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A GENERALIZATION OF AMENABILITY AND INNER AMENABILITY OF GROUPS

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Abstract. Let G be a locally compact group. We continue our work [A. Ghaffari: Γ-amenability of locally compact groups, Acta Math. Sinica, English Series, 26 (2010), 2313–2324] in the study of Γ-amenability of a locally compact group G defined with respect to a closed subgroup Γ of $G \times G$. In this paper, among other things, we introduce and study a closed subspace $A_{\Gamma}^{p}(G)$ of $L^{\infty}(\Gamma)$ and then characterize the Γ-amenability of G using $A_{\Gamma}^{p}(G)$. Various necessary and sufficient conditions are found for a locally compact group to possess a Γ-invariant mean.

Keywords: amenability, Banach algebra, inner amenability, locally compact group

MSC 2010: 43A60, 22D15

1. Introduction

Throughout this paper G denotes a locally compact group with a fixed left Haar measure $\mathrm{d}x$. The spaces $L^p(G)$, $1 \leqslant p \leqslant \infty$, of measurable functions will be as defined in [8]. We set $P^p(G) = \{f \in L^p(G)\colon\ f \geqslant 0,\ \|f\|_p = 1\}$. Let Γ be any closed subgroup of the product group $G \times G$, with left Haar measure denoted by $\mathrm{d}(y,z)$, and modular function Δ_Γ. For $(y,z) \in \Gamma$, $1 \leqslant p < \infty$, let $\pi_p(y,z)$ be the operator on $L^p(G)$ defined by

$$\pi_p(y,z)f(x) = f(y^{-1}xz)\Delta(z)^{1/p}, \quad x \in G,\ f \in L^p(G),$$

where Δ is the modular function on G. The corresponding map, denoted by T_φ^p, of the group algebra $L^1(\Gamma)$ is given by

$$T_\varphi^p f(x) = \int f(y^{-1}xz)\Delta(z)^{1/p}\varphi(y,z)\,\mathrm{d}(y,z).$$

It is well known that $\|T_\varphi^p f\|_p \leqslant \|\varphi\|_1 \|f\|_p$ [16].

For locally compact group G, Li and Pier [16] have introduced the notion of Γ-amenability, if there exists a Γ-invariant mean on G. Recall that a locally compact group G is Γ-amenable if there exists $m \in L^\infty(G)^*$ such that $m \geqslant 0$, $\langle m, 1_G\rangle = 1$ and $\langle m, {}_y h_z\rangle = \langle m, h\rangle$ for every $(y, z) \in \Gamma$, $h \in L^\infty(G)$, where ${}_y h_z(x) = h(y^{-1}xz)$ [9]. As is well known, this is equivalent to the existence of a Γ-invariant mean on $U(G, \Gamma)$, the subspace of $L^\infty(G)$ consisting of all $h \in L^\infty(G)$ for which the mappings $(y, z) \mapsto {}_y h_z$ from Γ into $L^\infty(G)$ are continuous. This generalizes the concepts of amenability and inner amenability of G. All abelian groups and all compact groups are Γ-amenable. In case $\Gamma = G \times \{e\}$ or $\{e\} \times G$, Γ-amenability is amenability in the usual sense. Ample information about amenability can be found in the books [18] and [20]. In case $\Gamma = \{(x, x)\colon x \in G\}$, Γ-amenability is inner amenability. The study of inner invariant means was initiated by Effros [6] and pursued by Lau and Paterson [14]. Recently, several authors have studied means on $L^\infty(G)$ that are invariant under the inner automorphisms of G (see [10], [1], [15], [17], [23] and [24]).

In this paper, among other things, we characterize Γ-amenable groups by introducing the convolution operators which develop the techniques of the usual convolution operators. A number of equivalent conditions characterizing Γ-amenable groups are given.

2. Preliminaries and some basic results

Recall that $A_p(G)$ is the Banach algebra consisting of all functions u on G written as $u = \sum\limits_{n=1}^{\infty} f_n * g_n^*$ with $f_n \in L^p(G)$, $g_n \in L^q(G)$ and $\sum\limits_{n=1}^{\infty} \|f_n\|_p \|g_n\|_q < \infty$. It is known that $A_p(G)$ has a bounded approximate identity if and only if G is amenable (see Theorem 10.4 in [20]). First we need to introduce some new spaces which we shall denote by $A^p_\Gamma(G)$. Let G be an arbitrary locally compact group, $1 < p < \infty$ and $1/p + 1/q = 1$. We define $A^p_\Gamma(G)$ to be that subset of $L^\infty(\Gamma)$ consisting of the elements u which can be expressed as $u(y, z) = \sum\limits_{n=1}^{\infty} \langle \pi_p(y, z) f_n, g_n\rangle$ almost everywhere with respect to Haar measure $\mathrm{d}(y, z)$, where $(y, z) \in \Gamma$, $f_n \in L^p(G)$, $g_n \in L^q(G)$, $n = 1, 2, \ldots$ and $\sum\limits_{n=1}^{\infty} \|f_n\|_p \|g_n\|_q < \infty$. For any $f \in L^p(G)$ and any $g \in L^q(G)$, we have $u(y, z) = \langle \pi_p(y, z) f, g\rangle \in C_b(\Gamma)$. Indeed, given $\varepsilon > 0$, there exist neighborhoods V_y of y and V_z of z in G such that for all $(s, t) \in \Gamma$ satisfying $(s, t) \in V_y \times V_z$, the inequality $\|{}_s f_t \Delta(t)^{1/p} - {}_y f_z \Delta(z)^{1/p}\|_p < \varepsilon/\|g\|_q$ holds [11]. Hence we have

$$\begin{aligned} |u(s, t) - u(y, z)| &= |\langle \pi_p(s, t) f, g\rangle - \langle \pi_p(y, z) f, g\rangle| \\ &\leqslant \|{}_s f_t \Delta(t)^{1/p} - {}_y f_z \Delta(z)^{1/p}\|_p \|g\|_q < \varepsilon. \end{aligned}$$

