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Groupoids and the Associative Law VII. (Semigroup Distance of SH-Groupoids)

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Szász-Hájek groupoids (shortly SH-groupoids) are those groupoids that contain just one non-associative (ordered) triple of elements. These groupoids were studied by G. Szász (see [10] and [11]), P. Hájek (see [2] and [3]) and later in [6], [7], [8] and [9]. The present short note is concerned with semigroup distances of SH-groupoids of type (a, a, a).

1. Preliminaries

A groupoid G is called an SH-groupoid if the set $\{(a,b,c) \in G^{(3)} \mid a \cdot bc \neq ab \cdot c\}$ of non-associative triples contains just one element. Let G be an SH-groupoid and let (a,b,c) be the only non-associative triple. We shall say that G is of type:
- (a,a,a) if $a = b = c$;
- (a,a,b) if $a = b \neq c$;
- (a,b,a) if $a = c \neq b$;
- (a,b,b) if $a \neq b = c$;
- (a,b,c) if $a \neq b \neq c \neq a$;

Furthermore, G will be called minimal if G is generated by the set $\{a,b,c\}$. The following assertions are easy:

1.1 Proposition. *Let G be an SH-groupoids and let $a,b,c \in G$ be such that $a \cdot bc \neq ab \cdot c$. Then:*

(i) *G is of exactly one of the types (a, a, a), (a, a, b), (a, b, a), (a, b, b) and (a, b, c).*

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(ii) *If H is a subgroupoid of G, then either $\{a,b,c\} \subseteq H$ and H is an SH-groupoid (of the same type as G) or $\{a,b,c\} \nsubseteq H$ and H is a semigroup.*

(iii) *The subgroupoid $\langle a,b,c\rangle_G$ is a minimal SH-groupoid.*

(iv) *If $u, v \in G$ are such that $uv \in \{a,b,c\}$, then $uv \in \{u,v\}$.*

Let $G(*)$ and $G(\circ)$ be two groupoids having the same underlying set. We put $\operatorname{dist}(G(*), G(\circ)) = \operatorname{card}\{(u,v) \in G^{(2)} \mid u * v \neq u \circ v\}$.

Let G be an SH-groupoid. Then $\operatorname{sdist}(G)$ denotes the minimum of $\operatorname{dist}(G, G(*))$, $G(*)$ running through all semigroups with the same underlying set as G.

2. Semigroup distances of SH-groupoids of type (a, a, a)

2.1 Construction.

Let K denote the set of integers $k \geq 4$, M be a four-element set $\{a,b,c,d\}$ such that $M \cap K = \emptyset$ and let $H = K \cup M$. Define an operation $\circ$ on H in the following way: $a \circ a = b$, $a \circ b = c$, $b \circ a = d$, $a \circ c = c \circ a = a \circ d = d \circ a = b \circ b = 4$, $b \circ c = c \circ b = b \circ d = d \circ b = 5$, $c \circ c = c \circ d = d \circ c = d \circ d = 6$ and $a \circ k = k \circ a = k + 1$, $b \circ k = k \circ b = k + 2$, $c \circ k = k \circ c = d \circ k = k \circ d = k + 3$, $k \circ m = m \circ k = m + k$ for all $m, k \in K$. Furthermore, define a mapping σ of H onto the set of positive integers by $\sigma(a) = 1$, $\sigma(b) = 2$, $\sigma(c) = \sigma(d) = 3$ and $\sigma(k) = k$ for every $k \in K$.

2.1.1 Lemma. *$\sigma(x \circ y) = \sigma(x) + \sigma(x) + \sigma(y)$ for all $x, y \in H$.*

Proof. Easy to check.

2.1.2 Lemma. *Let $(x, y, z) \in H^{(3)}$ be such that $\sigma(x) + \sigma(y) + \sigma(z) \geq 4$. Then $x \circ (y \circ z) = (x \circ y) \circ z$.*

Proof. Easy to check.

2.1.3 Lema. *Let $(x, y, z) \in H^{(3)}$ be such that $\sigma(x) + \sigma(y) + \sigma(z) = 3$. Then $x = y = z = a$ and $x \circ (y \circ z) \neq (x \circ y) \circ z$.*

Proof. Easy to check.

2.1.4 Lema. *$H(\circ)$ is a minimal SH-groupoid of type (a,a,a) (i.e., $H(\circ)$ is generated by the one-element set $\{a\}$).*

Proof. Easy to check (the structure of SH-groupoids of type (a, a, a) is described in [6]).

