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Selfdistributive Groupoids Part D1: Left Distributive Semigroups

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In this paper, the essentials of the algebraic theory of left distributive semigroups are presented.

0. Introduction

Every semilattice (i.e., an idempotent commutative semigroup) is selfdistributive. An explicit formulation of this fact (perhaps for the first time) can be found already in C. S. Pierce [Pie,80]. A structural study of two-sided selfdistributive semigroups was initiated in M. Petrich [Pet,69] and that of one-sided selfdistributive semigroups ten years later in S. Markovski [Mar,79].

Altogether, there are only a few papers devoted to selfdistributive semigroups. The present article is a survey treatment on the topic.

As concerns the notation, terminology, references, comments, etc., used and related to but neither defined nor formulated in the following text, a kind reader is fully referred to [KepN,03] (also cited as A1. ⋯).

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I.1 Basic properties of left distributive semigroups

1.1 Proposition. *Let S be an LD-semigroup. Then, for all* $x, y, z \in S$:

(i) $xyz = xyxz = xy^2z$.
(ii) $x^ny = x^2y$ *for every* $n \geq 2$.
(iii) $(xy)^n = xy^n = xy^2 = (xy)^2$ *for every* $n \geq 2$.
(iv) $x^n = x^3$ *for every* $n \geq 3$.

Proof. (i) $xyz = xyxz = xyxyz = xy^2z$ by repeated use of the left distributive law.

(ii) For $n \geq 3$, $x^ny = xx^{n-2}xy = xx^{n-2}y = x^{n-1}y$
(iii) For $n \geq 3$, $(xy)^n = xy^n = xyxy^{n-1} = xyxyy^{n-2} = xyxy^{n-2} = xy^{n-1}$.
(iv) For $n \geq 4$, $x^n = xxxx^{n-3} = xxx^{n-3} = x^{n-1}$. □

1.2 Proposition. *Let S be an LD-semigroup. Then:*

(i) $\mathrm{Id}(S)$ *is a left ideal of S and* x^3, xy^2, $xyx \in \mathrm{Id}(S)$ *for all* $x, y \in S$.
(ii) *S is elastic.*
(iii) *For every* $n \geq 3$, $o_{n,S} = o_{3,S}$.

Proof. (i) First, $xy^2 \in \mathrm{Id}(S)$ by 1.1(iii) and $(xyx)^2 = xyx^2 = xyx$. Now, $\mathrm{Id}(S)$ is a left ideal of S (see also A1.II.1.5(i)).

(ii) Every semigroup is elastic.
(iii) This is an immediate consequence of 1.1(iv). □

1.3 Proposition. *The following three conditions are equivalent for an LD-semigroup S*:

(i) $\mathrm{Id}(S)$ *is an ideal of S.*
(ii) $S^3 \subseteq \mathrm{Id}(S)$.
(iii) *S satisfies the (semigroup) identity* $x^2y \approx x^2y^2$.

If these conditions are satisfied, then $S/\mathrm{Id}(S)$ *is an A-semigroup.*

Proof. (i) implies (ii). $xyz = xy^2z$ by 1.1(i), and $xy^2 \in \mathrm{Id}(S)$ by 1.2(i).

(ii) implies (iii). Since $x^2y \in \mathrm{Id}(S)$, we have $x^2y = x^2y \cdot x^2y = x^2y^2$.
(iii) implies (i). By 1.2(i), $\mathrm{Id}(S)$ is a left ideal. Let $x \in S$ and $a \in \mathrm{Id}(S)$. Then $ax = a^2x = a^2x^2 = a^2x \cdot a^2x = (ax)^2$. Thus $\mathrm{Id}(S)$ is a right ideal. □

1.4 Definition. An LD-semigroup satisfying the equivalent conditions of 1.3 will be called an *LDR-semigroup*.

1.5 Proposition. *The following four conditions are equivalent for an LD-semigroup S:*

(i) $S^2 \subseteq \mathrm{Id}(S)$.
(ii) $\mathrm{Id}(S)$ *is an ideal of S and* $S/\mathrm{Id}(S)$ *is a Z-semigroup.*

(iii) *S satisfies the identity* $xy \approx xy^2$.
(iv) S/q_S *is idempotent.*
If these conditions are satisfied, then S is an LDR-semigroup.

Proof. Easy. □

1.6 Definition By an *LDR_l-semigroup* we mean a semigroup satisfying $xy \approx xyx$. (Clearly, every LDR_l-semigroup is left distributive.)

1.7 Proposition. *Every LDR_l-semigroup satisfies the equivalent conditions of 1.5 (hence it is an LDR-semigroup).*

Proof. Let S be an LDR_l-semigroup. By 1.2(i), $xy = xyx \in \mathrm{Id}(S)$ for all $x, y \in S$. Thus $S^2 \subseteq \mathrm{Id}(S)$. □

1.8 Proposition. *Let S be an LD-semigroup. Then:*
(i) p_S *is a congruence of S.*
(ii) S/p_S *is an LDR_l-semigroup.*

Proof. (i) This is true for every semigroup
(ii) We have $xy \cdot z = xyx \cdot z$ for all $x, y, z \in S$. □

1.9 Proposition. *The following four conditions are equivalent for an LD-semigroup S:*
(i) $o_{2,S}$ *is an endomorphism of S.*
(ii) $o_{3,S}$ *is an endomorphism of S.*
(iii) *S satisfies the identity* $xy^2 \approx x^2y^2$.
(iv) *S is left semimedial.*

Proof. By 1.1(ii) annd 1.1(iii) we have $(xy)^3 = xy^3 = xy^2 = (xy)^2$ and $x^3y^3 = x^2y^2$ for all $x, y \in S$. Now it is clear that the first three conditions are equivalent.
If (iii) is satisfied, then $xx \cdot yz = x^2yz = x^2y^2z = xy^2z = xyz = xy \cdot xz$ (use 1.1). Conversely, if S is left semimedial, then $x^2y^2 = xyxy = xy^2$. □

1.10 Definition. Every LD-semigroup satisfying the equivalent conditions of 1.9 will be called an *LDT-semigroup*.

1.11 Proposition. *Let S be an LDT-semigroup. Then:*
(i) $o_{3,S}$ *is a homomorphism of S onto* $\mathrm{Id}(S)$.
(ii) *Every block of* $\ker(o_{3,S})$ *is an A-semigroup.*

Proof. Easy. □

1.12 Proposition. *The following conditions are equivalent for an LD-semigroup S:*
(i) *S satisfies the identity* $xy \approx x^2y$.
(ii) S/p_S *is idempotent.*

Proof. Easy. □

1.13 Definition. Every LD-semigroup satisfying the equivalent conditions of 1.12 will be called an *LDT_1-semigroup.*

1.14 Proposition. *Let S be an LDT_1-semigroup. Then:*

(i) *S is an LDT-semigroup.*

(ii) *o_S is a homomorphism of S onto* $\mathrm{Id}(S)$.

(iii) *Every block of* $\ker(o_S)$ *is a Z-semigroup.*

Proof. Easy. □

1.15 Proposition. *Let S be an LD-semigroup. Then S/q_S is an LDT_1-semigroup.*

Proof. We have $zxy = zx^2y$ for all $x, y, z \in S$. □

1.16 Proposition. *The following three conditions are equivalent for an LD-semigroup S:*

(i) *S satisfies the identity $x^2y \approx xy^2$ (i.e., S is delightful).*

(ii) *S satisfies the identities $x^2y \approx xy^2$ and $xyz \approx x^2yz$ (i.e., S is strongly delightful).*

(iii) *S is an LDTR-semigroup. (I.e., both LDR and LDT.)*

Proof. (i) implies (ii). We have $x^2yz = xy^2z = xyz$ by 1.1(i).

(ii) implies (iii). We have $x^2y = x \cdot x^2y = x^2y^2$ by 1.1(ii), so that S is an LDR-semigroup. Similarly, $xy^2 = xy^2 \cdot y = x^2y^2$ by 1.1(iii), so that S is an LDT-semigroup.

(iii) implies (i). This follows immediately from the definitions. □

1.17 Proposition. *Let S be an LDRT-semigroup. Then:*

(i) $\mathrm{Id}(S)$ *is an ideal of S and* $S/\mathrm{Id}(S)$ *is an A-semigroup.*

(ii) *$o_{3,S}$ is a homomorphism of S onto* $\mathrm{Id}(S)$ *and every block of* $\ker(o_{3,S})$ *is an A-semigroup.*

(iii) $\ker(o_{3,S}) \cap \equiv_{\mathrm{Id}(S)} = \mathrm{id}_S$ *and S is a subdirect product of* $\mathrm{Id}(S)$ *and* $S/\mathrm{Id}(S)$.

Proof. For (i) see 1.3; for (ii) see 1.11; (iii) is clear. □

1.18 Proposition. *Let S be an LDR_1-semigroup. Then there exists a congruence r of S such that S/r is commutative and every block of r containing at least two elements is a subsemigroup of S and an LZ-semigroup.*

Proof. Define r by $(a, b) \in r$ iff either $a = b$ or $a = cb$ and $b = da$ for some $c, d \in S$. Clearly, r is an equivalence and $(a, b) \in r$ implies $(ax, bx) \in r$ for any $x \in S$. On the other hand, using the left distributive law, one can see that $(a, b) \in r$ also implies $(xa, xb) \in r$. So, r is a congruence of S. Since S is an LDR_1-semigroup, we have $ab = aba$, $ba = bab$ and $(ab, ba) \in r$ for all $a, b \in S$. Thus S/r is commutative.

Now, let A be a block of r and $a, b \in A$, $a \neq b$. We have $a = cb$ and $b = da$ for some elements c, d. Then $ab = ada = ad = cbd = cdad = cda = cb = a$.

Further, $(a, b) \in r$ implies $(aa, ab) \in r$, so that $(aa, a) \in r$, and we get $aa \in A$. If $a \neq aa$, then $a = a^3$ according to the previous observation, so that $a \in \mathrm{Id}(S)$ by 1.2(i), a contradiction.

1.19 Proposition. *The following five conditions are equivalent for an LD-semigroup S:*

(i) *S is right semimedial.*
(ii) *S is middle semimedial.*
(iii) *S is medial.*
(iv) *S/p_S is right permutable.*
(v) *S/q_S is left permutable.*

Proof. (i) implies (iii). $xyuv = xyu^2v = xuyuv = xuyv$.
(ii) implies (iii). $xyuv = xyuxv = xuyxv = xuyv$. □

1.20 Proposition. *The following conditions are equivalent for a semigroup S:*

(i) *S is a medial LDR-semigroup.*
(ii) *S is a medial LDRT-semigroup.*
(iii) *S is a D-semigroup.*

Proof. (i) implies (iii). $xyz = xyxz = xxyz = x^2y^2z = x^2y^2z^2 = x^2yz^2 = x^2zyz = xzyz$.
(iii) implies (ii). $xyuv = xuyuv = xuyv$, $xxy = xyxy = x^2y^2$ and $xyy = xyxy = x^2y^2$. □

1.21 Proposition. *The following conditions are equivalent for a semigroup S:*

(i) *S is an LD-semigroup and* $\mathrm{card}(\mathrm{Id}(S)) = 1$.
(ii) *S is an A-semigroup.*

Proof. (i) implies (ii). Let $\mathrm{Id}(S) = \{0\}$.By 1.2(i), 0 is a right absorbing element of S and $xy^2 = 0 = xyx$ for all $x, y \in S$. Now, $0x = 0x0x = 0x^2 = 0$ and hence $xyz = xyxz = 0z = 0$ for all $x, y, z \in S$. □

1.22 Proposition. *Let S be an LD-semigroup, $C = \mathscr{C}_l(S)$ and $D = S - C$. Then:*

(i) *Every element of C is a left neutral element of S.*
(ii) *If C is nonempty, then $q_S = \mathrm{id}_S$, S is an LDT_l-semigroup and C is an RZ-semigroup.*
(iii) *If D is nonempty, then D is a prime ideal of S.*
(iv) *If C is nonempty and S is an LDR_l-semigroup, then $C = \{e\}$ is a singleton and e is a neutral element of S.*

Proof. (i) For $a \in C$ and $x \in S$, $aax = aaax$ implies $x = ax$.
(ii) $C \neq \emptyset$ implies immediately that $q_S = \mathrm{id}_S$, and then S is an LDT_1-semigroup by 1.15. Further, C is a subsemigroup of S (see also A1.II.4.1(i)) and C is an RZ-semigroup by (i).

