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APPROXIMATIONS BY THE CAUCHY-TYPE INTEGRALS WITH PIECEWISE LINEAR DENSITIES

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Abstract. The paper is a contribution to the complex variable boundary element method, shortly CVBEM. It is focused on Jordan regions having piecewise regular boundaries without cusps. Dini continuous densities whose modulus of continuity $\omega(\cdot)$ satisfies

$$\limsup_{s\downarrow 0} \omega(s) \ln \frac{1}{s} = 0$$

are considered on these boundaries. Functions satisfying the Hölder condition of order α, $0 < \alpha \leqslant 1$, belong to them. The statement that any Cauchy-type integral with such a density can be uniformly approximated by a Cauchy-type integral whose density is a piecewise linear interpolant of the original one is proved under the assumption that the mesh of the interpolation nodes is sufficiently fine and uniform. This result ensures the existence of approximate CVBEM solutions of some planar boundary value problems, especially of the Dirichlet ones.

Keywords: Cauchy-type integral, Dini continuous density, piecewise linear interpolation, uniform convergence, complex variable boundary element method

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1. Introduction

The exact solution of a planar boundary value problem can often be expressed by means of a Cauchy-type integral along the problem region boundary. For example, let Ω be a region in the complex plane $\mathbb{C}$, $[\Gamma]$ the boundary of Ω, $h\colon [\Gamma] \to \mathbb{R}$ a continuous function and u the exact solution of the Dirichlet boundary-value problem

$$\text{(D)} \qquad \Delta u = 0 \text{ in } \Omega, \quad u = h \text{ on } [\Gamma].$$

Since u is harmonic in Ω, it coincides with the real part of some holomorphic function there. Since u is continuous on $\Omega \cup [\Gamma]$, it is a continuous extension of the real part of this holomorphic function over $\Omega \cup [\Gamma]$. Any function that is holomorphic in some region and whose real part has a continuous extension over the closure of that region can be represented by a Cauchy-type integral along the boundary of that region ([7, p. 255]). Thus the exact solution of the problem (D) can be expressed by the real part of some Cauchy-type integral along $[\Gamma]$ in Ω.

Because most of the Cauchy-type integrals cannot be evaluated exactly, they are often replaced by suitable approximants (see [1] for summary) that can be evaluated exactly as elementary functions. The approximant that is sufficiently close to the approximated Cauchy-type integral determines an approximate solution of the boundary value problem.

Within the CVBEM ([2], [3]), a Cauchy-type integral whose density is a piecewise linear interpolant of the original one is the approximant. In [4] it was proved that any function holomorphic in the closure of some region can be approximated by means of this integral that is considered along the boundary of the region to any degree of accuracy. However, the class of the functions that can be approximated in this way appears to be much wider than it is observed in [4]. Actually, from [5, p. 450–452] it follows that this class contains all Cauchy-type integrals along the region boundary whose density satisfies the Hölder condition of order α, where $0 < \alpha \leqslant 1$. The purpose of the present work is to give some further generalization of the result from [4] by weakening the assumptions on density.

Let us note that these results involve the existence of the so called approximate CVBEM solution of the boundary value problem as close to the exact one as we wish.

2. Preliminaries

Definition 1. Let M be a non-empty subset of the complex plane $\mathbb{C}$ and let $g\colon M \to \mathbb{C}$ be a function. The mapping $\omega_g\colon \langle 0, +\infty) \to \langle 0, +\infty\rangle$ defined by

$$\omega_g(\delta) = \sup\{|g(z) - g(w)|\colon |z - w| \leqslant \delta;\ z, w \in M\}, \quad \delta \in \langle 0, +\infty),$$

is called the *modulus of continuity of the function* g. The function g is called *Dini continuous* if there exists $\tau > 0$ such that

$$\int_0^\tau \frac{\omega_g(s)}{s}\,\mathrm{d}s < +\infty.$$

Remark 1. We use the familiar facts that any modulus of continuity is a non-decreasing function and that $\lim\limits_{\delta \downarrow 0} \omega_g(\delta) = 0$ if and only if the function g is uniformly continuous. If we assume that g is Dini continuous, then it is clear that

$$\lim_{\tau\downarrow 0}\int_0^{\tau}\frac{\omega_g(s)}{s}\,\mathrm{d}s = 0 \tag{2.1}$$

and it is also easy to show that

$$\liminf_{\tau\downarrow 0}\omega_g(\tau)\ln\frac{1}{\tau} = 0. \tag{2.2}$$

Actually, if $\liminf\limits_{\tau\downarrow 0}\omega_g(\tau)\ln\frac{1}{\tau} > 0$, then $\omega_g(\tau)/\tau \geqslant \varepsilon/(\tau\ln\frac{1}{\tau})$ for sufficiently small positive numbers τ, ε and, consequently, $\int_0^{\tau}\varepsilon/(s\ln\frac{1}{s})\,\mathrm{d}s = +\infty$ implies $\int_0^{\tau}\frac{1}{s}\omega_g(s)\,\mathrm{d}s = +\infty$, which contradicts the Dini continuity of g.