2.1.5 Lemma. $(a \circ a) \circ (a \circ a) = a \circ ((a \circ a) \circ a)$.

Proof. Easy to check.

2.1.6 Lemma. $\operatorname{sdist}(H(\circ)) = 1$.

Proof. Put $b \nabla a = c$ and $x \nabla y = x \circ y$ whenever $(x,y) \neq (b,a)$. It is easy to check that $H(\nabla)$ is a groupoid satisfying the identity $\sigma(x \nabla y) = \sigma(x) + \sigma(y)$ for all $x, y \in H$ and all triples $(x,y,z) \in H^{(3)}$ are associative. Thus $H(\nabla)$ is a semigroup and $\mathrm{sdist}(H(\circ)) = 1$.

2.2 Construction.

Consider the groupoid $H(\circ)$ constructed in 2.1 and let a set $A = \{p, v, w, r, s, t\}$ be disjoint with the set H. Put $E = H \cup A$ and consider the mapping σ from 2.1. Further, put $\sigma(p) = 1$, $\sigma(v) = \sigma(w) = 2$, $\sigma(r) = \sigma(s) = \sigma(t) = 3$. Now, define a binary operation on E in the following way:

- $xy = x \circ y$ for all $x, y \in H$;
- $ap = b$, $pa = v$, $pp = w$;
- $aw = bp = c$, $av = d$, $pb = va = vp = r$, $pv = wa = s$, $pw = wp = t$;
- $ar = as = at = pc = pd = pr = ps = pt = bv = bw = vb = vv = vw =$ $= wb = wv = ww = cp = dp = ra = rp = sa = sp = ta = tp = 4$;
- $kp = pk = k + 1$ for each $k \in K$;
- $vk = kv = wk = kw = k + 2$ for each $k \in K$;
- $dk = kd = fk = kf = gk = kg = k + 3$ for each $k \in K$.

Then E is a groupoid containing $H(\circ)$ as a proper subgroupoid. Moreover, every triple $(x,y,z) \in E^{(3)}$ such that $\sigma(x) + \sigma(y) + \sigma(z) \geq 4$ is associative. The triple (a,a,a) is non-associative and it is easy to check that the triples (a,a,p), (a,p,a), (p,a,a), (a,p,p), (p,a,p), (p,p,a) and (p,p,p) are associative. The groupid E is an SH-groupoid of the type (a,a,a) and it is generated by the two-element set $\{a,p\}$.

2.2.1 Lemma. $\mathrm{sdist}(E) > 1$.

Proof. Suppose that the opposite case takes place. Then there exists at least one semigroup $(E, *)$ having the same underlying set E such that $\mathrm{dist}(E, E(*)) = 1$. Of course, the equality $a * (a * a) = (a * a) * a$ is true. Therefore either $aa \neq a * a$ or $ba \neq b * a$ or $ab \neq a * b$.

If $aa \neq a * a = z$, then we have $xz = x * z = x * (a * a) = (x * a) * a =$ $= xa * a = (xa)a$ for every $a \neq x \in K$. From this it follows immediately that $\sigma(z) = 2$ and therefore $z \in \{v,w\}$. But for $z = v$ we obtain $d = av = a * v =$ $= a * ap = a * (a * p) = (a * a) * p = v * p = vp = r$, a contradiction. Similarly, for $z = w$ we have $c = aw = a * w = a * pa = a * (p * a) = (a * p) * a =$ $= ap * a = vp = r$, a contradiction again.

If $ba \neq b * a = z$, then we have $z = b * a = (a * a) * a = a * (a * a) = a * aa =$ $= a * ab = c$. But we have $d = av = a * v = a * pa = a * (p * a) = (a * p) * a =$ $= ap * a = b * a = c$, a contradiction. The case $ab \neq a * b$ is similar. Thus $\mathrm{sdist}(E) > 1$.

2.2.2 Lemma. $\mathrm{sdist}(E) = 2$.

Proof. Define on E a new binary operation $*$ such that $c = b * a \neq ba$, $c = p * w \neq pw$ and $x * y = xy$ whenever $(b,a) \neq (x,y) \neq (p,w)$. It is obvious that $E(*)$ is a groupoid satisfying the identity $\sigma(x * y) = \sigma(x) + \sigma(y)$ for all $x, y \in E$. Therefore, it is easy to check that every triple $(x, y, z) \in E^{(3)}$ is associative. Thus $\operatorname{dist}(E, E(*)) = 2$ and $\operatorname{sdist}(E) \leq 2$. The rest follows from 2.2.1.