(iii) Since S is a semigroup, D is a left ideal of S. Let $a \in D$ and $x \in S$. Then $au = av$ for some $u, v \in S$, $u \neq v$, and we have $axu = axau = axav = axv$. Hence $ax \in D$ and we see that D is an ideal. Finally, if $ab \in D$, then $abu = abv$, $u \neq v$, and therefore either $a \in D$ or $b \in D$.

(iv) We have $ax = axa$ and $x = xa$ for all $a \in C$ and $x \in S$. The rest is clear by (i). □

I.2. Examples of left distributive semigroups

2.1 Example. There are (up to isomorphism) precisely four two-element LD-semigrops. They are:

$$D(1), D(2), D(3), D(4)$$

(see A1.IV.4). The first three of them are idempotent; the last one is not.

2.2 Example. There are (up to isomorphism) precisely sixteen three-element LD-semigroups. They are:

$$D(7), \ldots, D(14), D(20), D(24), \ldots, D(28), D(36), D(46)$$

(see A1.IV.10). All of them, except $D(20)$ and $D(28)$, are distributive. The idempotent ones are $D(7), \ldots, D(14)$ and $D(20)$.

2.3 Example. The following table shows the numbers of isomorphism types of at most five-element LD-semigroups and LDI-semigroups:

	1	2	3	4	5
LDS	1	4	16	93	682
LDIS	1	3	9	38	179

2.4 Example. Consider the following five-element groupoid S:

S	0	1	2	3	4
0	1	1	3	4	4
1	1	1	4	4	4
2	2	2	2	2	2
3	3	3	3	3	3
4	4	4	4	4	4

This groupoid is an LDR_1-saemigroup; it is not an LDT-semigroup and it does not satisfy the identity $xyx \approx x^2yx$.

2.5 Example. Consider the following four-element groupoid S:

S	0	1	2	3
0	2	3	2	2
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3

This groupoid is an LDR_1-semigroup; it is not an LDT-semigroup; it is subdirectly irreducible and satisfies $x^2 \approx x^2y$.

2.6 Example. Consider the following two three-element LD-semigroups:

$D(20)$	0	1	2
0	0	0	0
1	1	1	1
2	0	1	2

$D(28)$	0	1	2
0	0	0	0
1	0	1	2
2	0	0	0

$D(20)$ is an idempotent LDR_1-semigroup; it is not medial. $D(28)$ is an LDT_1-semigroup; it is medial and satisfies $xy^2 \approx yx^2$. Moreover, $\mathrm{Id}(D(28))$ is not an ideal and $D(28)$ is not an LDR-semigroup.

2.7 Example. Let f be a transformation of a nonempty set S and define multuplication on S by $xy = f(y)$ for all $x, y \in S$. Then S becomes a D-semigroup.

2.8 Proposition. *Let S be an LD-semigroup and $e \notin S$. Then:*

(i) *$S[e]$ is an LD-semigroup.*
(ii) *$S\{e]$ is an LD-semigroup.*
(iii) *$S[e\}$ is an LD-semigroup iff S is an LZ-semigroup.*
(iv) *$S\{e\}$ is an LD-semigroup iff S is an idempotent LDR_l-semigroup.*

Proof. Easy (see A1.IV.1.9). □

2.9 Proposition. *Let S be a D-semigroup and $e \notin S$. Then:*

(i) *$S[e]$ is a D-semigroup.*
(ii) *$S\{e]$ (resp. $S[e\}$) is a D-semigroup iff S is an RZ-semigroup (resp. LZ-semigroup).*
(iii) *$S\{e\}$ is a D-semigroup iff S is a semilattice.*

Proof. Use 2.8. □

I.3 Basic facts on subdirectly irreducible left distributive semigroups

3.1 Proposition. *Let S be a subdirectly irreducible LD-semigroup. Then just one of the following two cases takes place:*

(i) $\mathscr{C}_l(S) \neq \emptyset$, $q_S = \mathrm{id}_S$ *and S is an LDT$_l$-semigroup.*

(ii) $\mathscr{C}_l(S) = \emptyset$ *and* $q_S \neq \mathrm{id}_S$.

Proof. Suppose first $\mathscr{C}_l(S) = \emptyset$. Then, for every $x \in S$, L_x is not injective, so that $\omega_S \subseteq q_{x,S}$; but then $\omega_S \subseteq q_S$. On the other hand, if $\mathscr{C}_l(S) \neq \emptyset$, then (i) is true by 1.22(ii). □

3.2 Proposition. *Let S be a subdirectly irreducible LD-semigroup such that $C = \mathscr{C}_l(S) \neq \emptyset$; put $D = S - C$. Then just one of the following five cases takes place:*

(i) $S \simeq D(1)$.

(ii) $S \simeq D(2)$.

(iii) $S \simeq D(10)$.

(iv) *S is neither idempotent nor an LDR-semigroup and* $\mathrm{card}(D) \geq 2$ *(then* $p_S \neq \mathrm{id}_S$.*)*

(v) *S is an idempotent LDR$_l$-semigroup,* $\mathrm{card}(D) \geq 2$, $p_S = \mathrm{id}_S$, $C = \{e\}$ *for a neutral element e of S, D is subdirectly irreducible and* $p_D = \mathrm{id}_D \neq q_D$.

Proof. By 3.1, $q_S = \mathrm{id}_S$ and S is an LDT$_1$-semigroup. By 1.22, either $D = \emptyset$ or D is a prime ideal of S. Let $(a, b) \in \omega_S$, $a \neq b$. Obviously, $D = \{x \in S : xa = xb\}$. If $D = \emptyset$, then S is a RZ-semigroup by 1.22(ii) and one can readily see that $S \simeq D(2)$ in that case.

Next assume that $D = \{0\}$ is a singleton. Then 0 is an absorbing element of S, C is an RZ-semigroup and it is easy to see that $s \cup \mathrm{id}_S$ is a congruence of S for any congruence s of C. If $\mathrm{card}(C) = 1$, then $S \simeq D(1)$. If $\mathrm{card}(C) \geq 2$, then $a, b \in C$, $C \simeq D(2)$ and $S \simeq D(10)$.

Finally, assume that $\mathrm{card}(D) \geq 2$. Since D is an ideal, $\equiv_D$ is a congruence of S and thus a, b both belong to D. Then $aa = ab$ and $ba = bb$.

Let $p_S \neq \mathrm{id}_S$. Then $(a, b) \in p_S$, $ab = bb$, and therefore $aa = bb$. It follows that either $aa \neq a$ or $bb \neq b$ and we see that S is not idempotent. Suppose that S is an LDR-semigroup. Then $\mathrm{Id}(S)$ is an ideal and, since either $a \notin \mathrm{Id}(S)$ or $b \notin \mathrm{Id}(S)$, we must have $\mathrm{card}(\mathrm{Id}(S)) = 1$ by the subdirect irreducibility. Then by 1.21, S is an A-semigroup and thus $C = \emptyset$, a contradiction.

Let $p_S = \mathrm{id}_S$. Then, by 1.8, S is an LDR$_1$-semigroup; S is idempotent by 1.22(ii) and 1.17(iii). Thhe rest is clear from 1.22(iv). □

3.3 Proposition. *Let S be a subdirectly irreducible delightful LD-semigroup (see 1.16). Then just one of the following four cases takes place:*

(i) $S \simeq D(2)$.

(ii) $S \simeq D(10)$.

(iii) *S is an idempotent* LDR_1*-semigroup with* $p_S = \mathrm{id}_S$.
(iv) *S is an A-semigroup.*

Proof. With respect to 1.16(iii) and 1.17(iii), we can assume that S is idempotent. Further, with respect to 3.1 and 3.2, we can assume that $q_S \neq \mathrm{id}_S$. Let $(a,b) \in \omega_S$, $a \neq b$. We have $(a,b) \in q_S$, so that $a = aa = ab$ abd $b = bb = ba$. Thus $ab \neq ba$ and $(a,b) \notin p_S$. But then $p_S = \mathrm{id}_S$ and S is an LDR_1-semigroup by 1.8(ii). □

3.4 Proposition. *Let S be a subdirectly irreducible D-semigroup. Then just one of the following two cases takes place:*

(i) *S is idempotent and S is isomorphic to one of the five distributive semigroups* $D(1)$, $D(2)$, $D(3)$, $D(9)$ *and* $D(10)$.
(ii) *S is an A-semigroup.*

Proof. With respect to 3.3, we can assume that S is an idempotent LDR_1-semigroup, i.e., S satisfies $xy \approx xyx$. Dually, using the right hand form of 3.3, we can assume that S satisfies $xy \approx yxy$. However, then S is commutative, i.e., it is a semilattice. A subdirectly irreducible semilattice is isomorphic to $D(1)$. □

3.5 Remark. Let S be a subdirectly irreducible LD-semigroup. We have either $t_S \neq \mathrm{id}_S$ or $t_S = \mathrm{id}_S$.

If $t_S \neq \mathrm{id}_S$, then $t_S = \omega_S = \{(a,b),(b,a)\}$ for some $a, b \in S$, $a \neq b$. Then $a^2 = = ab = ba = b^2$, and so either $a \notin \mathrm{Id}(S)$ or $b \notin \mathrm{Id}(S)$.

If $t = \mathrm{id}_S$, then either $p_S = \mathrm{id}_S$ and S is an LDR_1-semigroup, or else $q_S = \mathrm{id}_S$ and S is an LDT_1-semigroup. In the latter case, 3.2 applies.

3.6 Proposition. *The groupoids* $D(1)$, $D(2)$, $D(3)$ *and* $D(4)$ *are (up to isomorphism) the only (congruence) simple LD-semigroups.*

Proof. The result follows easily from A1.II.7.4. □

I.4 Comments and open problems

The results of this section are of introductory character and are based on the paper [Kep, 81]. The main open problems concern a more detailed description of subdirectly irreducible LD-semigroups. In particular, their subsemigroups are not known (cf III.4 and IV.3,4,5).

II. Free left distributive semigroups

II.1 Construction of free left distributive semigroups

1.1 Construction. Let X be a nonempty set. Denote by $\mathbf{F}$ the (absolutely) free semigroup over X. Denote by F the union of the following four pairwise disjoint subset A, B, C, D of $\mathbf{F}$:

$A = \{x^i : x \in X,\ 1 \le i \le 3\}$
$B = \{x^i y^j : x, y \in X,\ x \ne y,\ 1 \le i, j \le 2\}$
$C = \{x_1^i x_2 \dots x_{n-1} x_n^j : x_1, \dots, x_n \in X$ pairwise different, $n \ge 3,\ 1 \le i, j \le 2\}$
$D = \{x_1^i x_2 \dots x_{n-1} x_n x_k : x_1, \dots, x_n \in X$ pairwise different, $n \ge 2,\ 1 \le k < n,$
$1 \le i \le 2\}$

For every element u of **F**, (uniquely) expressed as $u = x_1^{k_n} \dots x_n^{k_n}$ where $n \ge 1$, $x_i \in X$, $k_i \ge 1$ and $x_1 \ne x_2 \ne \dots \ne x_n$, we define an element $f(u)$ of F as follows:

(i) If $n = 1$, let $f(u) = x_1^k$ where $k = \min(3, k_1)$.

(ii) If $n = 2$, let $f(u) = x_1^k x_2^l$ where $k = \min(2, k_1)$ and $l = \min(2, k_2)$.