Definition 2. Let $\alpha\in\mathbb{R}$ and $T>0$. A continuous mapping $\Gamma\colon\langle\alpha,\alpha+T\rangle\to\mathbb{C}$ is called the *piecewise regular Jordan path without cusps* (briefly the *path*) if it has the following properties:

(i) $\Gamma(\alpha)=\Gamma(\alpha+T)$,

(ii) for arbitrary $t,t'\in\langle\alpha,\alpha+T\rangle$

$$0<|t-t'|<T \Longrightarrow \Gamma(t)\neq\Gamma(t'), \tag{2.3}$$

(iii) there exists a sequence $\{\tau_j\}_{j=0}^{n}$ such that

$$\alpha=\tau_0<\tau_1<\ldots<\tau_n=\alpha+T$$

and that the derivative Γ' of Γ is continuous and nonzero in every interval $\langle\tau_j,\tau_{j+1}\rangle$, where $j\in\{0,\ldots,n-1\}$,

(iv) for arbitrary $t\in\langle\alpha,\alpha+T\rangle$

$$\frac{\Gamma'_+(t)}{\Gamma'_-(t)}\in\mathbb{C}\setminus(-\infty,0\rangle, \tag{2.4}$$

where $\Gamma'_+(t)$ and $\Gamma'_-(t)$ are the derivatives on the right and left of Γ at the point t and $\Gamma'_-(\alpha),\Gamma'_+(\alpha+T)$ are to be defined by $\Gamma'_-(\alpha+T)$, $\Gamma'_+(\alpha)$, respectively.

In some cases, we denote by Γ the T-periodic extension of the path Γ to the set of real numbers. We set

$$[\Gamma]=\{\Gamma(t)\colon\ \alpha\leqslant t\leqslant\alpha+T\}$$

and denote by Ω the bounded component of the open set $\mathbb{C}\setminus[\Gamma]$ in the sense of the Jordan theorem. We extend the sequence $\{\tau_j\}_{j=0}^{n-1}$ to a collection $\{\tau_k\colon\ k\in\mathbb{Z}\}$ so that for every $k\in\mathbb{Z}$ we put $\tau_k:=\tau_j+pT$ whenever $j\in\{0,\ldots,n-1\}$ and the integer p satisfies $pn=k-j$.

Definition 3. Let a path Γ and $m \geqslant 2$ be given. We say that a sequence $\mathcal{P} = \{\Gamma_j\}_{j=0}^{m-1}$ is a *partition* of Γ when there exists a sequence $\{t_j\}_{j=0}^{m}$ such that

$$t_0 < t_1 < \ldots < t_m = t_0 + T$$

and $\Gamma_j = \Gamma|_{\langle t_j, t_{j+1}\rangle}$ for every $j \in \{0, \ldots, m-1\}$. We extend the sequence $\{t_j\}_{j=0}^{m-1}$ to a collection $\{t_k\colon\ k \in \mathbb{Z}\}$ so that for every $k \in \mathbb{Z}$ we put $t_k := t_j + pT$ whenever $j \in \{0, \ldots, m-1\}$ and the integer p satisfies $pm = k-j$. We extend the sequence $\{\Gamma_j\}_{j=0}^{m-1}$ to a collection $\{\Gamma_k\colon\ k \in \mathbb{Z}\}$ so that for every $k \in \mathbb{Z}$ we put $\Gamma_k := \Gamma_j$ whenever $j \in \{0, \ldots, m-1\}$ and $k - j$ is divisible by m. For every $k \in \mathbb{Z}$ we introduce

$$z_k = \Gamma(t_k), \quad [\Gamma_k] = \{\Gamma(t)\colon\ t_k \leqslant t \leqslant t_{k+1}\}.$$

Definition 4. We say that a function $l\colon\ [\Gamma] \to \mathbb{C}$ is *piecewise linear on* $[\Gamma]$ *with respect to a partition* $\mathcal{P} = \{\Gamma_j\}_{j=0}^{m-1}$ if l coincides with some linear function on $[\Gamma_j]$ for every $j \in \{0, \ldots, m-1\}$. By the symbol $g_{\mathcal{P}}$ we denote the piecewise linear interpolant of a function $g\colon\ [\Gamma] \to \mathbb{C}$ with respect to a partition $\mathcal{P} = \{\Gamma_j\}_{j=0}^{m-1}$, i.e., a function $g_{\mathcal{P}}\colon\ [\Gamma] \to \mathbb{C}$ that is piecewise linear on $[\Gamma]$ with respect to $\mathcal{P}$ and satisfies

$$g_{\mathcal{P}}(z_j) = g(z_j).$$

Remark 2. The derivative Γ' of Γ is continuous and nonzero in the interval $\langle \tau_k, \tau_{k+1}\rangle$ for every $k \in \mathbb{Z}$ and the conditions (2.3), (2.4) as well as

$$|\Gamma(t) - \Gamma(t')| = \left|\int_t^{t'} \Gamma'(s)\,\mathrm{d}s\right| \leqslant V|t - t'| \tag{2.5}$$

are satisfied for arbitrary $t, t' \in \mathbb{R}$. Here

$$V = \sup\{|\Gamma'(t)|\colon\ t \in \mathbb{R} \setminus \{\tau_k\colon\ k \in \mathbb{Z}\}\}.$$

Lemma 1. *Let l be a piecewise linear function on $[\Gamma]$ with respect to a partition $\mathcal{P} = \{\Gamma_j\}_{j=0}^{m-1}$ and let*

$$K = V \max_{0 \leqslant j \leqslant m-1}\left|\frac{l(z_{j+1}) - l(z_j)}{z_{j+1} - z_j}\right| \quad \left(= V \max_{k \in \mathbb{Z}}\left|\frac{l(z_{k+1}) - l(z_k)}{z_{k+1} - z_k}\right|\right).$$

Then

$$|l(\Gamma(t)) - l(\Gamma(t'))| \leqslant K|t - t'|, \quad t, t' \in \mathbb{R}.$$

