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ON A PHASE-FIELD MODEL WITH A LOGARITHMIC NONLINEARITY

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Abstract. Our aim in this paper is to study the existence of solutions to a phase-field system based on the Maxwell-Cattaneo heat conduction law, with a logarithmic nonlinearity. In particular, we prove, in one and two space dimensions, the existence of a solution which is separated from the singularities of the nonlinear term.

Keywords: phase field system, Maxwell-Cattaneo law, well-posedness, logarithmic potential

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1. Introduction

We consider in this paper the following initial and boundary value problem:

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = \frac{\partial \alpha}{\partial t}, \tag{1.1}$$

$$\frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \Delta \alpha = -u - \frac{\partial u}{\partial t}, \tag{1.2}$$

$$u = \alpha = 0 \ \text{ on } \ \partial\Omega, \tag{1.3}$$

$$u\big|_{t=0} = u_0, \quad \alpha\big|_{t=0} = \alpha_0, \quad \frac{\partial \alpha}{\partial t}\Big|_{t=0} = \alpha_1, \tag{1.4}$$

in a bounded and regular domain $\Omega \subset \mathbb{R}^n$, $n = 1$ or 2, with boundary $\partial\Omega$. We assume here that $f = F'$, where

$$\begin{aligned} F(s) = &-\kappa_0 s^2 + \kappa_1[(1+s)\ln(1+s) + (1-s)\ln(1-s)], \\ &s \in (-1,1),\ 0 < \kappa_1 < \kappa_0, \end{aligned} \tag{1.5}$$

i.e.,

$$f(s) = -2\kappa_0 s + \kappa_1 \ln \frac{1+s}{1-s}, \quad s \in (-1,1). \tag{1.6}$$

In particular, it follows from (1.6) that

$$f'(s) \geqslant -2\kappa_0, \quad s \in (-1,1) \tag{1.7}$$

and

$$F(s) \geqslant -c_0, \quad c_0 \geqslant 0,\ s \in (-1,1). \tag{1.8}$$

Also note that F is bounded. We further assume that

$$u_0 \in H_0^1(\Omega) \cap H^3(\Omega), \quad \alpha_0 \in H_0^1(\Omega) \cap H^3(\Omega), \quad \alpha_1 \in H_0^1(\Omega) \cap H^2(\Omega) \tag{1.9}$$

with

$$\|u_0\|_{L^\infty(\Omega)} < 1, \tag{1.10}$$

where $\|\cdot\|_X$ denotes the norm on the Banach space X, and that the following compatibility conditions hold:

$$\Delta u_0 = \Delta \alpha_0 = 0 \quad \text{on } \partial\Omega. \tag{1.11}$$

Equations (1.1)–(1.2) have been proposed in [23] (see also [20] and [21]) as a generalization of the Caginalp phase-field system (see [4]). In this context, u is the order parameter and α is the thermal displacement variable, defined by

$$\alpha = \int_0^t \theta \,\mathrm{d}\tau + \alpha_0, \tag{1.12}$$

where θ is the relative temperature.

The original Caginalp system, which reads

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = \theta, \tag{1.13}$$

$$\frac{\partial \theta}{\partial t} - \Delta \theta = -\frac{\partial u}{\partial t}, \tag{1.14}$$

has been introduced in [4] to model phase transition phenomena, e.g., melting-solidification phenomena, in certain classes of materials. This equation has been extensively studied (see, e.g., [1], [2], [3], [4], [9], [11], [13], [14], [27], and [31] for regular nonlinearities and [6], [7], [17], and [28] for singular ones).

The Caginalp system can be derived by considering the (total) Ginzburg-Landau free energy

$$\Psi(u, \nabla u, \theta) = \int_\Omega \Big(\frac{1}{2}|\nabla u|^2 + F(u) - \theta u\Big)\, \mathrm{d}x \tag{1.15}$$

and the enthalpy $H = u + \theta$ and by writing

$$\frac{1}{d}\frac{\partial u}{\partial t} = -\partial_u \Psi, \tag{1.16}$$

$$\frac{\partial H}{\partial t} = -\operatorname{div} q, \tag{1.17}$$

where $d > 0$ is a relaxation parameter, ∂ denotes a variational derivative and q is the thermal flux vector. The first equation means that one postulates, in the bulk Ω, a relaxation dynamics for the order parameter u (note that, at equilibrium, u minimizes the Ginzburg-Landau free energy), while the other corresponds to the heat balance. Setting $d = 1$ and taking into account the usual Fourier law

$$q = -\nabla\theta, \tag{1.18}$$

we find (1.13)–(1.14).