(iii) If $n \ge 3$ and $x_n \notin \{x_1, \dots, x_{n-1}\}$, let $f(u) = x_1^k y_1 \dots y_m x_n^l$ where $k = \min(2, k_1)$, $l = \min(2, k_n)$ and (by induction on i) y_i is the first member of $x_1, \dots, x_{n-1}$ not contained in $\{x_1, y_1, \dots y_{i-1}\}$.

(iv) If $n \ge 3$ and $x_n \in \{x_1, \dots, x_{n-2}\}$, let $f(u) = x_1^k y_1 \dots y_m x_n$ where $k = \min(2, k_1)$ and (by induction on i) y_i is the first member of $x_1, \dots, x_{n-1}$ not contained in $\{x_1, y_1, \dots, y_{i-1}\}$.

It is easy to see that $f(u) \in F$ in any case. Also, it is easy to see that $f(u) = u$ for $u \in F$. Let us define a binary operation $*$ on F in this way: $u * v = f(uv)$ for any $u, v \in F$. We are going to prove that $F(*)$ is a free LD-semigroup over X.

1.2 Lemma. *Let $u \in$ **F**. The identity $u \approx f(u)$ is satisfied in any LD-semigroup.*

Proof. It is easy; use I.1.1, I.1.2 and, of course, the left distributive law. □

1.3 Lemma. *Let $u, v \in F$ and $u \ne v$. Thhen there is an LD-semigroup not satisfying $u \approx v$.*

Proof. Suppose that $u \approx v$ is satisfied in all LD-semigroups. Since every LZ-semigroup is left distributive, the words u, v have the same first letters. Similarly, every RZ-semigroup is left distributive and hence u, v have the same last letters. Furthermore, every semilattice is distributive and we conclude that the set of letters occurring in u coincides with the set of letters occurring in v. Now, we distinguish the following cases.

Case 1: $u = x^i$ and $v = x^j$. The LD-semigroup $D(28)$ (see I.2.6) satisfies neither $x \approx x^2$ nor $x \approx x^3$. The LD-semigroup $D(46)$ (see A1.IV.8.1) does not satisfy $x^2 \approx x^3$. Using these observations, we conclude that $i = j$. Hence $u = v$, a contradiction.

Case 2: $u = x^i y^j$ and $v = x^k y^l$. The LD-semigroup S from I.2.4 satisfies none of the identities $xy \approx x^2 y$, $xy \approx x^2 y^2$, $xy^2 \approx x^2 y^2$ and $xy^2 \approx x^2 y$. The LD-semigroup $D(28)$ satisfies neither $xy \approx xy^2$ nor $x^2 y \approx x^2 y^2$. Consequently, $i = k$, $j = l$ and $u = v$, a contradiction.

Case 3: $u = x_1^i x_2 \dots x_{n-1} x_n^j \in C$ and $v = x_{p(1)}^k x_{p(2)} \dots x_{p(n-1)} x_{p(n)}^l \in C$ for a permutation p of $\{1, \dots, n\}$ with $p(1) = 1$ and $p(n) = n$. If $n \ge 4$, then every idempotent LD-semigroup satisfying $u \approx v$ is medial. However, $D(20)$ (see I.2.6) is a nonmedial LDI-semigroup. Consequently, $n = 3$. It is easy to see that either $xy^2 \approx x^2y^2$ or $x^2y \approx x^2y^2$ is a consequently of $u \approx v$, and we get a contradiction by Case 2.

Case 4: $u = x_1^i x_2 \dots x_{n-1} x_n^j \in C$ and $v = x_{p(1)}^k x_{p(2)} \dots x_{p(n-1)} x_{p(n)} x_{p(k)} \in D$ for a permutation p of $\{1, \dots, n\}$ with $p(1) = 1$ and $p(k) = n$. One can easily check that every LDI-semigroup satisfying $u \approx v$ is distributive. However, $D(20)$ is not distributive, a contradiction.

Case 5: $u = x_1^i x_2 \dots x_{n-1} x_n x_k \in D$ and $v = x_{p(1)}^j x_{p(2)} \dots x_{p(n-1)} x_{p(n)} x_{p(l)} \in D$ for a permutation p of $\{1, \dots, n\}$ with $p(1) = 1$ and $p(l) = k$. Since $D(20)$ is not middle semimedial, we have $p(2) = 2, \dots, p(n) = n$. However, the LD-semigroup from I.2.4 does not satisfy $xyx \approx x^2yx$. Thus $i = j$ and $u = v$, a contradiction. □

1.4 Theorem. *For a nonempty set X, the groupoid $F(*)$ constructed in 1.1 is a free LD-semigroup over X.*

Proof. Denote by $\sim$ the set of the ordered pairs (u, v) of elements of $\mathbf{F}$ such that the equation $u \approx v$ is satisfied in all LD-semigroups. So, $\sim$ is a (fully invariant) congruence of $\mathbf{F}$ and $\mathbf{F}/\sim$ is a free LD-semigroup over X. We know (by 1.2) that $f(u) \sim u$ for any $u \in \mathbf{F}$, so that (by 1.3) $u \sim v$ iff $f(u) = f(v)$ for any $u, v \in \mathbf{F}$ and $\sim$ is just the kernel of f. Now, f is a homomorphism of $\mathbf{F}$ onto $F(*)$: if $u, v \in \mathbf{F}$, then both $f(uv)$ and $f(u) * f(v)$ belong to F and are congruent modulo $\sim$ with uv. The result follows from the homomorphism theorem. (In particular, the operation $*$ is associative; this is not immediate from the definition.) □

1.5 Corollary. *Every finitely generated LD-semigroup is finite. The variety of LD-semigroups is locally finite.* □

1.6 Remark. Proceeding similarly, one can construct free LDI-semigroups. In that case we get words of two types only: words of the form $x_1 \dots x_n$ for $n \geq 1$ and words of the form $x_1 x_2 \dots x_n x_k$ for $n \geq 2$ and $1 \leq k < n$, where (in both cases) $x_1, \dots, x_n$ are pairwise distinct letters.

1.7 Remark. By I.1.20, every D-semigroup is a medial LDRT-semigroup. The words in a free D-semigroup are of the following types only: x, x^2, x^3, xy, x^2y, xyx, $x_1 x_2 \dots x_m$ and $x_1 x_2 \dots x_m x_1$ $(m \geq 3)$. Of course,

$$x_1 \dots x_m \sim x_1 x_{p(2)} \dots x_{p(m-1)} x_m \text{ and } x_1 x_2 \dots x_m x_1 \sim x_1 x_{q(2)} \dots x_{q(m)} x_1$$

for any permutation p of $\{x_2, \dots, x_{m-1}\}$ and any permutation q of $\{x_2, \dots, x_m\}$.

II.2 Auxiliary results on number-theoretic functions

2.1 Definition. Put
(i) $a(n, m) = n(n - 1) \dots (n - m)$,
(ii) $a(n) = \sum_{m=0}^{n} a(n, m)$,

(iii) $b(n) = \sum_{m=0}^{n} ma(n,m)$
for all nonnegative integers n, m.

2.2 Lemma. *Let $n, m \geq 0$. Then:*
(i) $a(n+1, m+1) = (n+1)a(n,m)$.
(ii) $a(n+1) = (n+1)(a(n)+1)$.
(iii) $b(n+1) = (n+1)(a(n)+b(n))$.
(iv) $b(n) = (n-2)a(n) + n$.

Proof. By induction on n. □

2.3 Lemma. *For every $n \geq 1$, $a(n) + c(n) + 1 = n!\mathbf{e}$, where $(n+1)^{-1} < c(n) < n^{-1}$ and $\mathbf{e} = \sum_{k=1}^{\infty} 1/(k!)$.*

Proof. Indeed, $n!\mathbf{e} - 1 = 2n! + 3\cdot 4\cdot \ldots \cdot n + 4\cdot 5\cdot \ldots \cdot n + \ldots + (n-1)n + n + c(n) = a(n) + c(n)$, where $c(n) = 1/(n+1) + 1/(n+1)(n+2) + 1/(n+1)(n+2)(n+3) + \ldots$. Clearly, $1/(n+1) < c(n) < 1/n$. □

2.4 Lemma. *For every $n \geq 1$, $na(n) = [nn!\mathbf{e}] - n$ (here, for a positive real number r, $[r]$ means the entire part of r).*

Proof. By 2.3, $na(n) = [nn!\mathbf{e}] - n - nc(n) + u$, where $0 < u < 1$. Then $-1 < u - nc(n) < (n+1)^{-1}$ and, since $u - nc(n)$ is a whole number, we must have $u - nc(n) = 0$. □

II.3 The number of elements of a free left distributive semigroup

3.1 Theorem. *The cardinality $f_1(n)$ of the free LD-semigroup of rank n and the cadinality $f_2(n)$ of the free LDI-semigroup of rank n are given by*

$$f_1(n) = 2[n!n\mathbf{e}] - n,$$
$$f_2(n) = [n!(n-1)\mathbf{e}] + 1.$$

Proof. By 1.4, 2.1 and 2.2 we have $f_1(n) = 4a(n) + 2b(n) - n = n + 2na(n)$. In order to compute $f_1(n)$, it remains to use 2.4. The other formula is clear from 1.6. □

3.2 Remark.
(i) $f_1(n) = \varepsilon(n)(n+1)!$, where $\varepsilon(n) \to 2\mathbf{e}$. Moreover, $f_1(n)/f_2(n) \to 2$.
(ii) Let S be a finitely generated LD-semigroup and $n = \sigma(S)$ (see A1.I.1.5). If $n = 0$, then $\mathrm{card}(S) = 1$. If $n \geq 1$, then

$$n \leq \mathrm{card}(S) \leq 2[n!n\mathbf{e}] - n.$$

3.3 Remark.
(i) The cardinality $f_3(n)$ of the free idempotent LDR_1-semigroup of rank n is given by

$$f_3(n) = [n!\mathbf{e}] - 1.$$

(ii) The cardinality $f_4(n)$ of the free DI-semigroup of rank n is given by

$$f_4(n) = n(n+1)2^{n-2}.$$

(iii) The cardinality $f_5(n)$ (resp. $f_6(n)$) of the free LDI-semigroup satisfying $xyz \approx xzy$ (resp. $xyz \approx yxz$) of rank n is given by

$$f_5(n) = f_6(n) = n2^{n-1}.$$

(iv) The cardinality $f_7(n)$ of the free semilattice of rank n is given by

$$f_7(n) = 2^n - 1.$$

(v) The cardinality $f_8(n)$ of the free idempotent semigroup satisfying $x \approx xyx$ of rank n is given by

$$f_8(n) = n^2.$$

(vi) The cardinality $f_9(n)$ (resp. $f_{10}(n)$) of the free LZ-semigroup (resp. RZ-semigroup) of rank n is given by

$$f_9(n) = f_{10}(n) = n.$$

3.4 Remark. Denote by $f_{11}(n)$ the cardinality of the free D-semigroup of rank n. According to 1.7, $f_{11}(n) = 3n + 2n(n-1) + n(n-1)\left(\binom{n-2}{1} + \dots + \binom{n-2}{n-2}\right) + n\left(\binom{n-1}{1} + \dots + \binom{n-1}{n-1}\right)$. After easy calculation, we find that

$$f_{11}(n) = n(n+1)(1 + 2^{n-2}).$$

3.5 Remark. Denote by $f_{12}(n)$ (resp. $f_{13}(n)$, $f_{14}(n)$, $f_{15}(n)$, $f_{16}(n)$ the cardinality of the free A-semigroup (resp. free unipotent A-semigroup, free commutative A-semigroup, free unipotent commutative A-semigroup, free Z-semigroup) of rank n. Then

$$\begin{aligned} f_{12}(n) &= n^2 + n + 1, \\ f_{13}(n) &= n^2 + 1 \\ f_{14}(n) &= (n^2 + 3n + 2)/2, \\ f_{15}(n) &= (n^2 + n + 2)/2, \\ f_{16}(n) &= n + 1. \end{aligned}$$

3.6 Table.

	1	2	3	4	5	6	7	8
$f_1(n)$	3	18	93	516	3255	23478	191793	1753608
$f_2(n)$	1	6	33	196	1305	9786	82201	762208
$f_3(n)$	1	4	15	64	325	1956	13694	109600
$f_4(n)$	1	6	24	80	240	672	1792	4608
$f_{5,6}(n)$	1	4	12	32	80	192	448	1024
$f_8(n)$	1	4	9	16	25	36	49	64

	1	2	3	4	5	6	7	8
$f_7(n)$	1	3	7	15	31	63	127	255
$f_{9,10}(n)$	1	2	3	4	5	6	7	8
$f_{11}(n)$	3	12	36	100	270	714	1848	4680
$f_{12}(n)$	3	7	13	21	31	43	57	73
$f_{13}(n)$	2	5	10	17	26	37	50	65
$f_{14}(n)$	3	6	10	15	21	28	36	45
$f_{15}(n)$	2	4	7	11	16	22	29	37
$f_{16}(n)$	2	3	4	5	6	7	8	9

II.4 Comments and open problems

The description 1.4 of free LD-semigroups is taken from [Mar,79] and [Zej,89b]. The numbers of elements of finitely generated free LD-semigroups (3.1) were computed in [KepZ,89].