Proof. Let $t, t' \in \mathbb{R}$. Without loss of generality we suppose that $t \leqslant t'$. Then

$$t \in \langle t_j, t_{j+1}\rangle, \quad t' \in \langle t_k, t_{k+1}\rangle$$

for some $j, k \in \mathbb{Z}$, where $j \leqslant k$. First, let $j = k$. Clearly

$$t_k \leqslant t \leqslant t' \leqslant t_{k+1}.$$

As l coincides with some linear function on $[\Gamma_k]$, we have

$$l(\Gamma(t)) - l(\Gamma(t')) = \frac{l(z_{k+1}) - l(z_k)}{z_{k+1} - z_k}(\Gamma(t) - \Gamma(t')).$$

This identity and (2.5) imply

$$|l(\Gamma(t)) - l(\Gamma(t'))| \leqslant K(t' - t).$$

Next, let $j < k$. Then

$$t_j \leqslant t \leqslant t_{j+1} \leqslant t_k \leqslant t' \leqslant t_{k+1}$$

and, by virtue of the foregoing case ($j = k$), we obtain

$$\begin{aligned} |l(\Gamma(t)) - l(\Gamma(t'))| &\leqslant |l(\Gamma(t)) - l(z_{j+1})| + |l(z_{j+1}) - l(z_{j+2})| + \dots \\ &\quad + |l(z_{k-1}) - l(z_k)| + |l(z_k) - l(\Gamma(t'))| \\ &\leqslant K((t_{j+1} - t) + (t_{j+2} - t_{j+1}) + \dots + (t_k - t_{k-1}) + (t' - t_k)) \\ &= K(t' - t). \end{aligned}$$

□

Definition 5. Let us assume that $g\colon [\Gamma] \to \mathbb{C}$ is such a function that the integral

$$\int_\Gamma \frac{g(\zeta) - g(z)}{\zeta - z}\, \mathrm{d}\zeta$$

absolutely converges for every $z \in [\Gamma]$. By means of Cauchy-type integrals, we define the function $\mathscr{C}^-(g)\colon \Omega \cup [\Gamma] \to \mathbb{C}$ by

$$\begin{aligned} \mathscr{C}^-(g)(z) &= \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{g(\zeta)}{\zeta - z}\, \mathrm{d}\zeta, && z \in \Omega, \\ \mathscr{C}^-(g)(z) &= \lim_{\substack{w \to z \\ w \in \Omega}} \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{g(\zeta)}{\zeta - w}\, \mathrm{d}\zeta, && z \in [\Gamma]. \end{aligned}$$

R e m a r k 3. The definition of $\mathscr{C}^-(g)$ is correct because the limit exists under the considered assumption (see [6, p. 191]). We use the familiar fact that the function $\mathscr{C}^-(g)$ is holomorphic in Ω.

Our aim is to generalize the following result obtained by R. J. Whitley and T. V. Hromadka II in [4, Theorem 1].

Theorem 1. *Let f be a function holomorphic in the closure of Ω and let $g = f|_{[\Gamma]}$. Then for any $\varepsilon > 0$ there exists a partition $\mathcal{P} = \{\Gamma_j\}_{j=0}^{m-1}$ such that*

$$|\mathscr{C}^-(g)(z) - \mathscr{C}^-(g_{\mathcal{P}})(z)| < \varepsilon, \quad z \in \Omega \cup [\Gamma].$$

(Note that f coincides with $\mathscr{C}^-(g)$ on $\Omega \cup [\Gamma]$ by virtue of the Cauchy integral formula.)

Remark 4. Due to [5, p. 450–452], the assumptions of Theorem 1 can be replaced by "let $g\colon [\Gamma] \to \mathbb{C}$ be a function satisfying the Hölder condition of order α, where $0 < \alpha \leqslant 1$".

Another generalization of Theorem 1 is provided by our main statement, Theorem 3.

3. Some estimates

In this section we deal with some properties of the path Γ in order to prove the forthcoming Theorem 2 that states some important estimates for Γ.

Definition 6. We define the function $r\colon \langle 0, \frac{1}{2}T\rangle \to \langle 0, +\infty)$ by

$$r(s) = \min |\Gamma(t) - \Gamma(t+s)|\colon\ 0 \leqslant t \leqslant T\}, \quad s \in \langle 0, \tfrac{1}{2}T\rangle.$$

Since for every $s \in \mathbb{R}$ the function $t \mapsto |\Gamma(t) - \Gamma(t+s)|$ is T-periodic, we have

$$r(s) = \min\{|\Gamma(t) - \Gamma(t')|\colon\ |t - t'| = s;\ t, t' \in \mathbb{R}\}, \quad s \in \langle 0, \tfrac{1}{2}T\rangle. \tag{3.1}$$

Lemma 2. *The function r is uniformly continuous on $\langle 0, \frac{1}{2}T\rangle$ and positive in $(0, \frac{1}{2}T\rangle$.*

Proof. We set

$$M = \{z \in \mathbb{C}\colon\ 0 \leqslant \operatorname{Re} z \leqslant T, 0 \leqslant \operatorname{Im} z \leqslant \tfrac{1}{2}T\}$$

and

$$f(z) = |\Gamma(\operatorname{Re} z) - \Gamma(\operatorname{Re} z + \operatorname{Im} z)|$$

for all $z \in M$. The function f is uniformly continuous on the set M and

$$r(s) = \min\{f(t + \mathrm{i}s)\colon\ 0 \leqslant t \leqslant T\}, \quad s \in \langle 0, \tfrac{1}{2}T\rangle.$$