Now, one drawback of the Fourier law is that it predicts that thermal signals propagate at infinite speed, which violates causality (the so-called "paradox of heat conduction", see [10]).

One possibility to correct this unrealistic feature is to replace the Fourier law by the Maxwell-Cattaneo law (see [10]; see also [24] and [25] for other possibilities, based on an alternative treatment for a thermomechanical theory of deformable media proposed by Green and Naghdi in [19] or on a three-phase-lag heat conduction law proposed by Roy Choudhuri in [30])

$$\Big(1 + \eta\frac{\partial}{\partial t}\Big) q = -\nabla\theta, \tag{1.19}$$

where η is a relaxation parameter (which is small (of the order of picoseconds) in most situations, although this may not be the case for some materials; see [10]); when $\eta = 0$, one recovers the Fourier law. This generalization of the Fourier law accounts for the finite speed of heat conduction by adding a term which is proportional to the time derivative of the thermal flux vector and is called thermal inertia. Furthermore, the thermal relaxation constant η represents the time lag which is required to establish steady heat conduction in a volume element once a temperature gradient has been imposed.

Taking, for simplicity, $\eta = 1$, it follows from (1.17) that

$$\Big(1 + \frac{\partial}{\partial t}\Big)\frac{\partial H}{\partial t} - \Delta\theta = 0,$$

hence we obtain the following second-order (in time) equation for the relative temperature:

$$\frac{\partial^2\theta}{\partial t^2} + \frac{\partial\theta}{\partial t} - \Delta\theta = -\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial t^2}. \tag{1.20}$$

Integrating finally (1.20) between 0 and t, we obtain the equation

$$\frac{\partial^2\alpha}{\partial t^2} + \frac{\partial\alpha}{\partial t} - \Delta\alpha = -u - \frac{\partial u}{\partial t} + g, \tag{1.21}$$

where g depends on the initial data (for u and θ), which reduces to (1.2) when g vanishes (note that this equation can also be obtained by considering the Caginalp model with the so-called Gurtin-Pipkin law

$$q(t) = -\int_0^\infty k(s)\nabla\theta(t-s)\,\mathrm{d}s,$$

accounting for memory effects, for an exponentially decaying kernel $k(s) = \mathrm{e}^{-s}$, see [18] and [23]). Furthermore, noting that $\theta = (\partial\alpha/\partial t)$, (1.13) can be rewritten in the equivalent form (1.1).

Equations (1.1)–(1.4) have been studied, for regular nonlinearities, in [23] (see also [20] and [21]); a typical choice of a regular potential is the double-well potential $F(s) = \frac{1}{4}(s^2-1)^2$, i.e., $f(s) = s^3 - s$.

Now, the above regular cubic nonlinear term is actually taken as an approximation of the logarithmic function (1.6) which is thermodynamically consistent (see [5]); in particular, the logarithmic terms are related with the entropy.

Our aim in this paper is to prove the existence of a solution in the case of the logarithmic nonlinearity (1.6). The main difficulty is to prove that the order parameter is separated from the singularities of f. In particular, we are only able to prove such a property in one and two space dimensions.

Throughout the paper, the same letter c (and, sometimes, c') denotes constants which may change from line to line.

2. A priori estimates

We a priori assume that

$$\|u\|_{L^\infty((0,T)\times\Omega)} < 1,$$

where $T > 0$ is an arbitrary final time.

We multiply (1.1) by $(\partial u/\partial t)$ and have, integrating over Ω and by parts,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\nabla u\|^2 + 2\int_\Omega F(u)\,\mathrm{d}x\Big) + 2\Big\|\frac{\partial u}{\partial t}\Big\|^2 = 2\Big(\Big(\frac{\partial \alpha}{\partial t}, \frac{\partial u}{\partial t}\Big)\Big), \tag{2.1}$$

where $\|\cdot\|$ denotes the usual L^2-norm, with associated scalar product $((\cdot,\cdot))$.