It is easy to see that $A^p_\Gamma(G)$ is a translation invariant subspace of $C_b(\Gamma)$. If $u \in A^p_\Gamma(G)$ we set

$$\|u\|_{A^p_\Gamma(G)} = \inf\Big\{\sum_{n=1}^{\infty} \|f_n\|_p \|g_n\|_q\Big\}$$

where the infimum is taken over all representations

$$u(y,z) = \sum_{n=1}^{\infty} \langle \pi_p(y,z) f_n, g_n\rangle, \;\; f_n \in L^p(G), \;\; g_n \in L^q(G), \;\; \sum_{n=1}^{\infty} \|f_n\|_p \|g_n\|_q < \infty.$$

In view of the preceding discussion, it is apparent that if G is an arbitrary locally compact group then $\|u\| \leqslant \|u\|_{A^p_\Gamma(G)}$. It is easy to see that $\|\cdot\|_{A^p_\Gamma(G)}$ is a complete norm for $A^p_\Gamma(G)$. We shall omit the details. When $p = 2$ and $\Gamma = G \times \{e\}$, then $A^p_\Gamma(G)$ is the Fourier algebra $A(G)$ as studied by Eymard [7], and Herz [12] for $\Gamma = G \times \{e\}$. In this case $A^p_\Gamma(G)$ is an F-algebra, see [13] and [19]. Let G be the discrete free group that is freely generated by the elements $a, b \in G$. Put $\Gamma = \{(x,x)\colon\, x \in G\}$. Then $A^2_\Gamma(G)$ is not necessarily closed under the pointwise multiplication. In fact, let $u(x,x) = \langle \pi_2(x,x)\delta_a, \delta_a\rangle$ and $v(x,x) = \langle \pi_2(x,x)\delta_b, \delta_b\rangle$. It is easy to see that $uv = \delta_{(e,e)} \notin A^2_\Gamma(G)$. This shows that $A^2_\Gamma(G)$ is not an F-algebra.

Lemma 2.1. *Let G be a locally compact group and let Γ be a closed subgroup of $G \times G$. Let $\varphi \in P^1(\Gamma)$. If there exists $1 < p < \infty$ such that $\|T^p_\varphi\| = 1$, then for every $1 < p' < \infty$, $\|T^{p'}_\varphi\| = 1$.*

P r o o f. By a form of the Riesz-Thorin Convexity Theorem ([4], VI.10.11), the function $\log \|T^{1/a}_\varphi\|$ is convex on $0 \leqslant a \leqslant 1$. As $T^\infty_\varphi 1(x) = \int \varphi(y,z)\,\mathrm{d}(y,z) = 1$, we have $\|T^\infty_\varphi\| = 1$. Now let $f \in L^1(G)$ and $f \geqslant 0$. We have

$$\begin{aligned}\|T^1_\varphi f\|_1 &= \iint f(y^{-1}xz)\Delta(z)\varphi(y,z)\,\mathrm{d}(y,z)\,\mathrm{d}x \\ &= \iint f(y^{-1}xz)\Delta(z)\varphi(y,z)\,\mathrm{d}x\,\mathrm{d}(y,z) = \|f\|_1.\end{aligned}$$

We conclude that $\|T^1_\varphi\| = \|T^p_\varphi\| = \|T^\infty_\varphi\| = 1$. On the other hand, $\|T^{p'}_\varphi\| \leqslant 1$ for every $p' \in [1,\infty]$. From this it is immediate that $\|T^{p'}_\varphi\| = 1$ for every $p' \geqslant 1$. $\square$

3. Main results

By the weak* operator topology on $\mathcal{B}(L^p(G))$, we shall mean the weak* topology of $\mathcal{B}(L^p(G))$ when it is identified with the dual space $(L^p(G)\otimes_\gamma L^q(G))^*$ in the obvious way [3]. Let $M_\Gamma^p(G)$ denote the closed algebra generated by $\{\pi_p(y,z)\colon (y,z)\in\Gamma\}$ in $\mathcal{B}(L^p(G))$. For $\varphi\in L^1(\Gamma)$, it is easy to see that $T_\varphi^p\in M_\Gamma^p(G)$ and $\{T_\varphi^p\colon \varphi\in L^1(\Gamma)\}$ is weak* operator topology dense in $M_\Gamma^p(G)$.

Theorem 3.1. *Let G be a locally compact group, $1<p,q<\infty$ and $1/p+1/q=1$. Then there exists an isometric linear isomorphism of $M_\Gamma^p(G)$ onto $A_\Gamma^p(G)^*$, the Banach space of continuous linear functionals on $A_\Gamma^p(G)$.*

Proof. Let $T\in M_\Gamma^p(G)$. If u has a representation $u(y,z)=\sum\limits_{n=1}^{\infty}\langle\pi(y,z)f_n,g_n\rangle$, then set $\Lambda(T)u=\sum\limits_{n=1}^{\infty}\langle Tf_n,g_n\rangle$. First we note that

$$|\Lambda(T)u|\leqslant\sum_{n=1}^{\infty}|\langle Tf_n,g_n\rangle|\leqslant\|T\|\sum_{n=1}^{\infty}\|f_n\|_p\|g_n\|_q<\infty.$$

Clearly $\Lambda(T)$ is linear. We show that $\Lambda(T)u$ is independent of the representation of u. Indeed, suppose $u\in A_\Gamma^p(G)$ and $u=0$. Let $\{\varphi_\alpha\}\subseteq L^1(\Gamma)$ be a net of functions such that $\|T_{\varphi_\alpha}^p\|\leqslant\|T\|$ and T_{φ_α} converges to T in the weak* operator topology. Then for each α, we have

$$\sum_{n=1}^{\infty}|\langle T_{\varphi_\alpha}^p f_n,g_n\rangle|\leqslant\sum_{n=1}^{\infty}\|T_{\varphi_\alpha}^p f_n\|_p\|g_n\|_q\leqslant\|T\|\sum_{n=1}^{\infty}\|f_n\|_p\|g_n\|_q<\infty.$$

