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G-nilpotent units of commutative group rings

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Abstract. Suppose R is a commutative unital ring and G is an abelian group. We give a general criterion only in terms of R and G when all normalized units in the commutative group ring RG are G-nilpotent. This extends recent results published in [Extracta Math., 2008–2009] and [Ann. Sci. Math. Québec, 2009].

Keywords: group rings, normalized units, nilpotents, idempotents, decompositions, abelian groups

Classification: 16S34, 16U60, 20K10, 20K20, 20K21

1. Introduction

Throughout the present paper, let it be agreed that all groups are mult tively written and abelian as is customary when studying group rings, and a are commutative with identity 1 (further called commutative *unital*). For ring R and a group G, suppose $N(R)$ is the nil-radical of R and G_t is the part of G with p-component G_p. Likewise, suppose RG is the group ring of R with group of normalized units $V(RG)$. Standardly, $I(LG;G)$ is the fund tal ideal of LG where $L \leq R$ and $I(RG;H)$ is the relative augmentation i RG with respect to $H \leq G$. As usual, imitating [13], $\mathrm{id}(R) = \{e \in R \,|\, e^2$ $\mathrm{inv}(R) = \{p \,|\, p \cdot 1 \in U(R)\}$, where p is a prime number, $U(R)$ is the unit g R, $\mathrm{zd}(R) = \{p \,|\, \exists r \in R \setminus \{0\} : pr = 0\}$, and $\mathrm{supp}(G) = \{p \,|\, G_p \neq 1_G\}$.

Following [8], [9] we define the idempotent subgroup $\mathrm{Id}(RG)$ as follows:

$$\mathrm{Id}(RG) = \Big\{ e_1 g_1 + \cdots + e_k g_k \mid e_i \in \mathrm{id}(R),$$
$$\sum_i e_i = 1, e_i e_j = 0 \ (i \neq j), g_i \in G; 1 \leq i, j \leq$$

It is self-evident that $\mathrm{Id}(RG)$ is a group and that $\mathrm{Id}(RG) \leq V(RG)$.

Definition. A normalized unit $v \in V(RG)$ is said to be G-nilpotent if v
uniquely expressed as $v = gw$ where $g \in G$ and $w \in 1 + I(N(R)G; G)$.

This is tantamount to ask when the decomposition

$$V(RG) = G \times (1 + I(N(R)G; G))$$

holds; note that $G \cap (1 + I(N(R)G; G)) = 1$.

Some explorations that are closely related to this theme are given in
and [5] (compare with Section 2). Here we shall amend our technique an
result, we will generalize the main assertions from these papers.

2. Preliminaries and main results

Before proving the chief statements, we need some technicalities.

Lemma 1. *For each ring R the following equality is fulfilled:*

$$U(R/N(R)) = \{r + N(R) \mid r \in U(R)\}.$$

Proof: Clearly the left hand-side contains the right one because there
$r, f \in R$ with $rf = 1$ and hence $(r + N(R))(f + N(R)) = rf + N(R) = 1 +$

As for the converse inclusion, let $x \in U(R/N(R))$ be given. Then, $x = r+$
for some $r \in R$ such that there exists $f \in R$ with $(r + N(R))(f + N($
$rf + N(R) = 1 + N(R)$. Consequently, $rf - 1 \in N(R)$ which means tha
$1 + N(R) \subseteq U(R)$. Therefore, it is easily seen that $r \in U(R)$ as required.

Lemma 2. *For any ring R the following equality holds:*

$$\mathrm{inv}(R) = \mathrm{inv}(R/N(R)).$$

Proof: Assume that $p \in \mathrm{inv}(R)$. Then $p{\cdot}1 \in U(R)$ and hence in view of Le
we have $p(1 + N(R)) = p \cdot 1 + N(R) \in U(R/N(R))$. Thus $p \in \mathrm{inv}(R/N($
the inclusion "$\subseteq$" is obtained.

As for the converse containment "$\supseteq$", choose $p \in \mathrm{inv}(R/N(R))$, whence
$N(R)) \in U(R/N(R))$. In accordance with Lemma 1 we may write $p \cdot 1 + N$
$\alpha + N(R)$ where $\alpha \in U(R)$. Furthermore, $p \cdot 1 \in U(R) + N(R) = U(R)$ s
$p \in \mathrm{inv}(R)$, as required.

Let R be a ring. Define $\mathrm{np}(R) = \{p \mid \exists r \in R \setminus N(R) : pr \in N(R)\}$
following claim is useful.