Multiplying then (1.2) by $(\partial\alpha/\partial t)$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\nabla \alpha\|^2 + \Big\|\frac{\partial \alpha}{\partial t}\Big\|^2\Big) + 2\Big\|\frac{\partial \alpha}{\partial t}\Big\|^2 = -2\Big(\Big(u, \frac{\partial \alpha}{\partial t}\Big)\Big) - 2\Big(\Big(\frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t}\Big)\Big). \tag{2.2}$$

Summing (2.1) and (2.2), we easily find

$$\frac{\mathrm{d}E}{\mathrm{d}t} + \Big\|\frac{\partial u}{\partial t}\Big\|^2 \leqslant c\|u\|^2, \tag{2.3}$$

where

$$E = \|\nabla u\|^2 + 2\int_\Omega F(u)\,\mathrm{d}x + \|\nabla\alpha\|^2 + \Big\|\frac{\partial \alpha}{\partial t}\Big\|^2,$$

which yields, recalling that F is bounded, estimates on u in $L^\infty(0,T;H_0^1(\Omega))$, on $(\partial u/\partial t)$ in $L^2(0,T;L^2(\Omega))$, on α in $L^\infty(0,T;H_0^1(\Omega))$ and on $(\partial\alpha/\partial t)$ in $L^\infty(0,T;L^2(\Omega))$.

Next, we differentiate (1.1) with respect to time to have, owing to (1.2),

$$\frac{\partial}{\partial t}\Big(\frac{\partial u}{\partial t}\Big) - \Delta\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial t} = -\frac{\partial \alpha}{\partial t} + \Delta\alpha - u - \frac{\partial u}{\partial t}. \tag{2.4}$$

Multiplying (2.4) by $(\partial u/\partial t)$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\|\frac{\partial u}{\partial t}\Big\|^2 &+ \Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2 + \Big(\Big(f'(u)\frac{\partial u}{\partial t}, \frac{\partial u}{\partial t}\Big)\Big)\\ &= -\Big(\Big(\frac{\partial \alpha}{\partial t}, \frac{\partial u}{\partial t}\Big)\Big) - \Big(\Big(\nabla\alpha, \nabla\frac{\partial u}{\partial t}\Big)\Big) - \Big(\Big(u, \frac{\partial u}{\partial t}\Big)\Big) - \Big\|\frac{\partial u}{\partial t}\Big\|^2,\end{aligned}$$

which yields, in view of (1.7),

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\|\frac{\partial u}{\partial t}\Big\|^2 + \Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2 \leqslant c\Big(\|u\|^2 + \|\nabla\alpha\|^2 + \Big\|\frac{\partial u}{\partial t}\Big\|^2 + \Big\|\frac{\partial \alpha}{\partial t}\Big\|^2\Big), \tag{2.5}$$

hence estimates on $(\partial u/\partial t)$ in $L^\infty(0,T;L^2(\Omega))$ and $L^2(0,T;H_0^1(\Omega))$ (note that $(\partial u/\partial t)(0)=\Delta u_0-f(u_0)+\alpha_1$).

We then multiply (1.1) by $-\Delta u$ and have, owing again to (1.7),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2+\|\Delta u\|^2\leqslant c\Big(\|\nabla u\|^2+\Big\|\frac{\partial\alpha}{\partial t}\Big\|^2\Big), \tag{2.6}$$

hence an estimate on u in $L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega))$.

We finally multiply (1.2) by $-(\partial\Delta\alpha/\partial t)$ and easily find

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\Big\|\nabla\frac{\partial\alpha}{\partial t}\Big\|^2+\|\Delta\alpha\|^2\Big)\leqslant c\Big(\|\nabla u\|^2+\Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2\Big), \tag{2.7}$$

hence estimates on α in $L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega))$ and on $(\partial\alpha/\partial t)$ in $L^\infty(0,T;H_0^1(\Omega))$.

Our aim now is to prove that u a priori satisfies

$$\|u(t)\|_{L^\infty(\Omega)}\leqslant 1-\delta,\quad t\in[0,T], \tag{2.8}$$

where $\delta\in(0,1)$ depends only on the initial data and the final time T.