An open problem is a characterization of subsemigroups of free LD-semigroups (LDI-semigroups, etc.).

III. A-semigroups and their varieties

III.1 Basic properties of A-semigroups

1.1. An A-semigroup is a groupoid satisfying $x \cdot yz \approx uv \cdot w$. It is apparent that A-semigroups are nothing else than semigroups nilpotent of class at most 3. Thus every A-semigroup S contains an absorbing element 0 $(= 0_S)$ such that $xyz = 0$ for all $x, y, z \in S$.

1.2 Proposition. *Let S be an A-semigroup and $Z(S) = \{a \in S : Sa = 0 = aS\}$. Then:*

(i) *$0, S^2$ and $Z(S)$ are ideals of S.*
(ii) $\mathrm{Id}(S) = \mathrm{Int}(S) = \{0\} = S^3 \subseteq S^2 \subseteq Z(S) \subseteq S$.
(iii) *$S^2, Z(S), S/S^2$ and $S/Z(S)$ are Z-semigroups.*
(iv) $Z(S) \times Z(S) \subseteq t_S$.
(v) $\sigma(S) = \mathrm{card}(S - S^2)$.

Proof. Easy. □

III.2 Varieties of A-semigroups

2.1 Notation. Denote by $\mathscr{A}_0$ the variety of trivial groupoids, by $\mathscr{A}_1$ the variety of Z-semigroups, by $\mathscr{A}_2$ the variety of commutative unipotent A-semigroups, by

$\mathscr{A}_3$ the variety of commutative A-semigroups, by $\mathscr{A}_4$ the variety of unipotent A-semigroups and by $\mathscr{A} = \mathscr{A}_5$ the variety of A-semigroups.

2.2 Theorem. *The varieties $\mathscr{A}_0$, $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{A}_3$, $\mathscr{A}_4$, and $\mathscr{A}_5$ are pairwise different varieties of A-semigroups and there are no other varieties of A-semigroups. We have*

$$\mathscr{A}_0 \subset \mathscr{A}_1 \subset \mathscr{A}_2 \subset \mathscr{A}_3 \subset \mathscr{A}_5, \qquad \mathscr{A}_2 \subset \mathscr{A}_4 \subset \mathscr{A}_5$$

and there are no other inclusions except those which follow by transitivity. The lattice of varieties of A-semigroups is given in Fig. 1.

Proof. Let V by a variety of A-semigroups determined by an identity $u \approx v$, where u, v are two semigroup words of lengths k and l, respectively. If $k \geq 3$ and $l \geq 3$, then $V = \mathscr{A}_5$. If $k \geq 3$ and $l = 2$, then V is either $\mathscr{A}_4$ or $\mathscr{A}_1$. If $k \geq 3$ and $l = 1$, then $V = \mathscr{A}_0$. If $k = l = 2$, then V is either $\mathscr{A}_5$ or $\mathscr{A}_4$ or $\mathscr{A}_3$ or $\mathscr{A}_1$. If $k = 2$ and $l = 1$, then $V = \mathscr{A}_0$. Finally, if $k = l = 1$, then V is either $\mathscr{A}_5$ or $\mathscr{A}_0$. Hence every one-based variety of A-semigroups can be found among $\mathscr{A}_0, \ldots, \mathscr{A}_5$. Since this collection is closed under intersection (we have $\mathscr{A}_3 \cap \mathscr{A}_4 = \mathscr{A}_2$), it follows that there are no other subvarieties of $\mathscr{A}$.

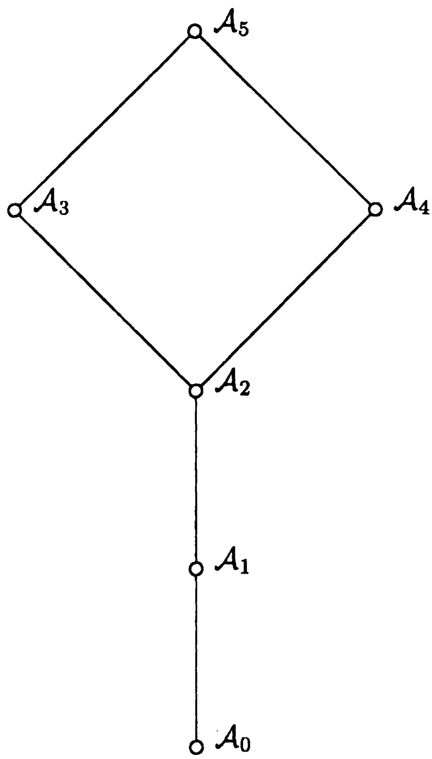

Fig. 1

All the inclusions are clear. The groupoid T given by

T	0	1	2	3
0	0	0	0	0
1	0	0	3	0
2	0	3	0	0
3	0	0	0	0

is in $\mathscr{A}_2$ but not in $\mathscr{A}_1$. The groupoid $D(46)$ see A1.IV.8.1) is in $\mathscr{A}_3$ but not in $\mathscr{A}_4$, and the groupoid S given by

S	0	1	2	3	4
0	0	0	0	0	0
1	0	0	3	0	0
2	0	4	0	0	0
3	0	0	0	0	0
4	0	0	0	0	0

is in $\mathscr{A}_4$ but not in $\mathscr{A}_3$. □

III.3 Free A-semigroups

3.1 Construction. Let X be a nonempty set and let $f: X \times X \rightarrow Y$ be a bijective mapping, where $X \cap Y = \emptyset$. Let 0 be an element not belonging to $X \cup Y$. Define a multiplication on $F = X \cup Y \cup \{0\}$ by $xy = f(x, y)$ for $x, y \in X$ and $xy = 0$ otherwise. Then F becomes a free A-semigroup over the set X.

3.2 Proposition. *An A-semigroup S is a free A-semigroup if and only if it satisfies the following four conditions:*

(i) *S is nontrivial;*
(ii) *If $x, y, u, v \in S$ are such that $xy = uv \neq 0$, then $x = u$ and $y = v$;*
(iii) *If $x, y \in S - Z(S)$, then $xy \neq 0$;*
(iv) *$Z(S) = S^2$.*

Proof. Easy. □

3.3 Proposition. *An A-semigroup S is a subsemigroup of a free A-semigroup if and only if it satisfies the conditions 3.2(ii) and 3.2(iii).*

Proof. The direct implication is clear from 3.2 (if $S \subseteq F$, then $S - Z(S) \subseteq$ $\subseteq F - Z(F)$). Now, assume that S satisfies both 3.2(ii) and 3.2(iii) and put $A = S - Z(S)$ and $B = Z(S) - S^2$. It follows from 3.2(iii) that $S = A \cup B \cup$ $\cup A^2 \cup \{0\}$ is a disjoint union. Further, let C be a set such that $C \cap S = \emptyset$ and $\operatorname{card}(C) = \operatorname{card}(B)$, and let $g: B \rightarrow C$ be a bijection. Put $X = A \cup C$ and define a mapping $h: S \to F$ (where F is as in 3.1) as follows: $h(a) = a$ for every $a \in A$; $h(b) = g(b)^2$ for every $b \in B$; $h(xy) = xy$ for all $x, y \in A$; $h(0) = 0$. It follows from 3.2(ii) that h is well defined and, by 3.2(iii), h is an injective homomorphism of S onto the free A-semigroup F. □

3.4 Corollary. *Every Z-semigroup is a subsemigroup of a free A-semigroup.* □

3.5 Remark. The A-semigroup T from the proof of 2.2 is not a subsemigroup of any free A-semigroup.

3.6 Remark. The number of elements of a free semigroup in any subvariety of $\mathscr{A}$ has been computed in II.3.5.

III.4 Subdirectly irreducible A-semigroups

4.1 Proposition. *Let S be an A-semigroup containing at least three elements. Then S is subdirectly irreducible if and only if the subsemigroup $T = S^2$ contains precisely two elemnts and $t_S = (T \times T) \cup \mathrm{id}_S$.*

Proof. Let S be subdirectly irreducible. As one can see easily, every subdirectly irreducible Z-semigroup contains only two elements. Consequently, S is not a Z-semigroup and $\mathrm{card}(T) \geq 2$. On the other hand, every nonempty subset M of T is an ideal of S, $(M \times M) \cup \mathrm{id}_S$ is a congruence, and it follows easily that $\mathrm{card}(T) = 2$ and $\omega_S = (T \times T) \cup \mathrm{id}_S$. Clearly, $\omega_S \subseteq t_S$. Conversely, if $(a, b) \in t_S$ and $a \neq b$, then $(\{a,b\} \times \{a,b\}) \cup \mathrm{id}_S$ is a congruence of S. Thus $\omega_S = t_S = (T \times T) \cup \mathrm{id}_S$.

Now assume that $T = \{0,a\}$ where $a \neq 0$, and that $t_S = (T \times T) \cup \mathrm{id}_S$. Let $r \neq \mathrm{id}_S$ be a congruence of S and let $(x, y) \in r$, $x \neq y$. If $xy \neq yz$ for some $z \in S$, then the elements xz and yz belong to T and we see that $(a, 0) \in r$. Similarly, $zx \neq zy$ implies $(a, 0) \in r$. If $xz = yz$ and $zx = zy$ for all $z \in S$, then $(x, y) \in t_S = (T \times T) \cup \mathrm{id}_S$. This proves $(a, 0) \in r$ in any case, so that S is subdirectly irreducible. □

4.2 Corollary. *Let S be a subdirectly irreducible A-semigroup containing at least three elements. Then $Z(S) = S^2$, $\omega_S = t_S$, $\sigma(S) = \mathrm{card}(S) - 2$ and every proper homomorphic image of S is a Z-semigroup.* □

4.3 Theorem. *An A-semigroup S is a subsemigroup of a subdirectly irreducible A-semigroup if and only if S^2 contains at most two elements.*

Proof. The direct implication follows from 4.1. Let S be an A-semigroup such that $S^2 \subseteq \{0,1\}$, where 0 is the absorbing element of S (and 1 is some other element); let S be not subdirectly irreducible. Put $K = S - \{0,1\}$. Let f be a bijection of K onto a set M with $S \cap M = \emptyset$. Put $G = S \cup M$ and define multiplication on G in the following way:

(i) S is a subsemigroup of G;
(ii) $x \cdot f(x) = f(x) \cdot x = 1$ and $f(x) \cdot f(x) = 0$ for all $x \in K$;
(iii) $f(x) \cdot y = y \cdot f(x) = 0$ and $f(x) \cdot f(y) = 1$ for all $x, y \in K$, $x \neq y$;
(iv) $z \cdot 0 = 0 \cdot z = z \cdot 1 = 1 \cdot z = 0$ for all $z \in G$.