Hence $\sum\limits_{n=1}^{\infty}|\langle T_{\varphi_\alpha}^p f_n,g_n\rangle|$ converges uniformly with respect to α. Thus

$$\lim_\alpha\sum_{n=1}^{\infty}\langle T_{\varphi_\alpha}^p f_n,g_n\rangle=\sum_{n=1}^{\infty}\lim_\alpha\langle T_{\varphi_\alpha}^p f_n,g_n\rangle=\sum_{n=1}^{\infty}\langle Tf_n,g_n\rangle$$

the interchange of limits being justified by the uniformity of the convergence of the series with respect to α. For every α, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\langle T_{\varphi_\alpha}^p f_n,g_n\rangle&=\sum_{n=1}^{\infty}\int\langle\pi_p(y,z)f_n,g_n\rangle\varphi_\alpha(y,z)\,\mathrm{d}(y,z)\\&=\int\sum_{n=1}^{\infty}\langle\pi_p(y,z)f_n,g_n\rangle\varphi_\alpha(y,z)\,\mathrm{d}(y,z)\\&=\int u(y,z)\varphi_\alpha(y,z)\,\mathrm{d}(y,z)=0.\end{aligned}$$

Consequently, if $u \in A^p_\Gamma(G)$ is such that $u = 0$ then $\sum\limits_{n=1}^{\infty} \langle Tf_n, g_n\rangle = 0$ for each representation $u(y,z) = \sum\limits_{n=1}^{\infty} \langle \pi_p(y,z)f_n, g_n\rangle$. Thus $\Lambda(T)$ is well defined linear functional on $A^p_\Gamma(G)$. Notice that

$$|\Lambda(T)u| = \left|\sum_{n=1}^{\infty}\langle Tf_n, g_n\rangle\right| \leqslant \|T\| \sum_{n=1}^{\infty} \|f_n\|_p\|g_n\|_q \tag{1}$$

and

$$\begin{aligned} \|T\| &= \sup\{|\langle Tf, g\rangle| \colon \ \|f\|_p \leqslant 1,\ \|g\|_q \leqslant 1\} \\ &= \sup\{|\Lambda(T)u| \colon \ u(y,z) = \langle \pi_p(y,z)f, g\rangle,\ \|f\|_p \leqslant 1,\ \|g\|_q \leqslant 1\} \\ &\leqslant \|\Lambda(T)\|. \end{aligned} \tag{2}$$

Now (1) and (2) imply that Λ is an isometry. It remains to show that Λ maps $M^p_\Gamma(G)$ onto $A^p_\Gamma(G)^*$. We now consider $F \in A^p_\Gamma(G)^*$ and define $T \in \mathcal{B}(L^p(G))$ by putting $\langle Tf, g\rangle = \langle F, u\rangle$ where $u(y,z) = \langle \pi_p(y,z)f, g\rangle$. Let $T \notin M^p_\Gamma(G)$. The Hahn-Banach theorem implies the existence of $f \in L^p(G)$ and $g \in L^q(G)$ such that for any $(y,z) \in \Gamma$, $u(y,z) = \langle \pi_p(y,z)f, g\rangle = 0$ whereas $\langle F, u\rangle = \langle Tf, g\rangle = 1$. This is a contradiction. Therefore Λ is surjective and the proof is complete. $\square$

Remark 3.2. If $\varphi \in L^1(\Gamma)$ and $u(y,z) = \sum\limits_{n=1}^{\infty} \langle \pi_p(y,z)f_n, g_n\rangle$ is any continuous mapping in $A^p_\Gamma(G)$, the mapping

$$\varphi * u(y,z) = \int \varphi(s,t)u(s^{-1}y, t^{-1}z)\,\mathrm{d}(s,t)$$

belongs to $A^p_\Gamma(G)$. Indeed, for every $(y,z) \in \Gamma$

$$\begin{aligned} \varphi * u(y,z) &= \int \varphi(s,t) \sum_{n=1}^{\infty} \langle \pi_p(s^{-1}y, t^{-1}z)f_n, g_n\rangle\,\mathrm{d}(s,t) \\ &= \int \varphi(s,t) \sum_{n=1}^{\infty} \int f_n((s^{-1}y)^{-1}xt^{-1}z)\Delta(t^{-1}z)^{1/p} g_n(x)\,\mathrm{d}x\,\mathrm{d}(s,t) \\ &= \int \varphi(s,t) \sum_{n=1}^{\infty} \int f_n(y^{-1}xz)\Delta(z)^{1/p} g_n(s^{-1}xt)\Delta(t)^{1/q}\,\mathrm{d}x\,\mathrm{d}(s,t) \\ &= \int \varphi(s,t) \sum_{n=1}^{\infty} \langle \pi_p(y,z)f_n, \pi_q(s,t)g_n\rangle\,\mathrm{d}(s,t) \\ &= \sum_{n=1}^{\infty} \langle \pi_p(y,z)f_n, T^q_\varphi g_n\rangle. \end{aligned}$$

Moreover

$$\|\varphi * u\|_{A^p_\Gamma(G)} \leqslant \sum_{n=1}^{\infty} \|f_n\|_p \|T^q_\varphi g_n\|_q \leqslant \|T^q_\varphi\| \sum_{n=1}^{\infty} \|f_n\|_p \|g_n\|_q.$$

Taking the inf of $\sum\limits_{n=1}^{\infty} \|f_n\|_p \|g_n\|_q$ over all such representations $\sum\limits_{n=1}^{\infty} \langle \pi_p(y,z) f_n, g_n\rangle$ for u, we obtain that $\|\varphi * u\|_{A^p_\Gamma(G)} \leqslant \|T^q_\varphi\| \|u\|_{A^p_\Gamma(G)}$.

Definition 3.3. We say that P_p-condition with bound $m \in \mathbb{N}$ is satisfied if for each $\varepsilon > 0$ and every compact subset $C \subseteq \Gamma$, there exists some $u \in A^p_\Gamma(G)$ such that $\|u\|_{A^p_\Gamma(G)} \leqslant m$ and $|u(y,z) - 1| < \varepsilon$ for all $(y,z) \in C$.