In one space dimension, we have, owing to the embedding $H^1(\Omega)\subset L^\infty(\Omega)$, an estimate on $(\partial\alpha/\partial t)$ in $L^\infty((0,T)\times\Omega)$. It is then not difficult to prove the separation property (2.8) for solutions to the parabolic equation

$$\frac{\partial u}{\partial t}-\Delta u+f(u)=g$$

with right-hand side $g\in L^\infty((0,T)\times\Omega)$.

Indeed, let $\delta\in(0,1)$ be such that

$$\|u_0\|_{L^\infty(\Omega)}\leqslant\delta,\quad \|g\|_{L^\infty((0,T)\times\Omega)}-f(\delta)\leqslant 0 \tag{2.9}$$

(note that $\lim_{s\to1^-}f(s)=\infty$).

We set $U=u-\delta$ and have

$$\frac{\partial U}{\partial t}-\Delta U+f(u)-f(\delta)=g-f(\delta). \tag{2.10}$$

We multiply (2.10) by $U^+=\max(U,0)$ and obtain, owing to (1.7) and (2.9),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|U^+\|^2\leqslant c\|U^+\|^2, \tag{2.11}$$

which yields, owing to Gronwall's lemma and noting that $U^+(0)=0$, that $u(t)\leqslant\delta$, $\forall\,t\in[0,T]$. Finally, (2.8) follows from the fact that f is odd and by proceeding similarly for a lower bound.

We now turn to the two-dimensional case. To this end, we first prove

Lemma 2.1. *We have, for every $L > 0$,*

$$\int_{(0,T)\times\Omega} \mathrm{e}^{L|f(u(x,t))|}\,\mathrm{d}x\,\mathrm{d}t \leqslant c, \tag{2.12}$$

where $c = c(L)$ depends only on the initial data and the final time T.

Proof. We proceed as in [28].

We rewrite (1.1) in the form

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = g, \tag{2.13}$$

where

$$\|g(t)\|_{H_0^1(\Omega)} \leqslant c, \quad t \in [0,T], \tag{2.14}$$

where c depends only on the initial data and T. We can also assume, without loss of generality, that

$$f'(s) \geqslant 0, \quad s \in (-1,1) \tag{2.15}$$

(i.e., $\kappa_0 = 0$ in (1.6); indeed, $f + 2\kappa_0 I$ satisfies (2.15) and $u \in L^\infty(0,T;H_0^1(\Omega))$).

We fix $L > 0$ and multiply (2.13) by $f(u)\mathrm{e}^{L|f(u)|}$ to have (note that $f(0) = 0$)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega F_L(u)\,\mathrm{d}x + \int_\Omega |\nabla u|^2 f'(u)(1 + L|f(u)|)\mathrm{e}^{L|f(u)|}\,\mathrm{d}x + \int_\Omega |f(u)|^2\mathrm{e}^{L|f(u)|}\,\mathrm{d}x \\ = \int_\Omega gf(u)\mathrm{e}^{L|f(u)|}\,\mathrm{d}x,\end{aligned}$$

where

$$F_L(s) = \int_0^s \tau\mathrm{e}^{L|\tau|}\,\mathrm{d}\tau,$$

which yields, by integrating between 0 and T,

$$\begin{aligned}\int_\Omega F_L(u(T))\,\mathrm{d}x + \int_{(0,T)\times\Omega} |\nabla u|^2 f'(u)(1 + L|f(u)|)\mathrm{e}^{L|f(u)|}\,\mathrm{d}x\,\mathrm{d}t \\ + \int_{(0,T)\times\Omega} |f(u)|^2\mathrm{e}^{L|f(u)|}\,\mathrm{d}x\,\mathrm{d}t \\ = \int_\Omega F_L(u_0)\,\mathrm{d}x + \int_{(0,T)\times\Omega} gf(u)\mathrm{e}^{L|f(u)|}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.16}$$

We thus deduce from (1.10), (2.15), and (2.16) that

$$\int_{(0,T)\times\Omega} |f(u)|^2 \mathrm{e}^{L|f(u)|}\,\mathrm{d}x\,\mathrm{d}t \leqslant c + \int_{(0,T)\times\Omega} |g||f(u)|\mathrm{e}^{L|f(u)|}\,\mathrm{d}x\,\mathrm{d}t, \tag{2.17}$$

where c depends on the initial data.