It is easy to check that G is an A-semigroup. Of course, S is a subsemigroup of G. We have $G^2 = \{0,1\}$, so that, according to 4.1, it remains to show that $t_G = (\{a,b\} \times \{a,b\}) \cup \mathrm{id}_G$.

Let $(a, b) \in t_G$, $a \neq b$. We are going to show that $a, b \in \{0,1\}$. If $a, b \in M$, then $0 = aa = ab = 1$, a contradiction. Therefore, we can assume that $a \in S$.

Suppose $a \in K$. If $b \notin M$, then $1 = a \cdot f(a) = b \cdot f(a) = 0$, a contradiction. Thus $b \in M$ and we have $b = f(c)$ for some $c \in K$. If there exists an element d of K different from both a and c, then $0 = a \cdot f(d) = b \cdot f(d) = 1$, a contradiction. Thus $K = \{a,c\}$. If $a = c$, then $b = f(a)$ and $1 = a \cdot f(a) = b \cdot f(a) = 0$, a contradiction. If $ac = 0$, then $0 = ac = bc = 1$, which is not true; if $ca = 0$, we get a contradiction similarly. Thus $ac = 1 = ca$. Similarly $aa = 0$, and S is subdirectly irreducible by 4.1, a contradiction.

This proves that $a \in \{0,1\}$. In this case, $xb = 0 = bx$ for every $x \in G$ and $b \in \{0,1\}$. The rest is clear. □

4.4 Corollary. *Every Z-semigroup is a subsemigroup of a (commutative and unipotent) subdirectly irreducible A-semigroup.* □

4.5 Remark. The subdirectly irreducible A-semigroup G constructed in the proof of 4.3 is commutative (resp. unipotent), provided that S is commutative (resp. unipotent). Hence, the analogue of 4.3 remains true for commutative (resp. unipotent) A-semigroups.

III.5 Comments

The theory of A-semigroups (i.e., semigroups nilpotent of class at most 3) is more or less of folklore character. Anyway, the results presented here are taken from [JezKN,81].

IV. Idempotent left distributive semigroups and their varieties

IV.1 Basic properties of idempotent left distributive semigroups

1.1 Proposition. *The following conditions are equivalent for an idempotent semigroup S:*
(i) *S is middle semimedial.*
(ii) *S is medial.*
(iii) *S is distributive.*

Proof. (i) implies (ii). We have $abcd = abcd \cdot abcd = a \cdot b \cdot cd \cdot a \cdot bcd = = a \cdot cd \cdot b \cdot a \cdot bcd = a \cdot c \cdot d \cdot bab \cdot c \cdot d = a \cdot c \cdot bab \cdot d \cdot c \cdot d = a \cdot c \cdot ba \cdot bd \cdot \cdot c \cdot d = a \cdot c \cdot bd \cdot ba \cdot c \cdot d = acb \cdot d \cdot b \cdot ac \cdot d = acb \cdot d \cdot ac \cdot b \cdot d = acbd \cdot acbd = = acbd$ for all $a, b, c, d \in S$.

(ii) implies (iii). We have $abc = aabc = abac$ and $cba = cbaa = caba$ for all $a, b, c \in S$.

(iii) implies (i). We have $abca = abcba = acba$ for all $a, b, c \in S$. □

1.2 Proposition. *The five pairwise nonisomorphic DI-semigroups $D(1)$, $D(2)$, $D(3)$, $D(9)$ and $D(10)$ are (up to isomorphism) the only subdirectly irreducible DI-semigroups. Moreover, $D(9)$ is right but not left permutable and $D(10)$ is left but not right permutable.*

Proof. See I.3.4. □

1.3 Proposition. *Let S be a rectangular band, i.e., an idempotent semigroup satisfying the identity $x \approx xyx$. Then:*

(i) *S is a DI-semigroup.*

(ii) *S/p_S is an LZ-semigroup and S/q_S is an RZ-semigroup.*

(iii) *$S \simeq S/p_S \times S/q_S$.*

Proof. (i) We have $abcd = aca \cdot bcd = a \cdot cabc \cdot d = acd = a \cdot cbc \cdot d = ac \cdot \cdot bdb \cdot cd = acb \cdot dbcd = acbd$ for all $a, b, c, d \in S$. Thus S is medial, and hence distributive by 1.1.

(ii) By (i), $xy = xzxy = xzy$ for all $x, y, z \in S$ and it follows that $(y, zy) \in q_S$ and S/q_S in an RZ-semigroup. Quite similarly, S/p_S is an LZ-semigroup.

(iii) Since S is idempotent, we have $t_S = p_S \cap q_S = \mathrm{id}_S$. On the other hand, by (ii), $a/p = ab/p$ and $b/q = ab/q$ for all $a, b \in S$. □

1.4 Proposition. *Let S be a subdirectly irreducible LDI-semigroup. Then either S is a DI-semigroups (and so S is isomorphic to one of $D(1)$, $D(2)$, $D(3)$, $D(9)$, $D(10)$) or S is an idempotent LDR_1-semigroup such that $p_S = \mathrm{id}_S$.*

Proof. See I.3.3 and 1.2. □

IV.2 Varieties of idempotent LD-semigroups

2.1 Notation. Consider the following varieties of idempotent semigroups:

$\mathscr{I}_0$ … trivial semigroups;

$\mathscr{I}_1$ … semigroups satisfying $xy \approx x$;

$\mathscr{I}_2$ … semilattices;

$\mathscr{I}_3$ … semigroups satisfying $xy \approx y$;

$\mathscr{I}_4$ … left permutable idempotent semigroups;

$\mathscr{I}_5$ … rectangular bands (idempotent semigroups satisfying $x \approx xyx$);

$\mathscr{I}_6$ … right permutable idempotent semigroups;

$\mathscr{I}_7$ ... normal bands (idempotent medial semigroups or DI-semigroups, see 1.1);
$\mathscr{I}_8$ ... idempotent LDR_1-semigroups (idempotent semigroups satisfying $xy \approx$
$\approx xyx$);
$\mathscr{I}_9 = \mathscr{I}$ ... LDI-semigroups.

2.2 Theorem. *The ten pairwise different varieties* $\mathscr{I}_0, \ldots, \mathscr{I}_9$ *are just all subvarieties of the variety* $\mathscr{I}$ *of LDI-semigroups. We have*

$$\mathscr{I}_0 \subset \mathscr{I}_1 \subset \mathscr{I}_4 \subset \mathscr{I}_8 \subset \mathscr{I}_9, \quad \mathscr{I}_1 \subset \mathscr{I}_5 \subset \mathscr{I}_7, \quad \mathscr{I}_2 \subset \mathscr{I}_6 \subset \mathscr{I}_7,$$
$$\mathscr{I}_0 \subset \mathscr{I}_2 \subset \mathscr{I}_4 \subset \mathscr{I}_7 \subset \mathscr{I}_9, \quad \mathscr{I}_0 \subset \mathscr{I}_3 \subset \mathscr{I}_5, \quad \mathscr{I}_3 \subset \mathscr{I}_6$$

and there are no other inclusions (except those that follow by transitivity). The lattice of subvarieties of $\mathscr{I}$ *is given in Fig. 2.*

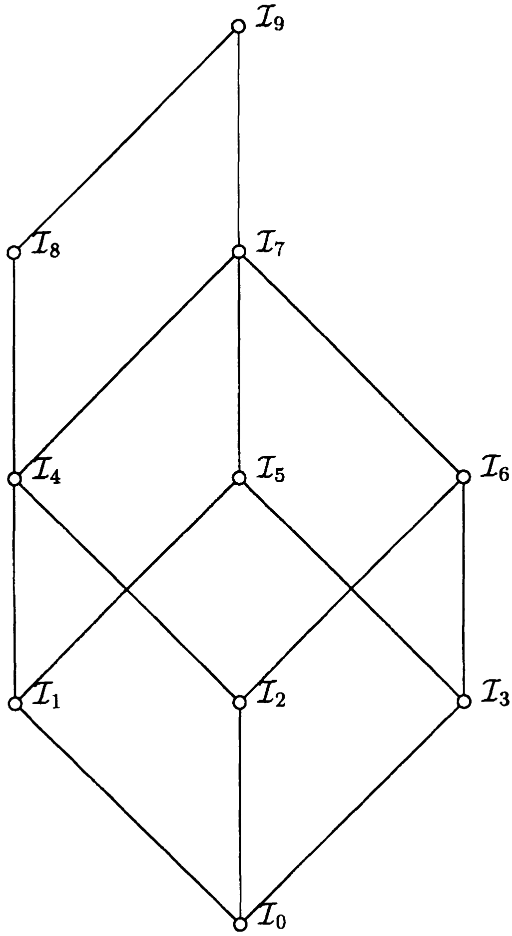

Fig. 2

Proof. All the non-sharp versions of the indicated inclusions are clear (use 1.1 and 1.3).

No nontrivial RZ-semigroup is in $\mathscr{I}_8$. Therefore, $\mathscr{I}_3 \nsubseteq \mathscr{I}_8$.

No nontrivial semilattice is in $\mathscr{I}_5$. Therefore, $\mathscr{I}_2 \nsubseteq \mathscr{I}_5$.

No nontrivial LZ-semigroup is in $\mathscr{I}_6$. Therefore, $\mathscr{I}_1 \nsubseteq \mathscr{I}_6$.

We have $D(20) \in \mathscr{I}_8 - \mathscr{I}_7$. This completes the inclusions part of the proof.

Now let V be a variety of LDI-semigroups determined (in $\mathscr{I}$) by a single indentity $u \approx v$.

Assume first that $V \subseteq \mathscr{I}_7$. The variety V is generated by its subdirectly irreducible members. Using 1.2, we easily conclude that V is one of the varieties $\mathscr{I}_0$, $\mathscr{I}_1$, $\mathscr{I}_2$, $\mathscr{I}_3$, $\mathscr{I}_4$, $\mathscr{I}_5$, $\mathscr{I}_6$, $\mathscr{I}_7$.

Let $V \subseteq \mathscr{I}_8$. We can restrict ourselves to the case when $u = x_1 \dots x_n$ and $v = y_1 \dots y_m$ where $x_1, \dots, x_n$ are pairwise different and also $y_1, \dots, y_m$ are pairwise different. If $\operatorname{var}(u) \neq \operatorname{var}(v)$, then $V \subseteq \mathscr{I}_5$ and, in fact, V is either $\mathscr{I}_0$ or $\mathscr{I}_1$. So, assume that $\operatorname{var}(u) = \operatorname{var}(v)$. Then $n = m$ and there is a permutation p of $\{1,2,\dots,n\}$ such that $y_i = x_{p(i)}$. If $p(1) \neq 1$, then V is either $\mathscr{I}_0$ or $\mathscr{I}_2$. Let $p(1) = 1$, $p \neq \mathrm{id}$, and let $2 \leq k \leq n - 1$ be the smallest number with $p(k) \neq k$. Using the substitution $x_1, \dots, x_{k-1} \to x$, $x_k \to y$ and $x_{k+1}, \dots, x_n \to z$, we can show that the identity $xyz \approx xzy$ is satisfied in V, and so $V \subseteq \mathscr{I}_4$. Thus V is either $\mathscr{I}_0$ or $\mathscr{I}_1$ or $\mathscr{I}_2$ or $\mathscr{I}_4$.

Assume, finally, that $V \nsubseteq \mathscr{I}_7$ and $V \nsubseteq \mathscr{I}_8$. By 1.4, every subdirectly irreducible member of V is either in $\mathscr{I}_7$ or in $\mathscr{I}_8$. Consequently, $V = \mathscr{I}_9$. □

IV.3 Subdirectly irreducible idempotent LDR$_1$-semigroups

3.1 Remark. According to 1.4, there exist (up to isomorphism) only two subdirectly irreducible LDI-semigroups that are not LDR$_1$-semigroups, namely, $D(2)$ and $D(10)$.