Theorem 3.4. *Let G be a locally compact group and let Γ be a closed subgroup of $G \times G$. Then the following conditions are equivalent:*

(i) *G is Γ-amenable;*
(ii) *for every $1 < p < \infty$, the P_p-condition is satisfied;*
(iii) *for every $1 < p < \infty$, the mapping $T^p_\varphi \mapsto \int \varphi(y,z)\,\mathrm{d}(y,z)$ from $\{T^p_\varphi \colon \varphi \in L^1(\Gamma)\}$ into $\mathbb{C}$ is a linear functional of norm one.*

P r o o f. (i) $\Longrightarrow$ (ii) We begin by showing that if G is Γ-amenable, the P_p-condition holds for every $1 < p < \infty$. Let a compact subset C of Γ and $\varepsilon > 0$ be given. By hypothesis there exists $f \in P^p(G)$ such that, for every $(y,z) \in C$, $\|\pi_p(y,z) f - f\|_p < \varepsilon$ [16]. Let $g = f^{p/q}$, then $g^q = f^p$. Therefore $g \in P^q(G)$ and $\langle f, g\rangle = 1$. For any $(y,z) \in C$, by Holder's inequality,

$$\begin{aligned} |\langle \pi_p(y,z) f, g\rangle - 1| &= |\langle \pi_p(y,z) f, g\rangle - \langle f, g\rangle| \\ &\leqslant \|\pi_p(y,z) f - f\|_p \|g\|_q < \varepsilon. \end{aligned}$$

Therefore the P_p-condition holds.

(ii) $\Longrightarrow$ (iii) Suppose $\varphi \in C_c(\Gamma)$ (the space of all complex-valued continuous functions on Γ having compact support) with compact support C. Assume that $c > 1$ and $n \in \mathbb{N}$. Choose $g \in C_c(\Gamma)$ such that $\int g(y,z)\,\mathrm{d}(y,z) = 1$. By assumption, there exists $u \in A^p_\Gamma(G)$ with $\|u\|_{A^p_\Gamma(G)} \leqslant m$ such that

$$\begin{aligned} \left| \int \varphi(y,z)\,\mathrm{d}(y,z) \right|^n &= \left| \int \varphi(y,z)\,\mathrm{d}(y,z) \right|^n |\langle 1_\Gamma, g\rangle| = |\langle \varphi * \ldots * \varphi * 1_\Gamma, g\rangle| \\ &= |\langle 1_\Gamma, \widetilde{(\varphi * \ldots * \varphi)} * g\rangle| \leqslant c |\langle u, \widetilde{(\varphi * \ldots * \varphi)} * g\rangle| \\ &= c |\langle \varphi * \ldots * \varphi * u, g\rangle| = c |\langle L^n_\varphi(u), g\rangle| \\ &\leqslant c \|L_\varphi\|^n \|u\|_{A^p_\Gamma(G)} \|g\|_1 \leqslant cm \|L_\varphi\|^n \|g\|_1, \end{aligned}$$

here $L_\varphi\colon A^p_\Gamma(G) \to A^p_\Gamma(G)$ is the operator of multiplication by φ on the left and $\tilde{\varphi}(y,z) = \varphi(y^{-1},z^{-1})\Delta_\Gamma(y^{-1},z^{-1})$ for $(y,z) \in \Gamma$. Since this holds for all n, we conclude that $|\int \varphi(y,z)\,\mathrm{d}(y,z)| \leqslant \|L_\varphi\|$. Since $C_c(\Gamma)$ is norm dense in $L^1(\Gamma)$, we deduce that for each $\varphi \in L^1(\Gamma)$, $|\int \varphi(y,z)\,\mathrm{d}(y,z)| \leqslant \|L_\varphi\|$. On the other hand, by Remark 3.2 $\|\varphi * u\|_{A^p_\Gamma(G)} \leqslant \|T^q_\varphi\|\|u\|_{A^p_\Gamma(G)}$. We conclude that $|\int \varphi(y,z)\,\mathrm{d}(y,z)| \leqslant \|T^q_\varphi\|$. But obviously, $\|T^q_\varphi\| \leqslant \int \varphi(y,z)\,\mathrm{d}(y,z)$ for every $\varphi \in L^1(\Gamma)$ with $\varphi \geqslant 0$. Thus (ii) implies (iii).

(iii) $\Longrightarrow$ (i) Our hypothesis permits us to define a linear functional ω_1 on $\{T^2_\varphi\colon\ \varphi \in L^1(\Gamma)\}$ by $\omega_1(T^2_\varphi) = \int \varphi(y,z)\,\mathrm{d}(y,z)$ with norm one. By the Hahn-Banach theorem for states (see Proposition 2.3.24 in [2]), we may extend ω_1 to a state ω on the algebra $\mathcal{B}(L^2(G))$ of bounded operators on $L^2(G)$. Therefore G is Γ-amenable by Proposition 3.7 in [16]. $\square$

Let $I = \{\varphi \in L^1(\Gamma)\colon\ \int \varphi(y,z)\,\mathrm{d}(y,z) = 0\}$. Clearly I is a closed two sided ideal in $L^1(\Gamma)$. Let ω be a state on $M^2_\Gamma(G)$ such that $\omega(T^2_\varphi) = 0$ for all $\varphi \in I$ and $\omega(T^2_{\varphi_0}) = 1$ for some $\varphi_0 \in P^1(\Gamma)$. Then G is Γ-amenable. Indeed, for every $\psi \in L^1(\Gamma)$,

$$\psi = \left(\psi - \int \psi(y,z)\,\mathrm{d}(y,z)\varphi_0\right) + \int \psi(y,z)\,\mathrm{d}(y,z)\varphi_0.$$

It follows that $\omega(T^2_\psi) = \int \psi(y,z)\,\mathrm{d}(y,z)$. Extend ω to a state on $\mathcal{B}(L^2(G))$. By Proposition 3.7 in [16], G is Γ-amenable.

Theorem 3.5. *Let G be a locally compact group and let Γ be a closed subgroup of $G \times G$. Let $1 \leqslant p < \infty$ and $1/p + 1/q = 1$. Then the following conditions are equivalent:*

(i) *G is Γ-amenable;*
(ii) *there exists a net $\{f_\alpha\}$ in $P^p(G)$ such that, for every weakly compact subset A of $L^1(\Gamma)$, $\lim\limits_\alpha \|T^p_\varphi f_\alpha - (\int \varphi(y,z)\,\mathrm{d}(y,z))f_\alpha\|_p = 0$ uniformly for every $\varphi \in A$.*