2.2.3 Corollary. *There is at least one SH-groupid E of type (a, a, a) containing a proper SH-subgroupoid H such that* $\operatorname{sdist}(H) < \operatorname{sdist}(E)$.

2.3 Construction.

Let K and M be the same sets as in 2.1 and consider the groupoid $H(\circ)$ constructed in 2.1. Put $B = \{r, s, t\}$ and let I be an arbitrary set of indexes. For every $i \in I$ consider a three-element set $A_i = \{p_i, v_i, w_i\}$ and denote $A = \bigcup_{i \in I} A_i$. Further, put $C = \{q\}$ and suppose that the sets K, M, A, B, C are pair-wise disjoint. Finally, put $G_I = A \cup B \cup C \cup K \cup M$ and denote $E_i = A_i \cup B \cup K \cup M$ for each $i \in I$. On each set E_i, let us define a binary operation in the way described in 2.2. Now, define a binary operation on G_I such that E_i is a subgroupoid of G_I for each $i \in I$. Further, for every $i, k \in I$, $i \neq k$, put:

- $p_i p_k = q$;
- $aq = c$, $qa = s = p_i v_k$, $v_i p_k = r$ and $qp_i = p_i q = p_i w_k = w_i p_k = t$;
- $bq = qb = qq = qv_i = qw_i = qw_i = v_i q = w_i q = v_i v_k = v_i w_k = w_i v_k =$ $= w_i w_k = 4$;
- $qc = cq = qd = dq = qr = rq = qs = sq = tq = at = 5$.

Finally, put $mq = qm = m + 2$ for every $m \in K$ and $\sigma(q) = 2$. Then G_I becomes a groupoid containing each of SH-groupoids E_i as a proper subgroupoid and the equation $\sigma(xy) = \sigma(x) + \sigma(y)$ holds for all $x, y \in G_I$.

2.3.1 Lemma. *G_I is an SH-groupoid of type (a, a, a) satisfying the condition $(aa)(aa) = a((aa)a)$.*

Proof. It is tedious but not difficult to check that G_I contains just one non-associative triple, namely (a, a, a).

2.3.2 Lemma. $\operatorname{sdist}(G_I) \leq 1 + \operatorname{card}(I)$.

Proof. Define on G_I a new binary operation $*$ such that $c = b * a \neq ba$, $c = a * v_i \neq av_i = d$ for each $i \in I$ and $x * y = xy$ whenever $(b, a) \neq (x, y) \neq$ $\neq (p, w_i)$ for every $i \in I$. Then $G_I(*)$ becomes a groupoid satisfying the identity $\sigma(x * y) = \sigma(x) + \sigma(y)$ for all $x, y \in G_I$. It is obvious that every triple (x, y, z) having $\sigma(x) + \sigma(y) + \sigma(z) \geq 4$ is associative. There is a finite number of triples (x, y, z) having $\sigma(x) + \sigma(y) + \sigma(z) \leq 3$. It is tedious but possible to check that all of them are associative. Thus $G_I(*)$ is a semigroup and the rest is clear.

2.4 Semigroup distance of the groupoid G_I.

In this section, let G_I be the groupoid from 2.3, let $\operatorname{card}(I) = \kappa$ and let $G_I(*)$ be a semigroup having the same underlying set G_I such that $\operatorname{dist}(G_I, G_I(*)) = \operatorname{sdist}(G_I)$. Further, for every $i \in I$ consider the following sets: $L(p_i) = \{x \in G_I \mid x * p_i \neq xp_i\}$,

$L(v_i) = \{x \in G_I \mid x * v_i \neq xv_i\}$, $L(w_i) = \{x \in G_I \mid x * w_i \neq xw_i\}$, $R(p_i) = \{x \in G_I \mid p_i * x \neq$ $\neq p_i x\}$, $R(v_i) = \{x \in G_I \mid v_i * x \neq v_i x\}$, $R(w_i) = \{x \in G_I \mid w_i * x \neq w_i x\}$.

2.4.1 Lemma. *If $ba = b * a$, $aa = a * a$ and $a * b \neq ab$, then* $\operatorname{dist}(G_I, G_I(*)) \geq$ $\geq 1 + \kappa$.

Proof. Suppose that $L(p_i) = \emptyset$ for some $i \in I$. Then $c = bp_i = b * p_i =$ $= aa * p_i = (a * a) * p_i = a * (a * p_i) = a * ap_i = a * b \neq c$, a contradiction. Therefore, $L(p_i) \neq \emptyset$ for every $i \in I$.