3.2 Proposition. *Let S be a subdirectly irreducible LDR$_1$I-semigroup such that $q_S = \mathrm{id}_S$. Then just one of the following two cases takes places:*

(i) *$S \simeq D(1)$;*

(ii) *S possesses at least three elements, among them a neutral element e, such that $T = S - \{e\}$ is a subsemigroup of S, $q_T \neq \mathrm{id}_T$ and T is a subdirectly irreducible LDR$_1$I-semigroup possessing no neutral element.*

Proof. See I.3.2. □

3.3 Proposition. *Let T be a nontrivial semigroup and e be an element not belonging to T. Then $T\{e\}$ is a subdirectly irreducible LDR$_1$I-semigroup if and only if T is a subdirectly irreducible LDR$_1$I-semigroup possessing no neutral element.*

Proof. See I.2.8(iv). □

3.4 Proposition. *Let T be a nontrivial semigroup and o be an element not belonging to T. Then $T[o]$ is a subdirectly irreducible LDR$_l$I-semigroup if and only if T is a subdirectly irreducible LDR$_l$I-semigroup possessing no absorbing element.*

Proof. Easy. □

3.5 Proposition. *Let S be a subdirectly irreducible LDR$_l$I-semigroup possessing an absorbing element o. Then just one of the following two cases takes place:*

(i) *$S \simeq D(1)$;*

(ii) *S contains at least three elements, $T = S - \{o\}$ is a subsemigroup of S, T is a subdirectly irreducible LDR$_l$I-semigroup and T contains no absorbing element.*

Proof. Assume that $\mathrm{card}(S) \geq 3$ and that $(a, b) \in \omega_S$, $a \neq b$, $a \neq o$. Let $u \in T$; put $I = \{x \in S : xu = o\}$ and $J = Su$. Then both I and J are ideals of S and $\mathrm{card}(J) \geq 2$; we have $o, u \in J$. Consequently, $\omega_S \subseteq (J \times J) \cup \mathrm{id}_S$ and $a = vu$ for some $v \in S$. We have $a = vu = vuu = au$, and so $a \notin I$. Thus $\omega_S \nsubseteq (I \times I) \cup \mathrm{id}_S$, $\mathrm{card}(I) = 1$ and $I = \{o\}$. We have proved that T is a subsemigroup of S and the rest is clear from 3.4. □

3.6 Definition. A subdirectly irreducible LDR$_l$I-semigroup S will be called primary if S contains no neutral element and no absorbing element either.

3.7 Theorem. *Let S be a subdirectly irreducible LDR$_l$I-semigroup. Then just one of the following five cases takes place:*

(i) *$S \simeq D(1)$.*

(ii) *S is primary.*

(iii) *S contains at least three elements, among them a neutral element e, no absorbing element, $T = S - \{e\}$ is a subsemigroup of $S = T\{e\}$ and T is a primary subdirectly irreducible LDR$_l$I-semigroup.*

(iv) *S contains at least three elements, among them an absorbing element o, no neutral element, $T = S - \{o\}$ is a subsemigroup of $S = T[o]$ and T is a primary subdirectly irreducible LDR$_l$I-semigroup.*

(v) *S contains at least four elements, among them both a neutral element e and an absorbing element o, $T = S - \{e, o\}$ is a subsemigroup of $S = (T\{e\})[o] = (T[o])\{e\}$ and T is a primary subdirectly irreducible LDR$_l$I-semigroup.*

Proof. Combine 3.2, 3.3, 3.4 and 3.5 □

3.8 Notation. For a semigroup S, let $\mathrm{LA}(S)$ denote the set of left absorbing elements of S, i.e., $\mathrm{LA}(S) = \{a \in S : aS = \{a\}\}$. If $L = \mathrm{LA}(S)$ is nonempty, then L is an ideal of S and $L = \mathrm{Int}(S)$. Moreover, L is equal to the intersection of all left ideals of S and every nonempty subset of L is a right ideal of S.

3.9 Lemma. *Let S be an idempotent semigroup and I be a right ideal of S. Then $I \subseteq \mathrm{LA}(S)$ iff I is an LZ-semigroup.*

Proof. If I is an LZ-semigroup and if $a \in I$ and $x \in S$, then $ax \in I$ and $ax = a \cdot ax = a$. $\square$

IV.4 Subdirectly irreducible semigroups in $\mathscr{I}_8$

4.1 Remark. Recall that $\mathscr{I}_8$ is the variety of LDR_1I-semigroups, i.e., the variety of idempotent semigroups satisfying $xyx \approx xy$. The aim of this section is to prove that every semigroup from $\mathscr{I}_8$ can be embedded into a subdirectly irreducible semigroup from $\mathscr{I}_8$. This is a special case of a more general result by Goralčík and Koubek [GorK,82]. The proof contained in [GorK,82] contains several inaccuracies, making it almost unreadable.

4.2 Definition. We fix two distinct elements α, β. A semigroup $S \in \mathscr{I}_8$ will be called admissible if $\{\alpha,\beta\} \subseteq \mathrm{LA}(S)$ and $s\alpha = s\beta \in \{\alpha,\beta\}$ for all $s \in S - \mathrm{LA}(S)$.

An admissible semigroup $S \in \mathscr{I}_8$ will be called reductive if for every pair u, v of distinct elements of S there exists an element $s \in \mathrm{LA}(S)$ with $us \neq vs$.

4.3 Proposition. *Every semigroup $S \in \mathscr{I}_8$ containing neither α nor β can be extended to an admissible semigroup in $\mathscr{I}_8$.*

Proof. Put $T = S \cup \{\alpha,\beta\}$ and define multiplication on T as follows: S is a subsemigroup of T; $\alpha s = \alpha$ and $\beta s = \beta$ for all $s \in T$; $s\alpha = s\beta = \alpha$ for all $s \in S$. It is easy to see that $T \in \mathscr{I}_8$, $\mathrm{LA}(T) = \{\alpha,\beta\}$ and T is admissible. $\square$

4.4 Proposition. *Every admissible semigroup $S \in \mathscr{I}_8$ can be extended to a reductive admissible semigroup in $\mathscr{I}_8$.*

Proof. Take an element $e \notin S$ and put $R = S\{e\}$. Let $x \to x'$ be a bijection of R onto a set R' with $R \cap R' = \{\alpha,\beta\}$, such that $\alpha' = \alpha$ and $\beta' = \beta$. Put $T = S \cup R'$ and define multiplication on T as follows:

(i) S is a subsemigroup of T;
(ii) $st' = (st)'$ for $s, t \in S$;
(iii) $se' = s'$ for $s \in S$;
(iv) $s'w = s'$ for $s \in S$, $w \in T$;
(v) $e'w = e'$ for $w \in T$.

It is easy to see that the multiplication is correctly defined, $T \in \mathscr{I}_8$, $\mathrm{LA}(T) = R'$, and T is admissible. It remains to prove tat T is reductive. Let $s,t \in T$, $s \neq t$. If $s,t \in S$, then $se' = s' \neq t' = te'$. If $s,t \in R'$, then $ss = s \neq t = ts$. Finally, if $s \in S$ and $t \in R' - \{\alpha,\beta\}$, then $s\alpha \neq t = t\alpha$. $\square$

4.5 Notation. In the next lemmas we suppose that $S \in \mathscr{I}_8$ is a given admissible reductive semigroup and c, d is a pair of distinct elements of $\mathrm{LA}(S)$ with $d \notin \{\alpha,\beta\}$.

Take two distinct elements x, y not belonging to S and denote by Z the LZ-semigroup with the underlying set $\{x,y\}$. Denote by F the free product of S and

Z in $\mathscr{I}_8$, so that S and Z are disjoint subsemigroups of F, F is generated by $S \cup Z$ and for any $A \in \mathscr{I}_8$, any pair of homomorphism $S \to A$, $Z \to A$ can be extended to a homomorphism $F \to A$.

By a canonical form of an element $u \in F$ we mean an expression $u = u_1 \dots u_n$, where

(i) $1 \leq n \leq 3$,

(ii) if $n = 2$, then either $u_1 \in Z$, $u_2 \in S$ or $u_1 \in S$, $u_2 \in Z$,

(iii) if $n = 3$, then $u_1 \in S$, $u_2 \in Z$, $u_3 \in S$ and $u_1 u_3 \neq u_1$.

Observe that for $n = 3$, $u_1 \in S - LA(S)$ (in particular, if $n = 3$, then $u_1 \notin \{\alpha, \beta\}$).

4.6 Lemma. *Every element of F can be expressed in a canonical form.*

Proof. As this is clear for the elements of $S \cup Z$, it is sufficient to show that the set of the elements expressible in a canonical form is a subsemigroup of F. For this sake, it is certainly sufficient to show that if $u = u_1 \dots u_n$ canonically, then each of the elements ux, uy and us (for $s \in S$) also has a canonical form. This can be done easily by considering the possible cases. For example, $xsy = xsxy = xsx = xs$. Also, if $st = s$, then $sxt = sxst = sxs = sx$. □

4.7 Lemma. *Let $u = u_1 \dots u_n$ and $u = v_1 \dots v_m$ be two canonical expressions of the same elements $u \in F$. Then $n = m$ and either $u_1 = v_1, \dots, u_n = v_n$ or else $n = 3$, $u_1 = v_1$, $u_2 = v_2$ and $u_1 u_3 = v_1 v_3$.*

Proof. Denote by h_1 the homomorphism of F onto the two-element semilattice $\{0,1\}$ (where $01 = 0$) such that $h_1(S) = \{1\}$ and $h_1(Z) = \{0\}$; define h_2 similarly, but setting $h_2(S) = \{0\}$ and $h_2(Z) = \{1\}$. Clearly, $h_1(u_1 \dots u_n) = 0$ iff $Z \cap \{u_1, \dots, u_n\} \neq \emptyset$; also, $h_2(u_1 \dots u_n) = 0$ iff $S \cap \{u_1, \dots, u_n\} \neq \emptyset$. From this it follows that it is sufficient to consider the case when $n \geq 2$ and $m \geq 2$.

For every $e \in \mathrm{LA}(S)$ denote by h_e the homomorphism of F into S extending the identity on S and the constant homomorphism of Z onto $\{e\}$. If $u_1 \in S$, then $h_e(u_1 \dots u_n) = u_1 e$. If $v_1 \in Z$, then $h_e(v_1 \dots v_m) = e$. So, if $u_1 \in S$ and $v_1 \in Z$, then $u_1 e = e$ for any $e \in \mathrm{LA}(S)$; in particular, $u_1 \alpha = \alpha$ and $u_1 \beta = \beta$, contradicting the admissibility of S. We conclude that u_1, v_1 either belong both to S or belong both to Z. In the case when $u_1, v_1 \in S$, we get $u_1 e = v_1 e$ for all $e \in \mathrm{LA}(S)$, so that $u_1 = v_1$ by the reductivity of S.

Denote by h_3 the homomorphism of F into $Z\{1\}$ extending the constant homomorphism of S onto $\{1\}$ and the identity on Z. If $u_1 = v_1 \in S$, then $h_3(u_1 \dots u_n) = u_2$ and $h_3(v_1 \dots v_m) = v_2$, so that $u_2 = v_2$. If $u_1, v_1 \in Z$, then $h_3(u_1 \dots u_n) = u_1$ and $h_3(v_1 \dots v_m) = v_1$, so that $u_1 = v_1$.

So far we have proved that $u_1 = v_1$ and if $u_1 = v_1 \in S$, then $u_2 = v_2$.

Denote by h_4 the homomorphism of F into $S\{1\}$ extending the identity on S and the constant homomorphism of Z onto $\{1\}$. If $u_1 = v_1 \in Z$, then $h_4(u_1 \dots u_n) = u_2$ and $h_4(v_1 \dots v_m) = v_2$. So, $u_2 = v_2$.