In order to estimate the second term on the right-hand side of (2.17), we use the following Young's inequality (see [12] and [22]):

$$ab \leqslant \varphi(a) + \psi(b), \quad a, b \geqslant 0, \tag{2.18}$$

where

$$\varphi(s) = \mathrm{e}^s - s - 1, \quad \psi(s) = (1+s)\ln(1+s) - s, \ \ s \geqslant 0. \tag{2.19}$$

Taking $a = N|g|$ and $b = N^{-1}|f(u)|\mathrm{e}^{L|f(u)|}$, where $N > 0$ is to be fixed later, in (2.18), we obtain

$$|g||f(u)|\mathrm{e}^{L|f(u)|} \leqslant \mathrm{e}^{N|g|} + (1 + N^{-1}|f(u)|\mathrm{e}^{L|f(u)|})\ln(1 + N^{-1}|f(u)|\mathrm{e}^{L|f(u)|}).$$

Now, if $|f(u)| \leqslant 1$, then

$$|g||f(u)|\mathrm{e}^{L|f(u)|} \leqslant \mathrm{e}^{N|g|} + (1 + N^{-1}\mathrm{e}^{L})\ln(1 + N^{-1}\mathrm{e}^{L}).$$

Furthermore, if $|f(u)| \geqslant 1$, then $|f(u)|\mathrm{e}^{L|f(u)|} \geqslant 1$ and

$$\begin{aligned}
&|g||f(u)|\mathrm{e}^{L|f(u)|} \\
&\quad \leqslant \mathrm{e}^{N|g|} + (1 + N^{-1}|f(u)|\mathrm{e}^{L|f(u)|})\ln((1 + N^{-1})|f(u)|\mathrm{e}^{L|f(u)|}) \\
&\quad = \mathrm{e}^{N|g|} + LN^{-1}|f(u)|^2\mathrm{e}^{L|f(u)|} + N^{-1}\ln(1+N^{-1})|f(u)|\mathrm{e}^{L|f(u)|} \\
&\qquad + N^{-1}|f(u)|\ln(|f(u)|)\mathrm{e}^{L|f(u)|} + L|f(u)| + \ln(|f(u)|) + \ln(1+N^{-1}) \\
&\quad \leqslant \mathrm{e}^{N|g|} + N^{-1}(L + 1 + \ln(1+N^{-1}))|f(u)|^2\mathrm{e}^{L|f(u)|} \\
&\qquad + (1+L)|f(u)| + \ln(1+N^{-1}) \\
&\quad \leqslant \mathrm{e}^{N|g|} + N^{-1}(L + 1 + \ln(1+N^{-1}))|f(u)|^2\mathrm{e}^{L|f(u)|} + \frac{1}{4}|f(u)|^2\mathrm{e}^{L|f(u)|} + c,
\end{aligned}$$

because $(1+L)|f(u)| \leqslant \frac{1}{4}|f(u)|^2 + (1+L)^2 \leqslant \frac{1}{4}|f(u)|^2\mathrm{e}^{L|f(u)|} + (1+L)^2$, where c depends on N and L. Choosing finally $N = N(L)$ large enough, we find, in both cases,

$$|g||f(u)|\mathrm{e}^{L|f(u)|} \leqslant \mathrm{e}^{N|g|} + \frac{1}{2}|f(u)|^2\mathrm{e}^{L|f(u)|} + c, \tag{2.20}$$

where c depends only on L. We thus deduce from (2.17) and (2.20) the following inequality:

$$(2.21) \qquad \int_{(0,T)\times\Omega} |f(u)|^2 \mathrm{e}^{L|f(u)|}\,\mathrm{d}x\,\mathrm{d}t \leqslant c + 2\int_{(0,T)\times\Omega} \mathrm{e}^{N|g|}\,\mathrm{d}x\,\mathrm{d}t,$$

where c depends only on the initial data, T and L.