PROOF: Given $p \in \mathrm{zd}(R/N(R))$, there is $r \notin N(R)$ such that $p(r + N($
$pr + N(R) = N(R)$. Thus $pr \in N(R)$ and $p \in \mathrm{np}(R)$.

Conversely, let $p \in \mathrm{np}(R)$. Then there is $r \in R \setminus N(R)$ with $pr \in$
Consequently, $p(r + N(R)) = N(R)$ and $r + N(R) \neq N(R)$ which implie
$p \in \mathrm{zd}(R/N(R))$.

Lemma 4. *Suppose R is a ring. Then*

$$\mathrm{id}(R) = \{0, 1\} \iff \mathrm{id}(R/N(R)) = \{0, 1\}.$$

PROOF: "$\Rightarrow$". Because of the classical fact that idempotents can always be
through $N(R)$ (see, e.g., [1]) if $R/N(R)$ has a non-trivial idempotent, th
same must be true of R, a contradiction.

"$\Leftarrow$". Choose an arbitrary element $r \in R$ with $r^2 = r$, hence $r + N$
$r^2 + N(R) = (r + N(R))^2$. Therefore, either $r + N(R) = N(R)$, whence $r \in$
and thus $r = 0$, or $r + N(R) = 1 + N(R)$, whence $r \in 1 + N(R) \subseteq U(R$
then $r(1 - r) = 0$ ensures that $1 - r = 0$ that is $r = 1$, as required.

Another topological approach in proving the above can be based on the
ing two standard facts in commutative ring theory:

Let A be any commutative unital ring. Then the following are true (e.g.,

(i) A has no non-trivial idempotents if and only if $\mathrm{Spec}(A)$, the set of
ideals of A equipped with the Zariski topology, is a connected topo
space;

(ii) the canonical surjection from $\mathrm{Spec}(A/N(A))$ to $\mathrm{Spec}(A)$, sendin
$N(A)$ to P, is a homeomorphism (relative to the Zariski topology o
space).

Proposition 5. *Suppose R is a ring and $\phi : R \to R/N(R)$ is the*
map. Define $\Phi : RG \to (R/N(R))G$ and its restriction $\Phi_{V(RG)} : V(R$
$V((R/N(R))G)$ by $\Phi(\sum_{g\in G} r_g g) = \sum_{g\in G} \phi(r_g)g = \sum_{g\in G}(r_g + N(R))g$.
the following relations are valid:

(a) *Φ is a surjective homomorphism;*

(b) *$\ker \Phi = N(R)G$ and $\ker \Phi_{V(RG)} = 1 + I(N(R)G; G)$.*

PROOF: (a) That Φ is a ring (and hence a group) homomorphism follows
since so is ϕ.

As for the epimorphism (= surjection), we will restrict our attention o
$V(RG)$ because for RG this is evident. And so, choose $x \in V((R/N($
whence there is $y \in RG$ with $\Phi(y) = x$. Moreover, there are $x' \in (R/N$
such that $xx' = 1$ and $y' \in RG$ such that $\Phi(y') = x'$. Therefore, $1 = \Phi(y)\Phi$
$\Phi(yy')$, so that $\Phi(yy' - 1) = 0$ and point (b) below applies to write that yy

(b) Clearly, $1 + I(N(R)G; G) \subseteq V(RG)$ because $I(N(R)G; G) \subseteq N($
$N(RG)$.

On the other hand, it is plainly seen that $\ker \Phi = N(R)G$. Moreove
checks that $\ker \Phi_{V(RG)} = (1 + I(RG; G)) \cap (1 + N(R)G) = 1 + I(N(R)G$
asserted.

Remark 1. Actually, the pre-image y can be chosen with augmentation
therefore $y \in U(RG)$ directly implies that $y \in V(RG)$. In fact, if $x =$
$N(R))g_1 + \cdots + (r_s + N(R))g_s$ with $r_1 + \cdots + r_s - 1 = \alpha \in N(R)$, the
$r_1g_1 + \cdots + r_sg_s - \alpha 1_G$ satisfies the required property that $\Phi(y) = x$ and aug(

Proposition 6. *Suppose G is a group and R is a ring. Then the fol*
equivalence holds:

$$V(RG) = G \times (1 + I(N(R)G; G)) \iff V((R/N(R))G) = G.$$

Proof: "$\Rightarrow$". Applying Proposition 5(a) and taking Φ in the both sides
given equality, we derive that $\Phi(V(RG)) = \Phi(G)\Phi(1 + I(N(R)G; G))$.
equivalent to $V((R/N(R))G) = G$ because $\Phi(G) = G$ and $\Phi(1 + I(N(R)G;$
1, as stated.