Proof. (i) $\Longrightarrow$ (ii) We denote by $\mathcal{A}$ the family of all weakly compact subsets in $L^1(\Gamma)$. We consider the directed set $I = \mathcal{A} \times \mathbb{R}^+$ where for $\alpha = (A,\varepsilon) \in I$, $\alpha' = (A',\varepsilon') \in I$, $\alpha \prec \alpha'$ in case $A \subseteq A'$ and $\varepsilon' < \varepsilon$. Now let $\alpha = (A,\varepsilon) \in I$. Since A is weakly compact, A is weakly bounded. By Theorem 3.18 in [21], A is norm bounded. Let $c_A = \sup\{\|\varphi\|_1\colon\ \varphi \in A\}$. By Theorem 4.21.2 in [5], there exists a compact subset C in Γ such that $\int_{\Gamma\setminus C} |\varphi(y,z)|\,\mathrm{d}(y,z) < \varepsilon/4$ for every $\varphi \in A$. By Proposition 3.1 in [16], there exist $f_\alpha \in P^p(G)$ such that $\|\pi_p(y,z)f_\alpha - f_\alpha\|_p < \varepsilon/(2c_A)$

for every $(y,z)\in C$. Hence for all $\varphi\in A$ and $g\in C_c(\Gamma)$,

$$\begin{aligned}
\varepsilon\|g\|_q &> 2\|g\|_q\frac{\varepsilon}{4}+\frac{\varepsilon}{2c_A}\|g\|_q\|\varphi\|_1\\
&\geqslant \int_{\Gamma\setminus C}+\int_C|\varphi(y,z)|\|\pi_p(y,z)f_\alpha-f_\alpha\|_p\|g\|_q\,\mathrm{d}(y,z)\\
&\geqslant \left|\int\varphi(y,z)\int\Big(f_\alpha(y^{-1}xz)\Delta(z)^{1/p}-f_\alpha(x)\Big)g(x)\,\mathrm{d}x\,\mathrm{d}(y,z)\right|\\
&\geqslant \left|\langle T^p_\varphi f_\alpha-\Big(\int\varphi(y,z)\,\mathrm{d}(y,z)\Big)f_\alpha,g\rangle\right|.
\end{aligned}$$

Since this holds for all $g\in C_c(\Gamma)$, by Theorem 12.13 in [11] we have

$$\left\|T^p_\varphi f_\alpha-\Big(\int\varphi(y,z)\,\mathrm{d}(y,z)\Big)f_\alpha\right\|_p\leqslant\varepsilon$$

for every $\varphi\in A$. Let A_0 be a weakly compact subset of $L^1(\Gamma)$ and $\varepsilon_0\in\mathbb{R}^+$. We consider $\alpha_0=(A_0,\varepsilon_0)$. If $(A_0,\varepsilon_0)=\alpha_0\prec\alpha=(A,\varepsilon)$, then

$$\left\|T^p_\varphi f_\alpha-\Big(\int\varphi(y,z)\,\mathrm{d}(y,z)\Big)f_\alpha\right\|_p\leqslant\varepsilon$$

for every $\varphi\in A$. This shows that the limit is uniform on compacta.

(ii) $\Longrightarrow$ (i) Consider a fixed $\varphi\in P^1(\Gamma)$. Let C be a compact subset of Γ and $\varepsilon\in(0,1)$. Since the mapping $(s,t)\mapsto{}_{(s^{-1},t^{-1})}\varphi$ is weakly continuous [8], it follows that $\{{}_{(s^{-1},t^{-1})}\varphi\colon (s,t)\in C\}$ is weakly compact. Hence $A=\{{}_{(s^{-1},t^{-1})}\varphi-\varphi\colon (s,t)\in C\}\bigcup\{\varphi\}$ is weakly compact. By assumption, there exists $f_1\in P^p(G)$ such that

$$\|T^p_{{}_{(s^{-1},t^{-1})}\varphi-\varphi}f_1\Big\|_p=\left\|T^p_{{}_{(s^{-1},t^{-1})}\varphi-\varphi}f_1-\Big(\int{}_{(s^{-1},t^{-1})}\varphi-\varphi\,\mathrm{d}(y,z)\Big)f_1\right\|_p<\varepsilon$$

and $\|T^p_\varphi f_1-(\int\varphi(y,z)\,\mathrm{d}(y,z))f_1\|_p<\varepsilon$ for every $(s,t)\in C$. Clearly $1-\varepsilon<\|T^p_\varphi f_1\|_p$. Let $f=(T^p_\varphi f_1)/(\|T^p_\varphi f_1\|_p)\in P^p(G)$. For any $(s,t)\in C$, we have

$$\begin{aligned}
\|T^p_\varphi f_1\|_p\|\pi_p(y,z)f-f\|_p &= \|{}_yT^p_\varphi f_{1_z}\Delta(z)^{1/p}-T^p_\varphi f_1\|_p\\
&= \|T^p_{{}_{(s^{-1},t^{-1})}\varphi}f_1-T^p_\varphi f_1\|_p<\varepsilon.
\end{aligned}$$

This shows that $\|\pi_p(y,z)f-f\|_p<\varepsilon/(1-\varepsilon)$ for every $(s,t)\in C$. As $\varepsilon\in(0,1)$ may be chosen arbitrarily, by Proposition 3.1 in [16], G is Γ-amenable. $\square$

It is known that G is amenable if and only if, for all $f \in L^1(G)$,

$$\inf\{\|f * \varphi\|_1 \colon \varphi \in P^1(G)\} = \left|\int f(x)\,\mathrm{d}x\right|$$

(see Theorem 4.18 in [18]). As an immediate consequence of Theorem 3.5, we can prove that G is inner amenable if and only if, for all $\varphi \in L^1(G)$,

$$\inf\{\|T^1_\varphi f\|_1 \colon f \in P^1(G)\} = \left|\int \varphi(x)\,\mathrm{d}x\right|.$$

Definition 3.6. Let μ be a positive, regular Borel measure on a locally compact group G. For $f \in L^p(G)$, we define $T^p_\mu f(x) = \int f(y^{-1}xz)\Delta(z)^{\frac{1}{p}}\,\mathrm{d}\mu(y,z)$ whenever the latter integral exists. μ is p-admissible if $T^p_\mu(L^p(G)) \subseteq L^p(G)$.

Note that $T^p_\mu(L^p(G)) \subseteq L^p(G)$ means that if $f \in L^p(G)$, then the function $(y,z) \mapsto f(y^{-1}xz)\Delta(z)^{1/p}$ is in $L^1(\mu)$, and that the function $T^p_\mu f$ given by $T^p_\mu f(x) = \int f(y^{-1}xz)\Delta(z)^{1/p}\,\mathrm{d}\mu(y,z)$ belongs to $L^p(G)$.