To conclude, we use the following Orlicz's embedding inequality (see [12] and [22]):

$$(2.22) \qquad \int_\Omega \mathrm{e}^{N|v|}\,\mathrm{d}x \leqslant \mathrm{e}^{c(\|v\|^2_{H^1(\Omega)}+1)}, \quad \forall\, v \in H^1(\Omega),$$

where c depends only on Ω and N. It then follows from (2.14), (2.21), and (2.22) that

$$(2.23) \qquad \int_{(0,T)\times\Omega} |f(u)|^2 \mathrm{e}^{L|f(u)|}\,\mathrm{d}x\,\mathrm{d}t \leqslant c,$$

where c depends only on the initial data, T and L. Noting finally that

$$\begin{aligned}
\int_{(0,T)\times\Omega} \mathrm{e}^{L|f(u)|}\,\mathrm{d}x &\leqslant \int_{|f(u)|\leqslant 1} \mathrm{e}^{L|f(u)|}\,\mathrm{d}x + \int_{|f(u)|\geqslant 1} \mathrm{e}^{L|f(u)|}\,\mathrm{d}x \\
&\leqslant c + \int_{|f(u)|\geqslant 1} |f(u)|^2 \mathrm{e}^{L|f(u)|}\,\mathrm{d}x \\
&\leqslant c + \int_{(0,T)\times\Omega} |f(u)|^2 \mathrm{e}^{L|f(u)|}\,\mathrm{d}x,
\end{aligned}$$

where c depends on T and L, (2.23) yields the desired inequality (2.12). □

It is not difficult to show, by comparing growths, that the logarithmic function f satisfies

$$(2.24) \qquad |f'(s)| \leqslant \mathrm{e}^{c|f(s)|+c'}, \quad s \in (-1,1), \;\; c, c' \geqslant 0.$$

Therefore,

$$\int_{(0,T)\times\Omega} |f'(u)|^p\,\mathrm{d}x\,\mathrm{d}t \leqslant \int_{(0,T)\times\Omega} \mathrm{e}^{cp|f(u)|+c'p}\,\mathrm{d}x\,\mathrm{d}t,$$

whence, owing to (2.12),

$$(2.25) \qquad \|f'(u)\|_{L^p((0,T)\times\Omega)} \leqslant c, \quad \forall\, p \geqslant 1,$$

where c depends only on the initial data and T (and p).

We then rewrite (1.1) in the form

$$\frac{\partial u}{\partial t} - \Delta u = \frac{\partial \alpha}{\partial t} - f(u)$$

and have, differentiating with respect to time,

$$\frac{\partial}{\partial t}\Big(\frac{\partial u}{\partial t}\Big) - \Delta \frac{\partial u}{\partial t} = h, \tag{2.26}$$

where

$$h = -\frac{\partial \alpha}{\partial t} + \Delta\alpha - u - \frac{\partial u}{\partial t} - f'(u)\frac{\partial u}{\partial t} \tag{2.27}$$

satisfies, owing to (2.25) (for $p = 4$) and the above a priori estimates (which imply that $(\partial u/\partial t) \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \subset L^4(0,T;H^{1/2}(\Omega)) \subset L^4((0,T)\times\Omega))$,

$$\|h\|_{L^2((0,T)\times\Omega)} \leqslant c, \tag{2.28}$$

where c depends only on the initial data and T.

Multiplying (2.26) by $-\Delta(\partial u/\partial t)$, we find, owing to (2.28),

$$\Big\|\nabla\frac{\partial u}{\partial t}(t)\Big\|^2 + \int_{(0,T)\times\Omega}\Big\|\Delta\frac{\partial u}{\partial t}\Big\|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant c, \quad t\in[0,T], \tag{2.29}$$

where c depends only on the initial data and T (recall that $u_0 \in H^3(\Omega)$).

We finally rewrite (1.2) in the form

$$\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} + \frac{\mathrm{d}\alpha}{\mathrm{d}t} + A\alpha = -u - \frac{\mathrm{d}u}{\mathrm{d}t} \quad \text{in } L^2(\Omega), \tag{2.30}$$

where A denotes the minus Laplace operator with Dirichlet boundary conditions. Taking the scalar product (in $L^2(\Omega)$) of (2.30) by $A^2(\mathrm{d}\alpha/\mathrm{d}t)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|A^{3/2}\alpha\|^2 + \Big\|A\frac{\mathrm{d}\alpha}{\mathrm{d}t}\Big\|^2\Big) \leqslant c\Big(\|\Delta u\|^2 + \Big\|\Delta\frac{\partial u}{\partial t}\Big\|^2\Big) \tag{2.31}$$

and we deduce from (2.29) and (2.31) estimates on α in $L^\infty(0,T;H_0^1(\Omega)\cap H^3(\Omega))$ and on $(\partial\alpha/\partial t)$ in $L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega))$.

