde{A} = QAQ^{-1}$, we have $\exp(At) = Q^{-1}\exp(\tilde{A}t)Q$ and

$$\|\exp(At)\|_2 = \|Q^{-1}\exp(\tilde{A}t)Q\|_2 \leq \|Q^{-1}\|_2\|Q\|_2\|\exp(\tilde{A}t)\|_2. \tag{50}$$

Let $z = Q^{-1}x$. By virtue of Definition 2.6 and Lemma 1.3, we have

$$\begin{aligned}
(\|\exp(\tilde{A}t)\|_2)^2 &= \max_{x\neq 0} \frac{x^*(\exp(\tilde{A}t))^* \exp(\tilde{A}t)x}{x^*x} \\
&= \max_{x\neq 0} \frac{x^*(Q\exp(At)Q^{-1})^* Q\exp(At)Q^{-1}x}{x^*x} \\
&= \max_{z\neq 0} \frac{z^*(\exp(At))^* D\exp(At)z}{z^*Dz} = (\|\exp(At)\|_D)^2 \\
&\leq (\exp(\mu_{H_k}[A]t))^2 \leq (\exp(\alpha_k t))^2,
\end{aligned}$$

which means

$$\|\exp(\tilde{A}t)\|_2 \leq \exp(\alpha_k t). \tag{51}$$

On the other hand, since Q is a positive definite matrix,

$$\|Q^{-1}\|_2\|Q\|_2 = \sqrt{\frac{\lambda_{\max}(D)}{\lambda_{\min}(D)}} = \sqrt{\frac{\lambda_{\max}(H_k)}{\lambda_{\min}(H_k)}} = \eta_k \tag{52}$$

holds [12]. From (50) through (52), the inequality $\|\exp(At)\|_2 \leq \eta_k \exp(\alpha_k t)$ holds.

Now we will prove the first half of (48). This time, letting $L_k = D$, $Q = \sqrt{D}$ and $\tilde{A} = QAQ^{-1}$, we have

$$\|\exp(\tilde{A}t)\|_2 = \|Q\exp(At)Q^{-1}\|_2 \leq \|Q^{-1}\|_2\|Q\|_2\|\exp(At)\|_2,$$

which yields

$$\|\exp(At)\|_2 \geq \frac{1}{\|Q^{-1}\|_2\|Q\|_2\|}\|\exp(\tilde{A}t)\|_2. \tag{53}$$

Let $z = Q^{-1}x$. By virtue of Definition 2.6 and (47), we have

$$\begin{aligned}
(\|\exp(\tilde{A}t)\|_2)^2 &= \max_{x\neq 0} \frac{x^*(\exp(\tilde{A}t))^* \exp(\tilde{A}t)x}{x^*x} \\
&= \max_{x\neq 0} \frac{x^*(Q\exp(At)Q^{-1})^* Q\exp(At)Q^{-1}x}{x^*x} \\
&= \max_{z\neq 0} \frac{z^*(\exp(At))^* D\exp(At)z}{z^*Dz} = (\|\exp(At)\|_D)^2 \\
&\geq (\exp(-\mu_{L_k}[-A]t))^2 \geq (\exp(\beta_k t))^2,
\end{aligned}$$

which means

$$\|\exp(\tilde{A}t)\|_2 \geq \exp(\beta_k t). \tag{54}$$

On the other hand, since Q is a positive definite matrix,

$$\|Q^{-1}\|_2\|Q\|_2 = \sqrt{\frac{\lambda_{\max}(D)}{\lambda_{\min}(D)}} = \sqrt{\frac{\lambda_{\max}(L_k)}{\lambda_{\min}(L_k)}},$$

which means

$$\theta_k = \frac{1}{\|Q^{-1}\|_2 \|Q\|_2}. \tag{55}$$

From (53) through (55), the inequality

$$\| \exp{(At)} \|_2 \geq \theta_k \exp{(\beta_k t)}$$

holds. The proof is completed. □

Remark 4.2. The above theorem provides lower bounds of the matrix exponential, which can give more information on the matrix exponential. No results are known in the existing literature for lower bounds of the matrix exponential. Hence, Theorem 4.1 complements the results in [8] and in Chapter 11 of [1]. Since

$$\lim_{k\to\infty} \alpha_k = \max_i \{\Re\lambda_i(A)\}, \quad \text{and} \quad \lim_{k\to\infty} \beta_k = \min_i \{\Re\lambda_i(A)\},$$

Theorem 4.1 is sharper than the results in [8] and in Chapter 11 of [1].

5. NUMERICAL EXAMPLES

In this section, we will illustrate the main results through two numerical examples.

Example 5.1. Let the real matrix given by

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 0 & 3 \\ -1 & -3 & 0 \end{bmatrix},$$

which is unstable, for its eigenvalues are $0.6534, 0.1733 + 3.7072\,\imath$ and $0.1733 - 3.7072\,\imath$. In the complex plane, let $\alpha + \imath\gamma, \beta + \imath\gamma, \alpha + \imath\delta$ and $\beta + \imath\delta$ be the exact coordinate of the four vertices of the rectangle which accurately encloses all the eigenvalues. Then we have

$$\alpha = 0.6534, \quad \beta = 0.1733, \quad \gamma = 3.7072 \quad \text{and} \quad \delta = -3.7072.$$

The approximation process described in Theorems 3.1 and 3.2 runs as follows.

For $k = 2:$ $\alpha_2 = 0.6732,$ $\beta_2 = 0.1544,$ $\gamma_2 = 3.7110$ and $\delta_2 = -3.7110.$
For $k = 10:$ $\alpha_{10} = 0.6534,$ $\beta_{10} = 0.1733,$ $\gamma_{10} = 3.7072$ and $\delta_{10} = -3.7072.$

Thus ten times iteration can give a sufficient approximation.

Example 5.2. Let the complex matrix given by

$$A = \begin{bmatrix} 4+7i & -10-3i & 1+6i \\ -7+i & 4+6i & -2+3i \\ -5+2i & 4+11i & -3-6i \end{bmatrix},$$

which is unstable. In fact its eigenvalues are $11.3788 + 6.4586\,\imath, -4.7824 + 9.2995\,\imath$ and $-1.5965 - 8.7581\,\imath$. Under the same notations in Example 5.1 we can give as follows.

$$\alpha = 11.3788; \quad \beta = -4.7824 \quad \gamma = 9.2995 \quad \text{and} \quad \delta = -8.7581.$$

The approximating sequences described in Theorems 3.1 and 3.2 are given as follows.

For $k = 2$: $\alpha_2 = 11.8484$, $\beta_2 = -4.8314$, $\gamma_2 = 9.3467$ and $\delta_2 = -9.0875$.
For $k = 10$: $\alpha_{10} = 11.3791$, $\beta_{10} = -4.7824$, $\gamma_{10} = 9.2995$ and $\delta_{10} = -8.7583$.

6. CONCLUSION

Combining Lyapunov equation with the weighted logarithmic matrix norm technique, four sequences are presented to locate eigenvalues of a matrix. Based on the relations between the real parts of eigenvalues and the weighted logarithmic norms, both lower and upper bounds of the matrix exponential are derived which complement and improve the results in the literature.

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