"$\Leftarrow$". Choose an arbitrary element $x \in V(RG)$. We have $\Phi(x) \in V((R/N($
$= G$. Thus we may write $\Phi(x) = g = \Phi(g)$ for some $g \in G$. Furthe
$\Phi(x)[\Phi(g)]^{-1} = \Phi(x)\Phi(g^{-1}) = \Phi(xg^{-1}) = 1$. Hence $xg^{-1} \in \ker \Phi_{V(}$
$1 + I(N(R)G; G)$ utilizing Proposition 5(b). Finally, $x \in G \times (1 + I(N(R)$
as required.

The following statement is an amended version of [3, Proposition].

Proposition 7. *Suppose G is a group with $|G| = 3$ and R is a ring suc*
$3 \in \mathrm{inv}(R)$. Then $V(RG) = G$ if and only if $U(R) = 1$ and the equation r^2
$rf + r + f = 0$ has only trivial solutions in R.

Proof: "$\Rightarrow$". What we need to show is that $\mathrm{char}(R) = 2$. Assume the co
$2 \neq 0$. Then we observe that $\frac{2}{3} + \frac{2}{3}g - \frac{1}{3}g^2$ is a non-trivial unit with the i
$\frac{2}{3} - \frac{1}{3}g + \frac{2}{3}g^2$. This contradiction allows us to conclude that $2 = 0$. Furthe
we apply the proof of Proposition on p. 51 from [3] to deduce that $U(R) =$
$r^2 + f^2 + rf + r + f = 0$ is possible unique when $r = 0$, $f = 0$ or $r = 1$, f
$r = 0$, $f = 1$.

"$\Leftarrow$". Certainly $U(R) = 1$ implies that $-1 = 1$, i.e., $2 = 0$. Thus char(
and the further argument follows as that in [3, p. 51, Proposition].

Remark 2. Note also that $2 \notin U(R)$ since otherwise $\frac{1}{2} + \frac{1}{2}g \in V(RG)$ wi
inverse $1 - g + g^2$. Moreover, we point out that the equations here and in
the same, which follows via the substitutions $a = 1 + r$ and $b = 1 + f$.

(1) $G = G_t$;
(2) $G \neq G_t$, $\mathrm{supp}(G) \cap (\mathrm{inv}(R) \cup \mathrm{zd}(R)) = \emptyset$.

Now we are planning to give a new, more conceptual, proof of the fol
result from [3].

Theorem B. *Suppose G is a group and R is a ring such that* $\mathrm{supp}(G) \cap \mathrm{inv}$
$\emptyset$*. Then $V(RG) = G$ if and only if* $\mathrm{id}(R) = \{0, 1\}$, $N(R) = 0$ *and at most*
the following conditions holds:

(1) $|G| = |U(R)| = 2$;
(2) $|G| = 3$, $U(R) = 1$ *and the equation* $a^2 + b^2 + ab + 1 = 0$ *has only*
solutions in R for each pair (a, b).

Proof: "$\Rightarrow$". If either the set $\mathrm{id}(R)$ contains a non-trivial idempotent e
nil-ideal $N(R)$ contains a non-trivial nilpotent r, taking $g \in G$ we can co
one of the elements $x_e = eg + 1 - e$ or $x_r = 1 - r + rg$ — for each of them it is
verified that $x_e \in V(RG) \setminus G$ with inverse $x_e^{-1} = eg^{-1} + 1 - e$, or $x_r \in V(R$
as the sum of 1 and the nilpotent $-r + rg = r(g - 1)$, a contradiction in e
the two situations. That is why both $\mathrm{id}(R)$ and $N(R)$ are trivial.

Claim that G is finite of order 2 or 3. In fact, assume in a way of contra
that G is infinite. Since there is a prime, say q, such that $G_q \neq 1$ and $q \in$ i
it is well known that there exists an idempotent $e \in RF$ where $F \leq G$
finite subgroup. Choose $g \notin F$ (this choice is possible since G is infinite
F is finite) and in the same manner as above one can construct the e
$x_e = eg + 1 - e \in V(RG) \setminus G$. Thus G is necessarily finite. By the same rea
follows that G does not contain proper subgroups, that is, G is of prime card
— thereby $|G|$ is a prime, say q. Furthermore, we claim that G has cardin
or 3. To show this, we assume the contrary that $|G| \geq 5$ and consider the e
$u = (1 + g)^{q-1} - \frac{2^{q-1}-1}{q}(1 + g + \cdots + g^{q-1})$ where $G = \langle g \rangle$ with $g^q = 1$. It
known that u is a unit with augmentation 1 which does not lie in G (se
[12]). This contradiction shows that $|G| \leq 4$. Finally, either $|G| = 2$ or $|G|$
claimed.