Theorem 3.7. *Let G be a locally compact group and let Γ be a closed subgroup of $G \times G$. Then the following conditions are equivalent:*

(i) *G is Γ-amenable;*

(ii) *for all $p > 1$, every p-admissible measure μ on G is bounded.*

Proof. (i) $\Longrightarrow$ (ii) Let μ be a p-admissible measure on G and suppose that $T^p_\mu \colon L^p(G) \to L^p(G)$ is not continuous. For each positive integer n there would be an $f_n \in P^p(G)$ such that $\|T^p_\mu f_n\|_p \geqslant n^3$. It is apparent that $\sum\limits_{n=1}^{\infty} f_n/n^2$ converges in $L^p(G)$ to some $f \in L^p(G)$. For each positive integer n, $n^2 f \geqslant f_n$ and so $n^2 T^p_\mu f \geqslant T^p_\mu f_n$. Hence for each n, we have $n^2\|T^p_\mu f\|_p \geqslant \|T^p_\mu f_n\|_p \geqslant n^3$ which contradicts the fact that $T^p_\mu(L^p(G)) \subseteq L^p(G)$.

By Theorem 3.4, there would exist $f_\alpha \in P^p(G)$ and $g_\alpha \in P^q(G)$ such that $u_\alpha(y,z) = \langle \pi_p(y,z)f_\alpha, g_\alpha\rangle$ converges to 1 whenever $(y,z) \in \Gamma$. The limit is uniform on compacta. Let C be a compact subset of Γ and let $c > 1$. There exists α such that $1 \leqslant c\langle \pi_p(y,z)f_\alpha, g_\alpha\rangle$ whenever $(y,z) \in \Gamma$. We have

$$\begin{aligned}\mu(C) &\leqslant c\int_C \langle \pi_p(y,z)f_\alpha, g_\alpha\rangle\,\mathrm{d}\mu(y,z) \leqslant c\int \langle \pi_p(y,z)f_\alpha, g_\alpha\rangle\,\mathrm{d}\mu(y,z)\\ &= c\langle T^p_\mu f_\alpha, g_\alpha\rangle \leqslant c\|T^p_\mu f_\alpha\|_p\|g_\alpha\|_q \leqslant c\|T^p_\mu\|.\end{aligned}$$

Using the inner regularity of μ, it follows that μ is bounded.

(ii) $\Longrightarrow$ (i) By Theorem 3.4, we assume to the contrary that there exists $\varphi \in P^1(\Gamma)$ such that $\alpha = \|T^p_\varphi\| < 1$. If φ^i is the ith power of φ in $L^1(\Gamma)$, we have $\|\varphi^i\|_1 = \|\varphi\|_1^i = 1$, $\|\sum\limits_{i=1}^n \varphi^i\|_1 = \sum\limits_{i=1}^n \|\varphi^i\|_1 = n$, because all functions are positive. Hence $\Phi = \sum\limits_{i=1}^\infty \varphi^i$ is a positive measurable function which is not in $L^1(\Gamma)$ and so defines an unbounded measure $\Phi\,\mathrm{d}(y,z)$ on Γ. Let us prove that Φ is locally integrable on Γ. If $1/p + 1/q = 1$, we may choose $f \in L^p(G)^+$ and $g \in L^q(G)^+$ such that $\langle f, g\rangle > 1$. Let C be a compact subset of Γ. For every $(y,z) \in \Gamma$, we have $\langle \pi_p(y,z)f, \pi_q(y,z)g\rangle = \langle f, g\rangle > 1$. For every $(y_0,z_0) \in C$, the mapping $(y,z) \mapsto \langle \pi_p(y,z)f, \pi_q(y_0,z_0)g\rangle$ is continuous [8]. We may determine a subset $\{(y_1,z_1),\dots,(y_n,z_n)\}$ in C and open subsets $U_1,\dots,U_n$ such that $C \subseteq \bigcup\limits_{i=1}^n U_i$, $(y_i,z_i) \in U_i$ and $\langle \pi_p(y,z)f, \pi_q(y_i,z_i)g\rangle > 1$ for all $(y,z) \in U_i$ and $1 \leqslant i \leqslant n$. Put $h = \sum\limits_{i=1}^n \pi_q(y_i,z_i)g$. Hence $\langle \pi_p(y,z)f, h\rangle > 1$ for every $(y,z) \in C$. We have

$$\begin{aligned}
\int_C \Phi(y,z)\,\mathrm{d}(y,z) &\leqslant \int_C \Phi(y,z)\langle \pi_p(y,z)f, h\rangle\,\mathrm{d}(y,z)\\
&\leqslant \iint \Phi(y,z) f(y^{-1}xz)\Delta(z)^{\frac{1}{p}} h(x)\,\mathrm{d}x\,\mathrm{d}(y,z)\\
&= \int T^p_\Phi f(x)h(x)\,\mathrm{d}x = \langle T^p_\varphi f, h\rangle < \infty.
\end{aligned}$$

So $\Phi(y,z)\,\mathrm{d}(y,z)$ is an unbounded Borel measure on Γ. On the other hand, for every $f \in L^p(G)$ and every $n \in \mathbb{N}$,

$$\Big\|\sum_{i=1}^n T^p_{\varphi^i} f\Big\|_p \leqslant \sum_{i=1}^n \|T^p_{\varphi^i}\|\|f\|_p \leqslant \sum_{i=1}^n \alpha^i \|f\|_p \leqslant \frac{\|f\|_p}{1-\alpha}.$$

Hence $\|T^p_\Phi f\|_p \leqslant \|f\|_p/(1-\alpha)$ and so $\|T^p_\Phi\| \leqslant 1/(1-\alpha)$. Consequently $\Phi\,\mathrm{d}(y,z)$ defines a bounded linear operator on $L^p(G)$ and is unbounded. This is a contradiction. □

Following Li and Pier [16], we define $S_\varphi h(x) = \int \varphi(y,z)h(yxz^{-1})\,\mathrm{d}(y,z)$ where $\varphi \in L^1(\Gamma)$ and $h \in L^\infty(G)$. A mean m on $L^\infty(G)$ is called topologically Γ-invariant if $\langle m, S_\varphi h\rangle = \langle m, h\rangle$ whenever $h \in L^\infty(G)$ and $\varphi \in P^1(\Gamma)$. It is well known that any topologically Γ-invariant mean on $L^\infty(G)$ is also Γ-invariant (see Proposition 2.3 in [16]). Let G be a nondiscrete abelian group and $\Gamma = G \times \{e\}$. By Proposition 22.3 in [20], there exists a Γ-invariant mean m on $L^\infty(G)$ such that $\langle m, S_\varphi h\rangle \neq \langle m, h\rangle$ for some $h \in L^\infty(G)$ and $\varphi \in P^1(G)$. This shows that m can not be a topologically Γ-invariant mean.