Let $s, s' \in \langle 0, \frac{1}{2}T\rangle$. Then for $t \in \langle 0, T\rangle$ we have

$$r(s) - f(t + \mathrm{i}s') \leqslant f(t + \mathrm{i}s) - f(t + \mathrm{i}s') \leqslant \omega_f(|s - s'|),$$
$$r(s') - f(t + \mathrm{i}s) \leqslant f(t + \mathrm{i}s') - f(t + \mathrm{i}s) \leqslant \omega_f(|s - s'|),$$

whence

$$r(s) - r(s') = \max\{r(s) - f(t + \mathrm{i}s')\colon\ 0 \leqslant t \leqslant T\} \leqslant \omega_f(|s - s'|),$$
$$r(s') - r(s) = \max\{r(s') - f(t + \mathrm{i}s)\colon\ 0 \leqslant t \leqslant T\} \leqslant \omega_f(|s - s'|).$$

Thus, it is proved that

$$|r(s) - r(s')| \leqslant \omega_f(|s - s'|), \quad s, s' \in \langle 0, \tfrac{1}{2}T\rangle.$$

This inequality and Remark 1 say that r is uniformly continuous on $\langle 0, \frac{1}{2}T\rangle$.

The function r is nonnegative. If $r(s) = 0$, then, according to (3.1), there exist $t, t' \in \mathbb{R}$ such that $\Gamma(t) = \Gamma(t')$ and $|t - t'| = s$. It follows $s = 0$ due to (2.3), so that r is positive in $(0, \frac{1}{2}T\rangle$. □

Definition 7. Let $t \in \mathbb{R}$ and let s_t be an arbitrary solution of the problem

$$|(1 - s)\Gamma'_-(t) + s\Gamma'_+(t)| \to \min, \quad s \in \langle 0, 1\rangle.$$

We define

$$\Gamma^\circ(t) = (1 - s_t)\Gamma'_-(t) + s_t\Gamma'_+(t).$$

The geometrical meaning of $|\Gamma^\circ(t)|$ is the distance of the line segment, which has the endpoints $\Gamma'_-(t)$, $\Gamma'_+(t)$, from the origin. Since (2.4) ensures that the origin does not belong to this line segment, $|\Gamma^\circ(t)|$ is positive. Moreover,

$$|\Gamma^\circ(t)| \leqslant |(1 - s)\Gamma'_-(t) + s\Gamma'_+(t)|, \quad s \in \langle 0, 1\rangle, \tag{3.2}$$

whence especially

$$|\Gamma^\circ(t)| \leqslant \min\{|\Gamma'_-(t)|, |\Gamma'_+(t)|\}. \tag{3.3}$$

The mapping

$$\mathbb{R} \ni t \mapsto \Gamma^\circ(t)$$

is T-periodic and obviously

$$|\Gamma^\circ(t)| = |\Gamma'_-(t)| = |\Gamma'_+(t)|, \quad t \in \mathbb{R} \setminus \{\tau_k\colon\ k \in \mathbb{Z}\}. \tag{3.4}$$

Let us consider $k \in \mathbb{Z}$ and $t \in (\tau_k, \tau_{k+1})$. Since $\Gamma'_-(t) = \Gamma'_+(t) = \Gamma'(t)$, we have $|\Gamma^\circ(t)| = |\Gamma'(t)|$. Moreover, $|\Gamma^\circ(\tau_k)| \leqslant \Gamma'_+(\tau_k)|$ and $|\Gamma^\circ(\tau_{k+1})| \leqslant |\Gamma'_-(\tau_{k+1})|$ if we take into account (3.3). We see that the number

$$v_k = \min\{|\Gamma^\circ(t)|\colon\ \tau_k \leqslant t \leqslant \tau_{k+1}\}$$

exists. It is clear that v_k is positive.

Definition 8. We define the number

$$v^* = \min\{v_k\colon\ k \in \mathbb{Z}\} \quad (= \min\{v_j\colon\ 0 \leqslant j \leqslant n-1\})$$

and the set Q of all positive numbers q for which there exists $\delta \in (0, \frac{1}{2}T\rangle$ such that

$$0 \leqslant s \leqslant \delta \Longrightarrow r(s) \geqslant qs, \tag{3.5}$$

i.e.,

$$Q = \{q > 0\colon\ \exists\, \delta \in (0, \tfrac{1}{2}T\rangle \ \text{ satisfying } \ (3.5)\}.$$

Lemma 3. *We have* $(0, v^*) \subseteq Q \subseteq (0, v^*\rangle$.

P r o o f. For every $k \in \mathbb{Z}$ the function

$$\Gamma'_k = \Gamma'|_{\langle \tau_k, \tau_{k+1}\rangle}$$

is uniformly continuous on $\langle \tau_k, \tau_{k+1}\rangle$. Thus, if we take into account Remark 1, the function $\omega\colon\ \langle 0, +\infty) \to \langle 0, +\infty)$ defined by

$$\omega(s) = \max_{k\in\mathbb{Z}} \omega_{\Gamma'_k}(s) \quad (= \max_{0\leqslant j\leqslant n-1} \omega_{\Gamma'_j}(s)), \quad s \in \langle 0, +\infty),$$

is nondecreasing and

$$\lim_{s\downarrow 0} \omega(s) = 0. \tag{3.6}$$

Now we prove the inclusion $(0, v^*) \subseteq Q$. Let $q \in (0, v^*)$. According to (3.6), there exists $\delta_0 \in (0, \frac{1}{2}T\rangle$ such that