Rewriting again (1.1) in the form

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = g, \tag{2.32}$$

we have, owing to the above estimates,

$$g \in L^\infty((0,T)\times\Omega) \tag{2.33}$$

and the separation property follows as in the one-dimensional case.

3. Existence of solutions

We have

Theorem 3.1. *We assume that* (1.9)–(1.11) *hold. Then,* (1.1)–(1.4) *possesses at least one solution* (u,α) *such that* $u \in L^\infty(0,T;H_0^1(\Omega)\cap H^3(\Omega))$, $(\partial u/\partial t) \in L^\infty(0,T;H_0^1(\Omega)) \cap L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega))$, $\alpha \in L^\infty(0,T;H_0^1(\Omega)\cap H^3(\Omega))$ *and* $(\partial\alpha/\partial t) \in L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega))$, $\forall T>0$.

Proof. The proof of existence is standard, once we have the separation property (2.8), since the problem then reduces to one with a regular nonlinearity.

Indeed, we consider the same problem, in which the logarithmic function f is replaced by the $\mathcal{C}^1$ function

$$f_\delta(s) = \begin{cases} f(s), & |s| \leqslant \delta, \\ f(\delta) + f'(\delta)(s-\delta), & s > \delta, \\ f(-\delta) + f'(-\delta)(s+\delta), & s < -\delta, \end{cases}$$

where δ is the same constant as in (2.8).

This function meets all the requirements of [23] to have the existence of a regular solution $(u_\delta, \alpha_\delta)$.

Furthermore, it is not difficult to see that f and f_δ satisfy (1.7), (1.8), and (2.24) for the same constants (taking, if necessary, δ close enough to 1 so that f and f' are nonnegative on $[\delta, 1)$ and $|f_\delta| \leqslant |f|$). We can thus derive the same estimates as above, with the very same constants. Indeed, we can note that the bounds on $(\partial\alpha/\partial t)$ obtained there depend only on f through the constants in (1.7), (1.8), and (2.24) (recall that $\|u_0\|_{L^\infty(\Omega)} \leqslant \delta$).

Since f and f_δ coincide on $[-\delta,\delta]$, we finally deduce that u_δ is a solution to the original problem. □

Remark 3.2. Actually, in one space dimension, we can take less regular initial data, namely, $u_0 \in H_0^1(\Omega)\cap H^2(\Omega)$, $\theta_0 \in H_0^1(\Omega)\cap H^2(\Omega)$, and $\theta_1 \in H_0^1(\Omega)$.

Remark 3.3.

(i) In one space dimension, we can more generally consider a nonlinear function $f \in \mathcal{C}^2(-1,1)$ which satisfies

$$f(0)=0, \quad \lim_{\pm 1} f = \pm\infty, \quad \lim_{s\to\pm 1} f' = \infty. \tag{3.1}$$

(ii) In two space dimensions, we would need, in addition to (3.1), the following assumption:

$$|f'(s)| \leqslant \mathrm{e}^{c|f(s)|+c'}, \quad s \in (-1,1), \ c, c' \geqslant 0. \tag{3.2}$$

R e m a r k 3.4. The difficulty, in three space dimensions, is to obtain an estimate on $(\partial\alpha/\partial t)$ in $L^\infty((0,T)\times\Omega)$. We can nonetheless prove such an estimate if we assume that f has "strong" singularities at ± 1, which excludes logarithmic nonlinear terms (see also [6], [16], [28], and [29] for similar situations). To do so, we assume that f satisfies, in addition to (3.1),

$$|f'(s)| \leqslant c|f(s)|^{6/5} + c', \quad s\in(-1,1),\ \ c,c'\geqslant 0. \tag{3.3}$$