Theorem 3.8 below, in the special case where $\Gamma = G \times \{e\}$, is proved in ([20, p. 265]).

Theorem 3.8. *Let G be a locally compact group and let Γ be a closed subgroup of $G \times G$. If m is a Γ-invariant mean on $L^\infty(G)$ and there exists $\varphi_0 \in P^1(\Gamma)$ such that $\langle m, S_{\varphi_0} h\rangle = \langle m, h\rangle$ whenever $h \in L^\infty(G)$, then also m is a topologically Γ-invariant.*

P r o o f. Let m be a Γ-invariant mean on $L^\infty(G)$ and $\langle m, S_{\varphi_0} h\rangle = \langle m, h\rangle$ for all $h \in L^\infty(G)$. Let $h \in L^\infty(G)$ and $\varphi \in P^1(\Gamma)$. Since $C_c(\Gamma)$ is dense in $L^1(\Gamma)$, we may assume that $\varphi \in P^1(\Gamma) \cap C_c(\Gamma)$. By [16], we have

$$\langle m, S_\varphi h\rangle = \langle m, S_{\varphi_0}(S_\varphi h)\rangle = \langle m, S_{\varphi * \varphi_0} h\rangle. \tag{1}$$

Since the mapping $(s,t) \mapsto S_{{}_{(s,t)}\varphi_0} h$ from Γ into $L^\infty(G)$ is continuous, by [16] and Theorem 3.27 in [21], we have

$$\begin{aligned}\langle S_{\varphi*\varphi_0} h, \rangle &= \int f(x) \iint \varphi(s,t)_{(s^{-1},t^{-1})}\varphi_0(y,z) h(yxz^{-1})\,\mathrm{d}(s,t)\,\mathrm{d}(y,z)\,\mathrm{d}x \\ &= \int \varphi(s,t)\langle S_{{}_{(s^{-1},t^{-1})}\varphi_0} h, f\rangle\,\mathrm{d}(s,t) \\ &= \Big\langle \int \varphi(s,t) S_{{}_{(s^{-1},t^{-1})}\varphi_0} h d(s,t), f\Big\rangle.\end{aligned}$$

We conclude that

$$\begin{aligned}\langle m, S_{\varphi*\varphi_0} h\rangle &= \int \langle m, S_{{}_{(s^{-1},t^{-1})}\varphi_0} h\rangle \varphi(s,t)\,\mathrm{d}(s,t) \\ &= \int \langle m, S_{\varphi_0\, s^{-1}} h_{t^{-1}}\rangle \varphi(s,t)\,\mathrm{d}(s,t) \\ &= \int \langle m, {}_{s^{-1}}h_{t^{-1}}\rangle \varphi(s,t)\,\mathrm{d}(s,t) = \langle m, h\rangle.\end{aligned} \tag{2}$$

Now (1) and (2) imply that m is a topologically Γ-invariant mean on $L^\infty(G)$. $\square$

Definition 3.9. A net (A_α) of measurable subsets of G with $0 < |A_\alpha| < \infty$ is called *asymptotically* Γ-invariant if

$$\frac{|yA_\alpha \Delta A_\alpha z|}{|A_\alpha|} \to 0$$

for every $(y,z) \in \Gamma$.

The following result is the key to the proof of one of our results:

Theorem 3.10 (Theorem 2.3 in [9]). *Let G be a unimodular locally compact group and let Γ be a closed subgroup of $G \times G$. Then the following conditions are equivalent:*

(i) *G is Γ-amenable;*
(ii) *G has an asymptotically Γ-invariant net of subsets.*

Theorem 3.11. *Let G be a Γ-amenable unimodular locally compact group, and let $L^\infty(G)$ have a unique Γ-invariant mean, say m. If (A_α) is an asymptotically Γ-invariant net and $h \in L^\infty(G)$, then $\lim_\alpha |A_\alpha|^{-1} \int_{A_\alpha} h(x)\,\mathrm{d}x \in \mathbb{C}$ and*

$$\langle m, h\rangle = \lim_\alpha \frac{1}{|A_\alpha|}\int_{A_\alpha} h(x)\,\mathrm{d}x.$$

Proof. If $h \in L^\infty(G)_\mathbb{R}$ (subscripts $\mathbb{R}$ indicate that we consider real functions), let $p(h) = \limsup_\alpha |A_\alpha|^{-1}\int_{A_\alpha} h(x)\,\mathrm{d}x$. Obviously p is positively homogenous and subadditive. Since $p(0) = 0$, by the Hahn-Banach theorem there exists $m_0 \in L^\infty(G)^*$ such that $-p(-h) \leqslant \langle m_0, h\rangle \leqslant p(h)$ whenever $h \in L^\infty(G)_\mathbb{R}$. It is easy to see that m_0 is a mean on $L^\infty(G)$. For every $(y,z) \in \Gamma$, $h \in L^\infty(G)_\mathbb{R}$ and α,

$$\left|\frac{1}{|A_\alpha|}\int_{A_\alpha} {}_y h_z - h(x)\,\mathrm{d}x\right| \leqslant \frac{|yA_\alpha \Delta A_\alpha z|}{|A_\alpha|}\|h\|.$$

This shows that

$$\lim_\alpha \frac{1}{|A_\alpha|}\int_{A_\alpha} {}_y h_z - h(x)\,\mathrm{d}x = 0 \tag{1}$$

and so $\langle m_0, {}_y h_z\rangle = \langle m_0, h\rangle$. We conclude that $m_0 = m$. Let $h_1 \in L^\infty(G)_\mathbb{R}$, we will show that the net

$$\lim_\alpha \frac{1}{|A_\alpha|}\int_{A_\alpha} h_1(x)\,\mathrm{d}x$$

converges and

$$\langle m, h_1\rangle = \lim_\alpha \frac{1}{|A_\alpha|}\int_{A_\alpha} h_1(x)\,\mathrm{d}x.$$