$$\omega(\delta_0) \leqslant v^* - q.$$

We put

$$\delta := \min\{\delta_0, \min\{\tau_{k+1} - \tau_k\colon\ k \in \mathbb{Z}\}\}$$

and consider $t, t' \in \mathbb{R}$ satisfying $0 < t' - t \leqslant \delta$. There exists a unique $k \in \mathbb{Z}$ such that $t' \in (\tau_k, \tau_{k+1}\rangle$. If $t \geqslant \tau_k$, then $\tau_k \leqslant t < t' \leqslant \tau_{k+1}$ and

$$\Gamma(t') - \Gamma(t) = \int_t^{t'} \Gamma'(s)\,\mathrm{d}s = \Gamma'_+(t)(t'-t) + \int_t^{t'} (\Gamma'_k(s) - \Gamma'_k(t))\,\mathrm{d}s, \tag{3.7}$$

which, by virtue of (3.3), yields

$$|\Gamma(t) - \Gamma(t')| \geqslant |\Gamma^\circ(t)|(t'-t) - \omega(t'-t)\int_t^{t'} \mathrm{d}s \geqslant (v^* - \omega(\delta_0))(t'-t) \geqslant q(t'-t).$$

If $t < \tau_k$, then $\tau_{k-1} < t < \tau_k < t' < \tau_{k+1}$ and consequently

$$\begin{aligned}\Gamma(t') - \Gamma(t) &= \int_t^{\tau_k} \Gamma'(s)\,\mathrm{d}s + \int_{\tau_k}^{t'} \Gamma'(s)\,\mathrm{d}s \\ &= \Gamma'_-(\tau_k)(\tau_k - t) + \int_t^{\tau_k} (\Gamma'_{k-1}(s) - \Gamma'_{k-1}(\tau_k))\,\mathrm{d}s \\ &\quad + \Gamma'_+(\tau_k)(t' - \tau_k) + \int_{\tau_k}^{t'} (\Gamma'_k(s) - \Gamma'_k(\tau_k))\,\mathrm{d}s \\ &= \Big\{\Big(1 - \frac{t'-\tau_k}{t'-t}\Big)\Gamma'_-(\tau_k) + \frac{t'-\tau_k}{t'-t}\,\Gamma'_+(\tau_k)\Big\}(t'-t) \\ &\quad + \int_t^{\tau_k} (\Gamma'_{k-1}(s) - \Gamma'_{k-1}(\tau_k))\,\mathrm{d}s + \int_{\tau_k}^{t'} (\Gamma'_k(s) - \Gamma'_k(\tau_k))\,\mathrm{d}s,\end{aligned} \tag{3.8}$$

which, by virtue of (3.2), implies

$$\begin{aligned}|\Gamma(t) - \Gamma(t')| &\geqslant |\Gamma^\circ(\tau_k)|(t'-t) - \omega(\tau_k - t)\int_t^{\tau_k} \mathrm{d}s - \omega(t' - \tau_k)\int_{\tau_k}^{t'} \mathrm{d}s \\ &\geqslant (v^* - \omega(\delta_0))(t'-t) \geqslant q(t'-t).\end{aligned}$$

Analogously, the same result can be established in the case $0 < t - t' \leqslant \delta$ and it is evidently valid for $t = t'$. Hence,

$$|t - t'| \leqslant \delta \Longrightarrow |\Gamma(t) - \Gamma(t')| \geqslant q|t - t'|.$$

This is equivalent to (3.5) if we take into account (3.1). We see that $q \in Q$.

In order to prove the inclusion $Q \subseteq (0, v^*\rangle$, we prove the following equivalent statement:

$$q \notin (0, v^*\rangle \Longrightarrow q \notin Q.$$

If $q \leqslant 0$, then $q \notin Q$ holds trivially. Let $q > v^*$. We show that

$$(\forall\,\delta \in (0, \tfrac{1}{2}T\rangle)\ (\exists\, t, t' \in \mathbb{R})\colon\ |t - t'| \leqslant \delta \wedge |\Gamma(t) - \Gamma(t')| < q|t - t'|, \tag{3.9}$$

because it is equivalent to $q \notin Q$ due to (3.1). There exists $\varepsilon > 0$ such that $v^* + \varepsilon < q$ and, according to (3.6), there exists $\delta_1 > 0$ such that $\omega(\delta_1) < \varepsilon$. We denote by t^* an arbitrary real number satisfying $|\Gamma^\circ(t^*)| = v^*$. Then there exists a unique $k \in \mathbb{Z}$ such that $t^* \in \langle \tau_k, \tau_{k+1})$. Now let $\delta \in (0, \frac{1}{2}T\rangle$. If $t^* > \tau_k$, we put

$$c := \min\{\delta, \delta_1, \tau_{k+1} - t^*\}, \quad t := t^*, \quad t' := t^* + c.$$

Then $\tau_k < t < t' \leqslant \tau_{k+1}$. Consequently, (3.7) and (3.4) imply

$$\begin{aligned} |\Gamma(t) - \Gamma(t')| &\leqslant |\Gamma^\circ(t)|(t' - t) + \omega(t' - t) \int_t^{t'} \mathrm{d}s \\ &\leqslant (v^* + \omega(\delta_1))(t' - t) < (v^* + \varepsilon)(t' - t) < q(t' - t). \end{aligned}$$