Let s, t, t' be elements of S. If $sx = sxt$, then $xsx = xsxt$, i.e., $xs = xst$ and hence $s = st$, so that sxt is not a canonical form. If $sxt = sxt'$, then (similarly) $st = st'$. □

4.8 Notation. We have seen that every element $u \in F$ can be expressed canonically, $u = u_1 \dots u_n$, and u_1 is uniquely determined by u; we say that u begins with u_1.

Denote by R the relation, containing the following pairs of elements of F:

$$(\alpha, xc), (\beta, yc), (x\alpha, x\beta), (y\alpha, y\beta), (\alpha, \alpha x), (\alpha, \alpha y), (\beta, \beta x), (\beta, \beta y), (xd, yd).$$

Denote by ϱ the congruence of F generated by R.

Put $A_\alpha = \{s \in S : s\alpha = \alpha\}$ and $A_\beta = \{s \in S : s\beta = \beta\}$.

Put $B_\alpha = \{\alpha\} \cup \{xs : s \in S - \{d\}\} \cup A_\alpha ZS$ (notice that $A_\alpha Z \subseteq A_\alpha ZS$).

Put $B_\beta = \{\beta\} \cup \{ys : s \in S - \{d\}\} \cup A_\beta ZS$.

For $s \in \mathrm{LA}(S) - \{\alpha, \beta\}$ put $B_s = \{s, sx, sy\}$.

For $s \in S - \mathrm{LA}(S)$ put $B_s = \{s\}$.

4.9 Lemma. *Let $(v, w) \in R \cup R^{-1}$ and let p, q be two elements of $F\{1\}$ such that $pvq \in B_\alpha$ (or $pvq \in B_\beta$). Then $pwq \in B_\alpha$ (or $pwq \in B_\beta$, respectively).*

Proof. Let $pvq \in B_\alpha$ (the other case is similar). Consider first the case $pvq = \alpha$. Then clearly $p, q \in S\{1\}, v \in \{\alpha, \beta\}, w \in \{xc, yc, \alpha x, ay\}$. If $p \neq 1$, then $\alpha = pv = p\alpha$, so that $p \in A_\alpha$ and $pwq \in A_\alpha ZS$. If $p = 1$, then $\alpha = vq = v$, so that $w \in \{xc, \alpha x\}$ and we have either $pwq = xcq = xc$ or $pwq = \alpha xq = \alpha x$; in both cases, $pwq \in B_\alpha$.

Let $pvq \in \{xs : s \in S - \{d\}\} \cup A_\alpha ZS$. If $p \notin S\{1\}$, it follows easily from 4.7 that p, and then also pwq belong to $\{xs : s \in S - \{d\}\} \cup A_\alpha ZS$. So, let $p \in S\{1\}$.

Let $p \in S$. Then $pvq \in A_\alpha ZS$; since v either begins with an element of Z or belongs to $\{\alpha, \beta, \alpha x, \alpha y, \beta x, \beta y\}$, we get $p \in A_\alpha$. If w either begins with an element of Z or is one of the elements αx, αy, βx, βy, we get $pwq \in A_\alpha ZS$. So, let $w \in \{\alpha, \beta\}$. Then $pw = \alpha$. If $q \in S\{1\}$, we get $pwq = \alpha \in B_\alpha$. Otherwise, $pwq = \alpha q \in A_\alpha ZS \subseteq$ $\subseteq B_\alpha$.

Finally, let $p = 1$. Then $pvq = vq$, so that v does not begin with y and $v \notin$ $\notin \{xd, \beta, \beta x, \beta y\}$. Hence both v and w belong to $\{\alpha, xc, x\alpha, x\beta, \alpha x, \alpha y\}$. But then $pwq = wq \in B_\alpha$. □

4.10 Lemma. *Let $(v, w) \in R \cup R^{-1}$ and let p, q be two elements of $F\{1\}$ such that $pvq \in B_s$, where $s \in S - \{\alpha, \beta\}$. Then $pwq \in B_s$.*

Proof. Consider first the case $pvq = s$. Then $p, v, q \in S\{1\}$, $v \in \{\alpha, \beta\}$, $s =$ $= pv \notin \{\alpha, \beta\}$, so by the admissibility of S we get $p = s \in \mathrm{LA}(S) - \{\alpha, \beta\}$. Hence $pwq = swq \in \{s, sx, sy\} = B_s$.

It remains to consider the case $s \in \mathrm{LA}(S) - \{\alpha, \beta\}$, $pvq \in \{sx, sy\}$.

Let $p \notin S\{1\}$. It follows easily from 4.7 and from $s \in \mathrm{LA}(S)$ that $p = pvq$. Then $pwq = pvq \in B_s$.

Let $p \in S\{1\}$. If v begins with either x or y, then from $pvq \in \{sx, sy\}$ we get $p = s$ and then $pwq = swq \in \{s, sx, sy\}$. So, let $v \in \{\alpha, \beta, \alpha x, \alpha y, \beta x, \beta y\}$. Then either $p\alpha$ or $p\beta$ does not belong to $\{\alpha, \beta\}$, so $p \in \mathrm{LA}(S)$ and we again obtain $p = s$ and $pwq = swq \in \{s, sx, sy\}$. □

4.11 Lemma. *Let $(s,t) \in \varrho \cap (S \times S)$. Then $s = t$.*

Proof. Since $(s,t) \in \varrho$, there is a finite sequence $s_0, \dots s_n$ of elements of F such that $s_0 = s$, $s_n = t$ and for every $i = 1, \dots, n$ we have $s_{i-1} = pvq$, $s_i = pwq$ for some $p, q \in F\{1\}$ and $(v, w) \in R \cup R^{-1}$. It remains to use 4.9 and 4.10. □

4.12 Lemma. *Every congruence of F containing ϱ and containing the pair (c,d) contains (α, β).*

Proof. Let $\sim$ be a congruence containing ϱ and (c,d). We have $\alpha \sim xc \sim \sim xd \sim yd \sim yc \sim \beta$. □

4.13 Proposition. *Let S be a reductive admissible semigroup from $\mathscr{I}_8$ and let $c, d \in S$, $c \neq d$. Then S can be extended to an admissible semigroup $T \in \mathscr{I}_8$ such that $(\alpha, \beta) \in \theta_{c,d}$, where $\theta_{c,d}$ is the congruence of T generated by (c,d).*

Proof. Since S is reductive, it is sufficient to consider the case $\{c,d\} \subseteq \mathrm{LA}(S)$. If $\{c,d\} = \{\alpha,\beta\}$, we can put $T = S$. So, we can assume that $d \notin \{\alpha,\beta\}$.

Let us keep the notation introduced in 4.5 and 4.8. Denote by T the semigroup F/ϱ, in which we identify (or replace) every element s/ϱ (for $s \in S$) with s (this is possible according to 4.11). So, T is an extension of S. We have $T \in \mathscr{I}_8$, since $F \in \mathscr{I}_8$.

We have $\{\alpha,\beta\} \subseteq \mathrm{LA}(T)$: this follows from $(\alpha x, \alpha) \in \varrho$, $(\alpha y, \alpha) \in \varrho$, $(\beta x, \beta) \in \varrho$ and $(\beta y, \beta) \in \varrho$.

Let $s \in \mathrm{LA}(S)$. Then $(\alpha, \alpha x) \in \varrho$ implies $(s\alpha, s\alpha x) \in \varrho$, i.e., $(s, sx) \in \varrho$. Similarly, $(s, sy) \in \varrho$. From this it follows that $(s, st) \in \varrho$ for any $t \in F$, so that $s \in \mathrm{LA}(T)$. This proves $\mathrm{LA}(S) \subseteq \mathrm{LA}(T)$. Now it is easy to see that $\mathrm{LA}(T)$ also contains all the elements sx/ϱ, sy/ϱ, xs/ϱ and ys/ϱ with $s \in \mathrm{LA}(S)$.

Let $u = u_1 \dots u_n$ (canonically) be an element of F such that $u/\varrho \in T - \mathrm{L}(T)$. We have $u_i \notin \mathrm{LA}(S)$ for all i.

We have $(\alpha, xc) \in \varrho$, so that $(x\alpha, xxc) \in \varrho$, i.e., $(x\alpha, xc) \in \varrho$ and hence $(\alpha, x\alpha) \in \in \varrho$. Hence also $(\alpha, x\beta) \in \varrho$. Similarly, $(\beta, y\alpha) \in \varrho$ and $(\beta, y\beta) \in \varrho$. This shows that if $u_i \in \{x,y\}$, then $(u_i\alpha)/\varrho = (u_i\beta)/\varrho \in \{\alpha,\beta\}$. If $u_i \in S - \mathrm{LA}(S)$, then $u_i\alpha = = u_i\beta \in \{\alpha,\beta\}$ by the admissibility of S. Now it is easy to see that $(u\alpha)/\varrho = = (u\beta)/\varrho \in \{\alpha,\beta\}$.

We see that T is admissible. The rest follows from 4.12. □

4.14 Proposition. *Let S be an admissible semigroup from $\mathscr{I}_8$. Then S can be extended to an admissible semigroup $T \in \mathscr{I}_8$ such that for any $c, d \in S$ with $c \neq d$, the congruence of T generated by (c,d) contains (α, β).*

Proof. By 4.4 and 4.14, for every admissible semigroup $S \in \mathscr{I}_8$ and every $c, d \in S$ with $c \neq d$ there exists an admissible semigroup $T_{c,d} \in \mathscr{I}_8$ such that (α, β) belongs to the congruence of $T_{c,d}$ generated by (c, d). The result follows by a standard argument using transfinite construction; observe that the union of a chain of admissible semigroups from $\mathscr{I}_8$ is an admissible semigroup from $\mathscr{I}_8$. $\square$

4.15 Theorem. *Every semigroup $S \in \mathscr{I}_8$ can be extended to a subdirectly irreducible semigroup from $\mathscr{I}_8$.*

Proof. By 4.3, it is enough to consider the case when S is admissible. Define a countable chain of admissible semigroups $S_0, S_1, \ldots$ as follows: $S_0 = S$; S_{i+1} is an extension of S_i claimed by 4.14. The union of this chain is the desired semigroup. $\square$

IV.5 Comments and open problems

The first three sections of the chapter are based on [Kep,81]. The main result of the last section (4.15) is a special case of a more general result by Goralčík and Koubek [GorK,82]. The original proof contained in [GorK,82] is rather inaccurate and almost unreadable.

According to 1.4 and 4.15, a semigroup S is a subsemigroup of a subdirectly irreducible LDI-semigroup if and only if either S is an $\mathrm{LDR_1I}$-semigroup (i.e., S satisfies $xx \approx x$ and $xyx \approx xy$) or S is isomorphic to either $D(2)$ or $D(10)$. It is an open problem to determine which semigroups are available as subsemigroups of finite subdirectly irreducible LDI-semigroups.

V. The lattice of varieties of left distributive semigroups

V.1 The subvarieties of $\mathscr{T} \cap \mathscr{R}$

1.1. Notation. We denote by $\mathscr{L}$ the variety of LD-semigroups, by $\mathscr{I}$ the variety of idempotent LD-semigroups (so that $\mathscr{I} = \mathscr{I}_9$), by $\mathscr{R}$ the variety of LDR-semigroups and by $\mathscr{T}$ the variety of LDT-semigroups.