We then rewrite (1.1) in the form

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = g, \tag{3.4}$$

where, in three space dimensions, $g\in L^6((0,T)\times\Omega)$. Multiplying (3.4) by $f(u)^5$ we obtain (we assume, as above, that $f'\geqslant 0$, which is not restrictive)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathcal{F}(u)\,\mathrm{d}x + 5((f'(u)|f(u)|^4\nabla u,\nabla u)) + \int_\Omega |f(u)|^6\,\mathrm{d}x &= ((g,f(u)^5))\\ \leqslant \|g\|_{L^6(\Omega)}\|f(u)\|^5_{L^6(\Omega)} &\leqslant \frac12\int_\Omega |f(u)|^6\,\mathrm{d}x + c\|g\|^6_{L^6(\Omega)},\end{aligned}$$

where $\mathcal{F}(s)=\int_0^s f(\tau)^5\,\mathrm{d}\tau$, which yields that $f(u)\in L^6((0,T)\times\Omega)$, hence $f'(u)\in L^5((0,T)\times\Omega)$ owing to (3.3). Now we have, in three space dimensions,

$$\frac{\partial u}{\partial t}\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1_0(\Omega))\subset L^{10/3}(0,T;H^{3/5}(\Omega))\subset L^{10/3}((0,T)\times\Omega)$$

and we can conclude as above in two space dimensions (indeed, $f'(u)(\partial u/\partial t)\in L^2((0,T)\times\Omega)$). We can note that, in particular, (3.3) holds when $|f|$ has a growth of the form

$$\frac{c}{(1-s^2)^r},\quad r\geqslant 5,\ \ c>0,$$

close to ±1. Unfortunately, as already mentioned, the logarithmic functions (1.6) do not satisfy this condition.

R e m a r k 3.5. We can easily prove the uniqueness of solutions satisfying the separation property, i.e., those which are given by the above approximation procedure. Now, there may very well be other solutions, obtained by a different approximation argument, which do not satisfy the separation property (2.8). Actually, in order to obtain the full uniqueness result, we need to integrate (1.2) (written for the difference of two solutions with the same initial data) between 0 and t to have

$$\frac{\partial\theta}{\partial t} + \theta - \Delta\int_0^t \theta\,\mathrm{d}\tau = -\int_0^t u\,\mathrm{d}\tau - u. \tag{3.5}$$

We then need an estimate on $(\partial\theta/\partial t)$ in $L^2(0,T;L^2(\Omega))$, but, due to the hyperbolic nature of (3.5) (and (1.2)), we have not been able to derive such an estimate.

R e m a r k 3.6. We can also consider, as in [23], a quasistatic model for the order parameter (which can be justified by taking η small in (1.19)), i.e., we consider, instead of (1.2), the following equation for the thermal displacement:

$$\frac{\partial^2\alpha}{\partial t^2}+\frac{\partial\alpha}{\partial t}-\Delta\alpha=-u. \tag{3.6}$$

In that case, it is not difficult to derive (the same first) a priori estimates as in Section 1 and, in particular, an estimate on u in $L^2(0,T;H_0^1(\Omega)\times H^2(\Omega))$. Rewriting then (3.6) in the form

$$\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2}+\frac{\mathrm{d}\alpha}{\mathrm{d}t}+A\alpha=-u \quad \text{in } L^2(\Omega), \tag{3.7}$$

we have, taking the scalar product of (3.7) by $A^2(\mathrm{d}\alpha/\mathrm{d}t)$ in $L^2(\Omega)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|A^{3/2}\alpha\|^2+\Big\|A\frac{\mathrm{d}\alpha}{\mathrm{d}t}\Big\|^2\Big)\leqslant c\|\Delta u\|^2, \tag{3.8}$$

hence an estimate on $(\partial\alpha/\partial t)$ in $L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega))$. Finally, proceeding as above, we obtain the separation property (2.8), without any restriction on the space dimension. We can also note that, for this quasistatic model, the uniqueness is straightforward.

R e m a r k 3.7. We also considered in [26] a heat conduction law based on type III thermoelasticity (see [19]). In that case, we have the following "strongly" damped wave equation for α:

$$\frac{\partial^2\alpha}{\partial t^2}-\Delta\frac{\partial\alpha}{\partial t}-\Delta\alpha=-\frac{\partial u}{\partial t}. \tag{3.9}$$

Due to the higher dissipation given by the term $-\Delta(\partial\alpha/\partial t)$, it is easier to obtain an estimate on $(\partial\alpha/\partial t)$ in $L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega))$, hence the separation property (2.8) which, in particular, holds in three space dimensions. This term also allows to prove the uniqueness of solutions.

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