Moreover, another approach is to notice that there is a nontrivial idem
$e = \frac{1}{2}(1 + g)$ or $e = \frac{1}{3}(1 + g + g^2)$ where g is either of order 2 or 3. If g'
then $1 - e + eg'$ is a nontrivial unit.

Next, we consider separately these two possibilities:

Case 1. G is cyclic of order 2.

Firstly, note that $2 \in U(R)$. We claim that if $r \in U(R)$ is an arbitrary el
then either $r = 1$ or $r = -1$; so $2 = -1$ and hence $3 = 0$ since $2 = 1$ do
hold. In fact, consider the element $x_r = \frac{1}{2} - \frac{r}{2} + (\frac{1}{2} + \frac{r}{2})g$. It is simple c
that $x_r \in V(RG)$ with the inverse $x_{r^{-1}} = \frac{1}{2} - \frac{r^{-1}}{2} + (\frac{1}{2} + \frac{r^{-1}}{2})g$. Since ther

"$\Leftarrow$". (1) First, note that $1 \neq -1$ and $\mathrm{char}(R) = 3$ because $2 \in U(R) = \{$
and thus $2 = -1$; the equality $2 = 1$ is impossible since it yields that $1 =$
$x_r = 1 - r + rg$. Then, there is $f \in R$ such that $(1 - r + rg)(1 - f + fg) = 1$
is equivalent to $f(2r - 1) = r$. Since $2rf - r - f = 0$, we have $(2r - 1)(2f -$
and it must be that $2r - 1 \in U(R)$. Consequently, $2r - 1 = 1$ or $2r - 1$
Thus $2r = 2$, whence $r = 1$, or $2r = 0$, whence $r = 0$. Finally, either x_r
$x_r = g$. In both cases we observe that $V(RG) = G$, as expected.

(2) Follows by a direct application of Proposition 7.

Remark 3. First, notice that in clause (2) we must have $\mathrm{char}(R) = 2$ if ch
is a prime integer. In fact, always $-1 \in U(R)$ and since $U(R) = 1$, we hav
$-1 = 1$ which is tantamount to $2 = 0$ as asserted.

Certainly, in the Main Theorem from [3], point (1) $G = 1$ is not realist
cannot be happen since $\mathrm{supp}(G) \neq \emptyset$.

The question of the triviality of units in commutative group rings will b
pletely exhausted if the following can be settled:

Problem 1. Find a criterion only in terms associated with R and G
$V(RG) = G$ holds, provided that $G = G_t$ and $\mathrm{supp}(G) \cap \mathrm{inv}(R) = \emptyset$.

We have now at our disposal all the information needed to prove the foll

Theorem 8. *Suppose G is a group and R is a ring. Then $V(RG) = G$
$I(N(R)G; G))$ if and only if $\mathrm{id}(R) = \{0, 1\}$, $V(RG_t) = G_t \times (1 + I(N(R)G$
and at most one of the following conditions holds:*

(1) *$G = G_t$;*
(2) *$G \neq G_t$, $\mathrm{supp}(G) \cap (\mathrm{inv}(R) \cup \mathrm{np}(R)) = \emptyset$.*

Proof: Employing Proposition 6 we equivalently reduce the decomposit
$V(RG)$ to the equality $V((R/N(R))G) = G$. Next, we subsequently apply
rem A combined with Lemmas 2, 3 and 4.

Theorem 9. *Suppose G is a group and R is a ring such that $\mathrm{supp}(G) \cap \mathrm{inv}(R$
Then $V(RG) = G \times (1 + I(N(R)G; G))$ if and only if $\mathrm{id}(R) = \{0, 1\}$ and e
one of the following points is valid:*

(1) *$|G| = |U(R/N(R))| = 2$;*
(2) *$|G| = 3$, $U(R/N(R)) = 1$ and the relation $a^2 + b^2 + ab + 1 \in N($
only trivial solutions in $R/N(R)$ for every pair $(a, b) \in R$.*

Proof: By application of Proposition 6 we can write in an equivalent wa
$V((R/N(R))G) = G$. Hereafter we subsequently employ Theorem B to
with Lemma 2 and Lemma 4.

As a consequence, we deduce

(a) $G_t = 1$;
(b) $|G| = p = 2$, $R = L + N(R)$ *with* $|L| = 2$;
(c) $p = 3$, $|G| = 2$ *and* $U(R) = \pm 1 + N(R)$;
(d) $p = 2$, $|G| = 3$, $U(R) = 1 + N(R)$ *and the equation* $X^2 + XY +$
$1 + N(R)$ *possesses only trivial solutions in* $R/N(R)$.