If $t^* = \tau_k$, we put

$$c := \min\{\delta, \delta_1, \tau_{k+1} - \tau_k, \tau_k - \tau_{k-1}\}, \quad t := \tau_k - c(1 - s_{\tau_k}), \quad t' := \tau_k + cs_{\tau_k}.$$

Then $\tau_{k-1} \leqslant t \leqslant \tau_k \leqslant t' \leqslant \tau_{k+1}$ and

$$\Big(1 - \frac{t' - \tau_k}{t' - t}\Big)\Gamma'_-(\tau_k) + \frac{t' - \tau_k}{t' - t}\,\Gamma'_+(\tau_k) = (1 - s_{\tau_k})\Gamma'_-(\tau_k) + s_{\tau_k}\Gamma'_+(\tau_k) = \Gamma^\circ(\tau_k).$$

This identity and (3.8) imply

$$\begin{aligned} |\Gamma(t) - \Gamma(t')| &\leqslant |\Gamma^\circ(\tau_k)|(t' - t) + \omega(\tau_k - t) \int_t^{\tau_k} \mathrm{d}s + \omega(t' - \tau_k) \int_{\tau_k}^{t'} \mathrm{d}s \\ &\leqslant (v^* + \omega(\delta_1))(t' - t) < (v^* + \varepsilon)(t' - t) < q(t' - t). \end{aligned}$$

The proof of (3.9) and thus the whole proof is complete. □

The main result of this section is presented by the following theorem that contains two important estimates.

Theorem 2. *There exist $q \in Q$ and $\hat{\Delta} \in (0, \frac{1}{2}T\rangle$ such that for every $\Delta \in (0, \hat{\Delta}\rangle$ and $t, t' \in \mathbb{R}$ we have*

$$|t - t'| \leqslant \Delta \Longrightarrow |\Gamma(t) - \Gamma(t')| \geqslant q|t - t'|, \tag{3.10}$$

$$\Delta \leqslant |t - t'| \leqslant \tfrac{1}{2}T \Longrightarrow |\Gamma(t) - \Gamma(t')| \geqslant q\Delta. \tag{3.11}$$

P r o o f. The set Q is non-empty by Lemma 3. Let us consider an arbitrary $q \in Q$. Then there exists $\delta \in (0, \frac{1}{2}T\rangle$ satisfying (3.5) and, according to Lemma 2, the number

$$c = \min\{r(s)\colon\ \delta \leqslant s \leqslant \tfrac{1}{2}T\}$$

exists and is positive. We set

$$\hat{\Delta} = \min\{\delta, \tfrac{c}{q}\}.$$

Now let $\Delta \in (0, \hat{\Delta}\rangle$ and $t, t' \in \mathbb{R}$. Then (3.10) as well as

$$\Delta \leqslant |t - t'| \leqslant \delta \Longrightarrow |\Gamma(t) - \Gamma(t')| \geqslant q|t - t'| \geqslant q\Delta,$$
$$\delta \leqslant |t - t'| \leqslant \tfrac{1}{2}T \Longrightarrow |\Gamma(t) - \Gamma(t')| \geqslant c \geqslant q\hat{\Delta} \geqslant q\Delta$$

follow from (3.5) and (3.1). The last two implications yield (3.11). □

Remark 5. A function l that is piecewise linear on $[\Gamma]$ with respect to a partition $\mathcal{P} = \{\Gamma_j\}_{j=0}^{m-1}$ is Dini continuous, because

$$\omega_l(s) \leqslant K \sup\{|t - t'|\colon\ |\Gamma(t) - \Gamma(t')| \leqslant s\} \leqslant \frac{K}{q} s, \quad s \in (0, \hat{\Delta}\rangle,$$

due to Lemma 1 and (3.10).

4. Convergence

In this section we consider a Dini continuous function $g\colon [\Gamma] \to \mathbb{C}$, the quantity V from Remark 2 and numbers $q, \hat{\Delta}$ from Theorem 2. Further, we consider a fixed $\mu \in (0, 1\rangle$ and denote by $\mathscr{P}$ the set of all partitions $\mathcal{P} = \{\Gamma_j\}_{j=0}^{m-1}$ such that

$$t_{j+1} - t_j \geqslant \mu\nu(\mathcal{P}), \quad j = 0, \ldots, m-1,$$

with

$$\nu(\mathcal{P}) = \max\{t_{j+1} - t_j\colon\ 0 \leqslant j \leqslant m-1\}.$$

For any $\tau \in (0, +\infty)$ we set

$$\mathcal{A}(\tau) = \omega_{g\circ\Gamma}\Big(\frac{1}{2}\tau\Big) + \frac{V}{2q\mu}\,\omega_{g\circ\Gamma}(\tau),$$
$$\mathcal{B}(\tau) = \frac{2V}{q}\Big(\int_0^\tau \frac{\omega_{g\circ\Gamma}(s)}{s}\,\mathrm{d}s + \frac{V}{q\mu}\,\omega_{g\circ\Gamma}(\tau) + 2\mathcal{A}(\tau)\Big(\ln\frac{\hat{\Delta}}{\tau} + \frac{T}{2\hat{\Delta}} - 1\Big)\Big).$$

Lemma 4. *Let $\mathcal{P} \in \mathscr{P}$ and $\nu(\mathcal{P}) \leqslant \hat{\Delta}$. Then for every $z \in [\Gamma]$ we have*

$$|(g - g_{\mathcal{P}})(z)| \leqslant \mathcal{A}(\nu(\mathcal{P})), \tag{4.1}$$
$$\Big|\int_\Gamma \frac{(g - g_{\mathcal{P}})(\zeta) - (g - g_{\mathcal{P}})(z)}{\zeta - z}\,\mathrm{d}\zeta\Big| \leqslant \mathcal{B}(\nu(\mathcal{P})). \tag{4.2}$$