2.4.2 Lemma. *If $ba = b * a$, $b = aa \neq a * a$ and $a * b = ab$, then* $\operatorname{dist}(G_I,$ $G_I(*)) \geq 1 + \kappa$.

Proof. Suppose that $L(p_i) = \emptyset$ for some $i \in I$. If $y = a * a$ then $yp_i = y * p_i =$ $= (a * a) * p_i = a * (a * p_i) = a * ap_i = a * b = ab = c$. However, the equation $yp_i = c$ is solvable in G_I if and only if $y = b$, a contradiction. Therefore $L(p_i) \neq \emptyset$ for every $i \in I$.

2.4.3 Lemma. *If $ba \neq b * a$, then* $\operatorname{dist}(G_I, G_I(*)) \geq 1 + \kappa$.

Proof. Suppose that $L(p_i) = R(p_i) = R(v_i) = \emptyset$ for some $i \in I$. Then $d = av_i =$ $= a * v_i = a * (p_i a) = a * (p_i * a) = (a * p_i) * a = ap_i * a = b * a \neq d$, a contradiction. Therefore at least one of the sets $L(p_i)$, $R(p_i)$, $R(v_i)$ is non-empty for every $i \in I$.

2.4.4 Lemma. *If $aa \neq a * a = y$ and $\sigma(y) \geq 3$ then* $\operatorname{dist}(G_I, G_I(*)) \geq 1 + \kappa$.

Proof. Suppose that $R(p_i) = \emptyset = R(v_i)$ for some $i \in I$. Then we have $\sigma(p_i y) =$ $= \sigma(p_i) + \sigma(y) \geq 4$. But $p_i y = p_i * y = p_i * (a * a) = (p_i * a) * a = p_i a * a =$ $= v_i * a = v_i a$. Thus $\sigma(p_i y) = 3$, a contradiction. Therefore at least one of the sets $R(p_i)$, $R(v_i)$ is non-empty for every $i \in I$.

2.4.5 Lemma. *If $a = a * a$ then* $\operatorname{dist}(G_I, G_I(*)) \geq 1 + \kappa$.

Proof. Suppose that $R(p_i) = \emptyset = R(v_i)$ for some $i \in \mathrm{I}$. Then $v_i = p_i a =$ $= p_i * a = p_i * (a * a) = (p_i * a) * a = p_i a * a = v_i * a = v_i a = r$, a contradiction. Therefore at least one of the sets $R(p_i)$, $R(v_i)$ is non-empty for every $i \in I$.

2.4.6 Lemma. *If $p_k = a * a$ for some $k \in I$ and $b * a = ba$, then* $\operatorname{dist}(G_I, G_I(*)) \geq$ $\geq 1 + \kappa$.

Proof. Suppose that $p_k p_k = p_k * p_k$ and $ap_k = a * p_k$. Then $w_k = p_k p_k =$ $= p_k * p_k = (a * a) * (a * a) = ((a * a) * a) * a = (a * (a * a)) * a = (a * p_k) * a =$ $= ap_k * a = b * a = ba = d$, a contradiction. Therefore either $p_k p_k \neq p_k * p_k$ or $a * p_k \neq ap_k$. Further, suppose that $k \neq i \in I$ and $R(p_i) = R(v_i) = \emptyset$. Then $q =$ $= p_i p_k = p_i * p_k = p_i * (a * a) = (p_i * a) * a = p_i a * a = v_i * a = v_i a = r$, a contradiction. Therefore, at least one of the sets $R(p_i), R(v_i)$ is non-empty for every $k \neq i \in I$.

2.4.7 Lemma. *If* $a * a = v_k$ *for some* $k \in I$ *and* $b * a = ba$ *then* $\operatorname{dist}(G_I, G_I(*)) \geq 1 + \kappa$.

Proof. It is obvious if $L(P_i) \neq \emptyset$ for each $i \in I$. Suppose first that $a * p_k = ap_k$ and $a * p_i \neq ap_i$ for every $k \neq i \in I$. If $a * b = ab$ nd $v_k * p_k = v_k p_k$ then $c = = ab = a * b = a * (ap_k) = a * (a * p_k) = (a * a) * p_k = v_k p_k = r$, a contradiction. Thus we have $a * a \neq aa$ and either $a * b \neq ab$ or $v_k * p_k \neq v_k p_k$ in this case. Further, suppose that $L(P_j) = \emptyset$ for some $k \neq j \in I$. Then $a * b = a * (ap_j) = = a * (a * p_j) = (a * a) * p_j = v_k * p_j = v_k p_j = r \neq c = ab$. If $R(p_i) = \emptyset = R(v_i)$ for some $k \neq i \in I$, then $s = p_i v_k = p_i * v_k = p_i * (a * a) = (p_i * a) * a = p_i a * a = = v_i * a = v_i * a = v_i a = r$, a contradiction. Thus at least one of the sets $R(p_i)$, $R(v_i)$ is non-empty for every $k \neq i \in I$. Moreover, $a * b \neq ab$ and $a * a \neq aa$ in this case.