Proof: First of all, observe that $\mathrm{inv}(R)$ contains all primes but p. Tha indecomposable follows easily since $1-r+rg \in V(RG)$ is always a non-G-nil unit whenever $r \in \mathrm{id}(R) \setminus \{0,1\}$ and $g \in G \setminus \{1\}$. Moreover, if G is torsio everything was done in [6], [7] (see [8] and [9] as well). So, assume G Further, if $G_t \neq G_p$ we see that $\mathrm{supp}(G) \cap \mathrm{inv}(R) \neq \emptyset$ and hence Theo applies to get the result. If now G is p-mixed, i.e., $G_t = G_p$, it follow $V(RG) = G(1 + I(RG; G_p) + I(N(R)G; G))$. Hereafter, the proof goes arguments similar to these from [5] considering the cases $G = G_t$ and G The first one leads to $|G| = 2 = p$, while the second one is impossible.

Finally, we will apply the results alluded to above to derive a recent achie from [2]. First, we need the following technicality.

Lemma 11. *Let* $\mathrm{char}(R) = p$ *be a prime integer. Then*

$$V(RG) = GV_p(RG) \iff V(R(G/G_p)) = (G/G_p)V_p(R(G/G_p)).$$

Proof: Consider the natural map $\psi : G \to G/G_p$. It is well known can be linearly extended to the homomorphism $\Psi : V(RG) \to V(R(G$ with kernel $1 + I(RG; G_p)$. Since $1 + I(RG; G_p) \subseteq V_p(RG)$, it easily foll standard arguments that Ψ is actually an epimorphism (= surjective hom phism). Moreover, it is also clear that $\Psi(V_p(RG)) = V_p(R(G/G_p))$. So, the action of Ψ on the both sides of $V(RG) = GV_p(RG)$ we immediately that $V(R(G/G_p)) = (G/G_p)V_p(R(G/G_p))$ holds, as stated.

As for the sufficiency, choose an arbitrary element $x \in V(RG)$ and obser there is $y \in V(R(G/G_p))$ such that $\Psi(x) = y$. Write $y = g'v'$ where $g' \in$ and $v' \in V_p(R(G/G_p))$. Since by what we have shown above there exist and $v \in V_p(RG)$ such that $\Psi(g) = g'$ and $\Psi(v) = v'$, we get $\Psi(x) =$ Furthermore, $\Psi(xg^{-1}v^{-1}) = 1$ and thus $xg^{-1}v^{-1} \in \ker \Psi \subseteq V_p(RG)$ as pre noticed. This leads to $x \in GV_p(RG)$, as required.

So, we are ready to prove the following affirmation.

Proposition 12 ([2])**.** *Suppose* $\mathrm{char}(R) = p$ *is a prime natural number.* $V(RG) = GV_p(RG)$ *if and only if*

(1) $G = G_p$ *or*

(2.3) $p = 2$, $U(R) = 1 + N(R)$, *the equality* $X^2 + XY + Y^2 = 1 +$
has only trivial solutions in $R/N(R)$ *and* $G = G_p \times C$ *with* $|C$

Proof: By virtue of Lemma 11, we may with no harm of generality assum
$G_p = 1$. Since it is plainly checked that then $V_p(RG) = 1 + I(N(R)G;$
obviously deduce that $V(RG) = G \times (1 + I(N(R)G;G))$ — see also [5]. Henc
we employ the main theorem from [5] or, respectively, Corollary 10.

We close the work with the following:

Problem 2. Find a necessary and sufficient condition when the equality

$$V(RG) = G \times (1 + I(N(R)G;G))$$

holds, provided that $\mathrm{supp}(G) \cap \mathrm{inv}(R) = \emptyset$.

In particular, as an immediate consequences, we will extract the cases
(Karpilovsky) and $R = \mathbb{Z}$ (May).

In conclusion, one can expect that if $\mathrm{supp}(G) \cap \mathrm{zd}(R) \neq \emptyset$, then there is a
nilpotent unit. However, this is not generally true. For instance, a counterex
may be obtained for rings of characteristic 4 by taking $R = \mathbb{Z}_4 = \mathbb{Z}/(4)$ (
to be the ring of all integers modulo 4) and G is of order 2. There are onl
elements of augmentation 1, so that the computations are minimal. If
counterexample of a ring of characteristic 0 is desired, let G be of order 2
and let $R = \mathbb{Z}[x]$ be the polynomial ring of x over $\mathbb{Z}$ where the element x is s
to the relations $x^2 = 2x = 0$.

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