P r o o f. According to (3.10), for every $j \in \{0,\dots,m-1\}$ we have

$$\Big|\frac{g_{\mathcal P}(z_{j+1})-g_{\mathcal P}(z_j)}{z_{j+1}-z_j}\Big| = \Big|\frac{g(\Gamma(t_{j+1}))-g(\Gamma(t_j))}{\Gamma(t_{j+1})-\Gamma(t_j)}\Big| \leqslant \frac{\omega_{g\circ\Gamma}(t_{j+1}-t_j)}{q(t_{j+1}-t_j)} \leqslant \frac{\omega_{g\circ\Gamma}(\nu(\mathcal P))}{q\mu\nu(\mathcal P)}.$$

Then

$$V \max_{0\leqslant j\leqslant m-1}\Big|\frac{g_{\mathcal P}(z_{j+1})-g_{\mathcal P}(z_j)}{z_{j+1}-z_j}\Big| \leqslant \frac{V\omega_{g\circ\Gamma}(\nu(\mathcal P))}{q\mu\nu(\mathcal P)}$$

and from Lemma 1 it follows that

$$|g_{\mathcal P}(\Gamma(t))-g_{\mathcal P}(\Gamma(t'))| \leqslant \frac{V\omega_{g\circ\Gamma}(\nu(\mathcal P))}{q\mu\nu(\mathcal P)}\,|t-t'|, \quad t,t'\in\mathbb R. \tag{4.3}$$

Now let $z\in[\Gamma]$. There exists a unique $t\in\langle t_0,t_m)$ such that $z=\Gamma(t)$ and there exists a unique $j\in\{0,\dots,m-1\}$ such that $t\in\langle t_j,t_{j+1})$. Without loss of generality, let us suppose that $t-t_j\leqslant\frac12(t_{j+1}-t_j)$ (otherwise, we consider t_{j+1} instead of t_j in the following computation). If we make use of (4.3), we obtain

$$\begin{aligned}
|(g-g_{\mathcal P})(z)| &= |g(\Gamma(t))-g(\Gamma(t_j))+g_{\mathcal P}(\Gamma(t_j))-g_{\mathcal P}(\Gamma(t))|\\
&\leqslant |g(\Gamma(t))-g(\Gamma(t_j))|+|g_{\mathcal P}(\Gamma(t_j))-g_{\mathcal P}(\Gamma(t))|\\
&\leqslant \omega_{g\circ\Gamma}(t-t_j)+\frac{V\omega_{g\circ\Gamma}(\nu(\mathcal P))}{q\mu\nu(\mathcal P)}\,(t-t_j)\\
&\leqslant \omega_{g\circ\Gamma}\Big(\frac12\nu(\mathcal P)\Big)+\frac{V\omega_{g\circ\Gamma}(\nu(\mathcal P))}{2q\mu}=\mathcal A(\nu(\mathcal P)),
\end{aligned}$$

so that (4.1) holds. We set

$$L(s)=\frac{|(g-g_{\mathcal P})(\Gamma(s))-(g-g_{\mathcal P})(\Gamma(t))|}{|\Gamma(s)-\Gamma(t)|}\,|\Gamma'(s)|.$$

It is clear that

$$\Big|\int_\Gamma \frac{(g-g_{\mathcal P})(\zeta)-(g-g_{\mathcal P})(z)}{\zeta-z}\,\mathrm d\zeta\Big| \leqslant \int_{t-T/2}^{t+T/2} L(s)\,\mathrm ds. \tag{4.4}$$

Consequently,

$$\begin{aligned}
\int_t^{t+\nu(\mathcal P)} L(s)\,\mathrm ds &\leqslant \frac Vq\int_t^{t+\nu(\mathcal P)}\frac{|g(\Gamma(s))-g(\Gamma(t))|+|g_{\mathcal P}(\Gamma(s))-g_{\mathcal P}(\Gamma(t))|}{s-t}\,\mathrm ds\\
&\leqslant \frac Vq\Big(\int_t^{t+\nu(\mathcal P)}\frac{\omega_{g\circ\Gamma}(s-t)}{s-t}\,\mathrm ds+\frac{V\omega_{g\circ\Gamma}(\nu(\mathcal P))}{q\mu\nu(\mathcal P)}\int_t^{t+\nu(\mathcal P)}\mathrm ds\Big)\\
&= \frac Vq\Big(\int_0^{\nu(\mathcal P)}\frac{\omega_{g\circ\Gamma}(s)}{s}\,\mathrm ds+\frac{V}{q\mu}\,\omega_{g\circ\Gamma}(\nu(\mathcal P))\Big)
\end{aligned}$$

by (3.10) and (4.3),

$$\begin{aligned}\int_{t+\nu(\mathcal{P})}^{t+\hat{\Delta}} L(s)\,\mathrm{d}s &\leqslant \frac{V}{q}\int_{t+\nu(\mathcal{P})}^{t+\hat{\Delta}} \frac{|(g-g_{\mathcal{P}})(\Gamma(s))|+|(g-g_{\mathcal{P}})(\Gamma(t))|}{s-t}\,\mathrm{d}s\\ &\leqslant \frac{2V}{q}\mathcal{A}(\nu(\mathcal{P}))\ln\frac{\hat{\Delta}}{\nu(\mathcal{P})}\end{aligned}$$