If this net did not converge, there would exist $\lambda \neq \langle m, h_1\rangle$ such that

$$-p(-h_1) = \liminf_\alpha \frac{1}{|A_\alpha|}\int_{A_\alpha} h_1(x)\,\mathrm{d}x < \lambda < \limsup_\alpha \frac{1}{|A_\alpha|}\int_{A_\alpha} h_1(x)\,\mathrm{d}x = p(h_1).$$

Write N for the real closed subspace generated by $\{1_G, h_1\}$. By one form of the Hahn-Banach theorem, there exists a linear functional m_1 on N such that $\langle m_1, 1_G\rangle = 1$, $\langle m_1, h_1\rangle = \lambda$, and $-p(-h) \leqslant \langle m_1, h\rangle \leqslant p(h)$ for all $h \in N$ (see B.12 in [11]). By the Hahn-Banach theorem there exists $m_2 \in L^\infty(G)^*$ such that $m_2|_N = m_1$ and $-p(-h) \leqslant \langle m_2, h\rangle \leqslant p(h)$ for all $h \in L^\infty(G)_\mathbb{R}$. Then by (1), m_2 is a Γ-invariant mean on $L^\infty(G)_\mathbb{R}$. By assumption $\lambda = \langle m_1, h_1\rangle = \langle m_2, h_1\rangle = \langle m, h_1\rangle$ which is a contradiction. Clearly

$$\langle m, h\rangle = \lim_\alpha \frac{1}{|A_\alpha|}\int_{A_\alpha} h(x)\,\mathrm{d}x.$$

This completes the proof. $\square$

For functions $\varphi, \psi \in L^1(G)$ and $h \in L^\infty(G)$, we write

$$\varphi \star \psi(x) = \int \psi(y^{-1}xy)\Delta(y)\varphi(y)\,\mathrm{d}y, \quad \varphi \star h(x) = \int h(yxy^{-1})\varphi(y)\,\mathrm{d}y.$$

For more information on this multiplication, see [22]. For $\varphi, \psi \in L^1(G)$, $h \in L^\infty(G)$ and $m, n \in L^\infty(G)^*$, the elements $h \,.\, \varphi$ and $m \,.\, h$ of $L^\infty(G)$ and $m \,.\, n$ of $L^\infty(G)^*$ are defined by

$$\langle h \,.\, \varphi, \psi\rangle = \langle h, \varphi \star \psi\rangle, \quad \langle m \,.\, h, \varphi\rangle = \langle m, h \,.\, \varphi\rangle, \quad \langle m \,.\, n, h\rangle = \langle m, n \,.\, h\rangle,$$

respectively. This is the first Arens multiplication on $L^\infty(G)^*$ [3].

By Proposition 1.10 in [22], G is inner amenable if and only if there exists a mean m on $L^\infty(G)$ such that $\langle m, \varphi \star h\rangle = \langle m, h\rangle$ for all $h \in L^\infty(G)$ and $\varphi \in P^1(G)$.

Theorem 3.12. *For $h \in L^\infty(G)$, the following conditions are equivalent:*

(i) *there exists a mean m on $L^\infty(G)$ such that $\langle m, \varphi \star h\rangle = \langle m, h\rangle$ for all $\varphi \in P^1(G)$;*

(ii) *there exists a net $\{\varphi_\alpha\}$ in $P^1(G)$ such that $\{\varphi_\alpha \,.\, h\}$ converges to a constant function in the weak* topology of $L^\infty(G)$.*

Proof. (i) $\Longrightarrow$ (ii) Let $h \in L^\infty(G)$ and $\langle m, \varphi \star h\rangle = \langle m, h\rangle$ for all $\varphi \in P^1(G)$. By Proposition 3.3 in [20], there exists a net $\{\varphi_\alpha\}$ in $P^1(G)$ that is weak* convergent to m. For any $\varphi \in P^1(G)$,

$$\begin{aligned}\lim_\alpha \langle \varphi_\alpha . h, \varphi\rangle &= \lim_\alpha \langle \varphi_\alpha, h \,.\, \varphi\rangle = \langle m, \varphi \star h\rangle \\ &= \langle m, h\rangle = \langle m, h\rangle\langle 1_G, \varphi\rangle.\end{aligned}$$

As $P^1(G)$ spans $L^1(G)$, we conclude that $\langle m, h\rangle 1_G$ belongs to the weak* closure of $\{\varphi \,.\, h\colon \varphi \in P^1(G)\}$.

(ii) $\Longrightarrow$ (i) Let $h \in L^\infty(G)$. Assume that there exists a net $\{\varphi_\alpha\}$ in $P^1(G)$ such that $\{\varphi_\alpha \,.\, h\}$ converges to a constant function $c1_G$ in the weak* topology of $L^\infty(G)$. Let n be a weak* cluster point of the net $\{\varphi_\alpha\}$ in $L^\infty(G)^*$. Then n is a mean on $L^\infty(G)$. For every $\psi \in L^1(G)$,

$$\begin{aligned}\langle n \,.\, h, \psi\rangle = \langle n, h \,.\, \psi\rangle &= \lim_\alpha \langle \varphi_\alpha, h \,.\, \psi\rangle \\ &= \lim_\alpha \langle \varphi_\alpha \,.\, h, \psi\rangle = c\langle 1_G, \psi\rangle.\end{aligned}$$

This shows that $n \cdot h = c1_G$. Finally, let $m = n \,.\, n$. Then m is a mean on $L^\infty(G)$ and for every $\varphi \in P^1(G)$,

$$\begin{aligned}\langle m, \varphi \star h\rangle = \langle m, h \,.\, \varphi\rangle &= \langle n, n \,.\, (h \,.\, \varphi)\rangle = \langle n, (n \,.\, h) \,.\, \varphi\rangle \\ &= c\langle n, 1_G \,.\, \varphi\rangle = \langle n, n \,.\, h\rangle = \langle m, h\rangle.\end{aligned}$$

This completes the proof. $\square$

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