2.4.8 Lemma. *If* $a * a = w_k$ *for some* $k \in I$ *and* $b * a = ba$, *then* $\operatorname{dist}(G_I, G_I(*)) \geq 1 + \kappa$.

Proof. It is obvious if $L(p_i) \neq \emptyset$ for all $i \in I$. Suppose first that $a * p_k = ap_k$ and $a * p_i \neq ap_i$ for every $k \neq i \in I$. If $a * b = ab$ and $v_k * p_k = v_k p_k$ then $c = ab = = a * b = a * (a * p_k) = (a * a) * p_k = v_k * p_k = v_k p_k = r$, a contradiction. Thus we have $a * a \neq aa$ and either $a * b \neq ab$ or $v_k * p_k \neq v_k p_k$. Further, suppose that there is $k \neq j \in I$ such that $L(p_j) = \emptyset$. Then $a * b = a * (ap_j) = a * (a * p_j) = = (a * a) * p_j = v_k * p_j = v_k p_j = t$. Thus we have $a * a \neq aa$ and $a * b \neq ab$. If $R(p_i) = \emptyset = R(v_i)$ for some $k \neq i \in I$ then $t = p_i w_k = p_i * w_k = p_i * (a * a) = = (p_i * a) * a = p_i a * a = v_i * a = v_i a = r$, a contradiction. Therefore at least one of the sets $R(p_i)$, $r(v_i)$ is non-empty for every $k \neq i \in I$.

2.4.9 Lemma. *If* $a * a = q$ *and* $b * a = ba$ *then* $\operatorname{dist}(G_I, G_I(*)) \geq 1 + \kappa$.

Proof. Of course, $a * a \neq aa$ and the assertion is obvious if $a * p_i \neq ap_i$ for every $i \in I$. Now, let $k \in I$ be such that $a * p_k = ap_k$. If $q * p_k = qp_k$ and $a * b = ab$, then $c = ab = a * b = a * ap_k = a * (a * p_k) = (a * a) * p_k = q * p_k = = qp_k = t$, a contradiction. Hence we have either $q * p_k \neq qp_k$ or $a * b \neq ab$. Finally, let $k \neq i \in I$. Then either $a * p_i \neq ap_i$ (and then $L(p_i) \neq \emptyset$), or $a * p_i = ap_i$. In the second case, suppose that $R(p_i) = \emptyset = R(v_i)$. Then $t = p_i q = p_i * q = = p_i * (a * a) = (p_i * a) * a = p_i a * a = v_i a = r$, a contradiction. Therefore, at least one of the sets $R(p_i)$, $R(v_i)$ is non-empty.

2.4.10 Proposition. $\operatorname{sdist}(G_I) = 1 + \operatorname{card}(I)$.

Proof. With respect to 2.3.2, $\operatorname{dist}(G_I, G_I(*)) \leq 1 + \kappa$. Of course, at least one of the conditions $a * a \neq aa$, $a * b \neq ba$, $b * a \neq ba$ has to be valid (otherwise $c = ab = a * b = a * aa = a * (a * a) = (a * a) * a = aa * a = b * a = ba = d$, a contradiction). For $b * a \neq ba$ see 2.4.3, for $b * a = ba$, $a * b = ab$ and $a * a \neq aa$ see 2.4.2, for $b * a = ba$, $a * b \neq ab$ and $a * a = aa$ see 2.4.1. The remaining case depends on the value of $y = a * a \neq aa$ and the result follows from one of 2.4.4, 2.4.5, 2.4.6, 2.4.7, 2.4.8 and 2.4.9.

3. Conclusion

It was proved above that there exist SH-groupoids of type (a, a, a) satisfying the equation $aa \cdot aa = a(aa \cdot a)$ and having an arbitrary large semigroup distance. Is the same true also for SH-groupoids G of type (a, a, a) satisfying the condition $aa \cdot aa \neq a(aa \cdot a)$ for at least one $a \in G$? Is it true for S-groupoids of other types?

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