by (3.10) and (4.1),

$$\begin{aligned}\int_{t+\hat{\Delta}}^{t+T/2} L(s)\,\mathrm{d}s &\leqslant \frac{V}{q\hat{\Delta}}\int_{t+\hat{\Delta}}^{t+T/2} (|(g-g_{\mathcal{P}})(\Gamma(s))|+|(g-g_{\mathcal{P}})(\Gamma(t))|)\,\mathrm{d}s\\ &\leqslant \frac{2V}{q}\mathcal{A}(\nu(\mathcal{P}))\Big(\frac{T}{2\hat{\Delta}}-1\Big)\end{aligned}$$

by (3.11) and(4.1), so that $\int_t^{t+T/2} L(s)\,\mathrm{d}s \leqslant \frac{1}{2}\mathcal{B}(\nu(\mathcal{P}))$. Similarly, we can obtain $\int_{t-T/2}^{t} L(s)\,\mathrm{d}s \leqslant \frac{1}{2}\mathcal{B}(\nu(\mathcal{P}))$. Then the inequalities (4.4) and $\int_{t-T/2}^{t+T/2} L(s)\,\mathrm{d}s \leqslant \mathcal{B}(\nu(\mathcal{P}))$ imply (4.2). □

The main statement of this article follows. It provides a generalization of Theorem 1 and Remark 4.

Theorem 3. *Let $g\colon [\Gamma]\to\mathbb{C}$ be a Dini continuous function satisfying*

$$\limsup_{\tau\downarrow 0}\omega_g(\tau)\ln\frac{1}{\tau}=0. \tag{4.5}$$

Then $\lim\limits_{\substack{\nu(\mathcal{P})\downarrow 0\\ \mathcal{P}\in\mathscr{P}}}\mathscr{C}^-(g_{\mathcal{P}})=\mathscr{C}^-(g)$ uniformly on $\Omega\cup[\Gamma]$.

P r o o f. Since g is Dini continuous, the integral

$$\int_\Gamma \frac{g(\zeta)-g(z)}{\zeta-z}\,\mathrm{d}\zeta$$

absolutely converges for every $z\in[\Gamma]$. Then from [6, p. 191] it follows that

$$\mathscr{C}^-(g)(z)=g(z)+\frac{1}{2\pi\mathrm{i}}\int_\Gamma\frac{g(\zeta)-g(z)}{\zeta-z}\,\mathrm{d}\zeta,\quad z\in[\Gamma]. \tag{4.6}$$

Let $\mathcal{P}\in\mathscr{P}$ be given. The function $g_{\mathcal{P}}$ is Dini continuous by Remark 5 and thus (4.6) remains valid if we replace g by $g_{\mathcal{P}}$. Then

$$\begin{aligned}&\mathscr{C}^-(g)(z)-\mathscr{C}^-(g_{\mathcal{P}})(z)\\ &\qquad=(g-g_{\mathcal{P}})(z)+\frac{1}{2\pi\mathrm{i}}\int_\Gamma\frac{(g-g_{\mathcal{P}})(\zeta)-(g-g_{\mathcal{P}})(z)}{\zeta-z}\,\mathrm{d}\zeta,\quad z\in[\Gamma].\end{aligned}$$

Let us assume that $\nu(\mathcal{P}) \leqslant \hat{\Delta}$. Lemma 4 says that

$$|\mathscr{C}^{-}(g)(z) - \mathscr{C}^{-}(g_{\mathcal{P}})(z)| \leqslant \mathcal{A}(\nu(\mathcal{P})) + \frac{1}{2\pi}\mathcal{B}(\nu(\mathcal{P}))$$

for any $z \in [\Gamma]$. According to the maximum modulus principle for holomorphic functions, the same holds for any $z \in \Omega \cup [\Gamma]$. Now it is sufficient to prove that $\lim\limits_{\tau\downarrow 0}(\mathcal{A}(\tau) + \frac{1}{2\pi}\mathcal{B}(\tau)) = 0$. It is clear that $\lim\limits_{\tau\downarrow 0}\mathcal{A}(\tau) = 0$. Since $\omega_{g\circ\Gamma}(\tau) \leqslant \omega_g(V\tau)$ for every $\tau \in \langle 0, +\infty)$, we obtain

$$\begin{aligned}
\lim_{\tau\downarrow 0}\int_0^{\tau}\frac{\omega_{g\circ\Gamma}(s)}{s}\,\mathrm{d}s &\leqslant \lim_{\tau\downarrow 0}\int_0^{\tau}\frac{\omega_g(Vs)}{s}\,\mathrm{d}s = \lim_{\tau\downarrow 0}\int_0^{\tau}\frac{\omega_g(s)}{s}\,\mathrm{d}s = 0,\\
\lim_{\tau\downarrow 0}\omega_{g\circ\Gamma}(\tau)\ln\frac{1}{\tau} &\leqslant \lim_{\tau\downarrow 0}\omega_g(V\tau)\ln\frac{1}{\tau} = \lim_{\tau\downarrow 0}\omega_g(V\tau)\Big(\ln V + \ln\frac{1}{V\tau}\Big)\\
&= \lim_{\tau\downarrow 0}\omega_g(V\tau)\ln\frac{1}{V\tau} = 0
\end{aligned}$$

by virtue of (2.1), (2.2), and (4.5). Obviously, both limits on the left-hand side are equal to zero, so that $\lim\limits_{\tau\downarrow 0}\frac{1}{2\pi}\mathcal{B}(\tau) = 0$. □

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