1.2 Lemma. *$\mathscr{T} \cap \mathscr{R} = \mathscr{A} \vee \mathscr{I}$ and every subvariety of $\mathscr{T} \cap \mathscr{R}$ is equal to $\mathscr{A}_i \vee \mathscr{I}_j$ for some $0 \leq i \leq 5$ and $0 \leq j \leq 9$.*

Proof. By I.1.17, every semigroup in $\mathscr{T} \cap \mathscr{R}$ is a subdirect product of an A-semigroup and an idempotent LD-semigroup. Now, use Theorems III.2.2 and IV.2.2. $\square$

1.3 Lemma. *For $j \notin \{0,2\}$ we have $\mathscr{A}_2 \vee \mathscr{I}_j = \mathscr{A}_4 \vee \mathscr{I}_j$ and $\mathscr{A}_3 \vee \mathscr{I}_j = \mathscr{A}_5 \vee \mathscr{I}_j$.*

Proof. Let G be the free semigroup in $\mathcal{A}_3 \vee \mathcal{I}_j$ over two generators x and y. Clearly, $xy \neq yx$ in G and $xy\, yx \notin \mathrm{Id}(G)$. From this it follows that $G/\mathrm{Id}(G) \notin \mathcal{A}_3$ and hence $(\mathcal{A}_3 \vee \mathcal{I}_j) \cap \mathcal{A}_5 \nsubseteq \mathcal{A}_3$. Consequently, $(\mathcal{A}_3 \vee \mathcal{I}_j) \cap \mathcal{A}_5 = \mathcal{A}_5$, which means that $\mathcal{A}_3 \vee \mathcal{I}_j = = \mathcal{A}_5 \vee \mathcal{I}_j$. One can prove $\mathcal{A}_2 \vee \mathcal{I}_j = \mathcal{A}_4 \vee \mathcal{I}_j$ similarly. □

1.4 Lemma. *Let either $i \notin \{2,3\}$ or $j \in \{0,2\}$. Then a semigroup S belongs to $\mathcal{A}_i \vee \mathcal{I}_j$ if and only if $S \in \mathcal{T} \cap \mathcal{R}$, $\mathrm{Id}(S) \in \mathcal{I}_j$ and $S/\mathrm{Id}(S) \in \mathcal{A}_i$.*

Proof. Denote by V the class of all semigroups S with this property. It is easy to see that V is a variety, and hence $V = \mathcal{A}_i \vee \mathcal{I}_j$. □

1.5 Lemma. *Let (i,j) and (k,l) be two ordered pairs from $\{0,\dots, 5\} \times \{0,\dots, 9\}$. Then $\mathcal{A}_i \vee \mathcal{I}_j \subseteq \mathcal{A}_k \vee \mathcal{I}_l$ if and only if $\mathcal{I}_j \subseteq \mathcal{I}_l$ and one of the following three cases takes place: either $\mathcal{A}_i \subseteq \mathcal{A}_k$ or $l \notin \{0,2\}$, $i = 4$, $k = 2$ or $l \notin \{0,2\}$, $i = 5$, $k = 3$.*

Proof. Apply 1.2, 1.3 and 1.4. □

1.6 Lemma. *The variety $\mathcal{T} \cap \mathcal{R}$ has the following 44 subvarieties:*

$L_0 = \mathcal{A}_0 \vee \mathcal{I}_0 = \mathcal{A}_0 = \mathcal{I}_0$,
$L_1 = \mathcal{A}_0 \vee \mathcal{I}_1 = \mathcal{I}_1$,
...
$L_9 = \mathcal{A}_0 \vee \mathcal{I}_9 = \mathcal{I}_9$,
$L_{10} = \mathcal{A}_1 \vee \mathcal{I}_0 = \mathcal{A}_1$,
$L_{11} = \mathcal{A}_1 \vee \mathcal{I}_1$,
...
$L_{19} = \mathcal{A}_1 \vee \mathcal{I}_9$,
$L_{20} = \mathcal{A}_2 \vee \mathcal{I}_0$,
$L_{21} = \mathcal{A}_2 \vee \mathcal{I}_1 = \mathcal{A}_4 \vee \mathcal{I}_1$,
$L_{22} = \mathcal{A}_2 \vee \mathcal{I}_2$,
$L_{23} = \mathcal{A}_2 \vee \mathcal{I}_3 = \mathcal{A}_4 \vee \mathcal{I}_3$,
...
$L_{29} = \mathcal{A}_2 \vee \mathcal{I}_9 = \mathcal{A}_4 \vee \mathcal{I}_9$,
$L_{30} = \mathcal{A}_3 \vee \mathcal{I}_0$,
$L_{31} = \mathcal{A}_3 \vee \mathcal{I}_1 = \mathcal{A}_5 \vee \mathcal{I}_1$,
$L_{32} = \mathcal{A}_3 \vee \mathcal{I}_2$,
$L_{33} = \mathcal{A}_3 \vee \mathcal{I}_3 = \mathcal{A}_5 \vee \mathcal{I}_3$,
...
$L_{39} = \mathcal{A}_3 \vee \mathcal{I}_9 = \mathcal{A}_5 \vee \mathcal{I}_9 = \mathcal{T} \cap \mathcal{R}$,
$L_{40} = \mathcal{A}_4 \vee \mathcal{I}_0$,
$L_{41} = \mathcal{A}_4 \vee \mathcal{I}_2$,
$L_{42} = \mathcal{A}_5 \vee \mathcal{I}_0$,
$L_{43} = \mathcal{A}_5 \vee \mathcal{I}_2$.

Proof. It follows from 1.5. □

V.2 The varieties $S_{i,j}$, $R_{i,j}$ and $T_{i,j}$

2.1 Notation. We denote by $\mathrm{M}(u_1 \approx v_1, \ldots)$ the variety of LD-semigroups satisfying $u_1 \approx v_1, \ldots$ Put

$S_1 = \mathrm{M}(x^2 \approx x^3,\ xy^2 \approx xyx)$,
$S_2 = \mathrm{M}(x^2 \approx x^3)$,
$S_3 = \mathrm{M}(xy^2 \approx xyx)$,
$S_4 = \mathscr{L}$ (the variety of all LD-semigroups),
$S_{i,j} = \{S \in S_i : \mathrm{Id}(S) \in \mathscr{I}_j\}$ for $1 \le i \le 4$ and $0 \le j \le 9$,
$R_1 = \mathrm{M}(xy \approx xyx)$,
$R_2 = \mathrm{M}(xy \approx xy^2)$,
$R_3 = \mathrm{M}(x^2 \approx x^3,\ xy^2 \approx xyx,\ x^2y \approx x^2y^2) = \mathscr{R} \cap S_1$,
$R_4 = \mathrm{M}(x^2 \approx x^3, x^2y \approx x^2y^2) = \mathscr{R} \cap S_2$,
$R_5 = \mathrm{M}(x^2y \approx x^2y^2, xy^2 \approx xyx) = \mathscr{R} \cap S_3$,
$R_6 = \mathrm{M}(x^2y \approx x^2y^2) = \mathscr{R}$,
$R_{i,j} = R_i \cap S_{4,j}$ for $1 \le i \le 6$ and $0 \le j \le 9$,
$T_1 = \mathrm{M}(xy \approx x^2y)$,
$T_2 = \mathrm{M}(x^2 \approx x^3,\ xy^2 \approx x^2y^2) = \mathscr{T} \cap S_2$,
$T_3 = \mathrm{M}(xy^2 \approx x^2y^2) = \mathscr{T}$,
$T_{i,j} = T_i \cap S_{4,j}$ for $1 \le i \le 3$ and $0 \le j \le 9$.

2.2 Lemma. *The following are true:*
(i) $S_{i,j}$ *is a subvariety of* $\mathscr{L}$ *and* $S_{i,j} \cap \mathscr{I} = \mathscr{I}_j$.
(ii) $S_1 = S_2 \cap S_3$ *and* $S_2 \vee S_3 \subseteq S_4$.
(iii) $\mathscr{A}_5 \subseteq S_{3,j} \subseteq S_{4,j}$, $\mathscr{A}_5 \nsubseteq S_{1,j}$ *and* $\mathscr{A}_5 \nsubseteq S_{2,j}$.
(iv) $S_{1,j} = S_{2,j} \cap S_{3,j}$, $S_{1,0} = S_{2,0} = \mathscr{A}_4$ *and* $S_{3,0} = S_{4,0} = \mathscr{A}_5$.
(v) $R_1 = R_2 \cap R_3$, $R_3 = R_4 \cap R_5$, $R_2 \subseteq R_4$ *and* $R_4 \vee R_5 \subseteq R_6$.
(vi) $T_1 \subseteq T_2 \subseteq T_3$.

Proof. It is easy. □

V.3 Auxiliary results

3.1 Notation. Let X be a countably infinite set of variables. As before, we denote by $\mathbf{F}$ the free semigroup over X; the elements of $\mathbf{F}$ will be called words. Recall that F is a subset of $\mathbf{F}$, and every word is equivalent to a unique word from F with respect to the equational theory of LD-semigroups.

We denote by W_1 the set of the words t such that $f(t) \in \mathrm{Id}(S)$ for all LD-semigroups S and all homomorphisms f of $\mathbf{F}$ into S. Denote by W_2 the subsemigroup of $\mathbf{F}$ generated by $\{x^3 : x \in X\}$. Clearly, $W_2 \subseteq W_1$.

The first variable in a word t will be denoted by $o(t)$. Denote by $\mathrm{var}(t)$ the set of variables occurring in t.

3.2 Lemma. *Let r, s be two words with $o(r) \neq o(s)$ and let x be a variable such that $x \neq o(r)$. Then* $\mathrm{M}(xr \approx xs) \subseteq \mathscr{T}$.

Proof. Let y be a variable not occurring in xrs. Denote by y_1 the first variable in s. Consider the substitution f with $f(x) = f(y_1) = x$ and $f(z) = y$ for all variables $z \notin \{x, y_1\}$. Applying f to the equation $xry \approx xsy$ (which is a consequence of $xr \approx xs$), it is easy to see that either $xy^2 \approx x^2y$ or $xy^2 \approx$ $\approx x^2y^2$ is a consequence of $xr \approx xs$. However, $\mathrm{M}(xy^2 \approx x^2y) = \mathscr{T} \cap \mathscr{R}$ and $\mathrm{M}(xy^2 \approx x^2y^2) = \mathscr{T}$. □

3.3 Lemma. *Let r, s be two words.*

(i) *If $o(r) \neq o(s)$, then* $\mathrm{M}(r \approx s) \subseteq \mathscr{T}$.

(ii) *If $o(r) \neq o(s) = x$ and s starts with x^2 (i.e., either $s = x^2$ or $s = x^2t$ for some t), then* $\mathrm{M}(xr \approx s) \subseteq \mathscr{T}$.

(iii) *If x, y, z are variables and $y \neq z$, then* $\mathrm{M}(xyr \approx xzs) \subseteq \mathscr{T}$.

Proof. (i) Let x be a variable not occurring in rs. Then $\mathrm{M}(r \approx s) \subseteq$ $\subseteq \mathrm{M}(xr \approx xs) \subseteq \mathscr{T}$ by 3.2.

(ii) This follows from 3.2.

(iii) Let u be a variable not occurring in $xyzrs$. Consider the substitution f with $f(x) = f(z) = x$ and $f(v) = y$ for all variables $v \notin \{x, z\}$. Applying f to the equation $xyru \approx xzsu$, it is easy to see that either $xy^2 \approx x^2y$ or $xy^2 \approx x^2y^2$ is a consequence of $xyr \approx xzs$. □

3.4 Lemma. *Let r, s be two words.*

(i) *If x is a variable not occurring in r and if $s \notin \{x, x^2\}$ and $s \neq tx$ for any word t with $x \notin \mathrm{var}(t)$, then* $\mathrm{M}(rx \approx s) \subseteq \mathscr{R}$.

(ii) *If $\mathrm{var}(r) \neq \mathrm{var}(s)$, then* $\mathrm{M}(r \approx s) \subseteq \mathscr{R}$.

Proof. (i) Consider the substitution f with $f(x) = y$ and $f(v) = x$ for all variables $v \neq x$. Applying f to $rx \approx s$, we see that the equation $rx \approx s$ has a consequence $t \approx u$, where

$$t \in \{xy, x^2y\}$$

and

$$u \in \{x, x^2, x^3, y^3, xyx, x^2yx, xy^2, x^2y^2, yx, yx^2, y^2x, y^2x^2\}.$$

Every one of these 24 equations implies $x^2y = x^2y^2$.

(ii) By symmetry, we can assume that there is a variable $x \in \mathrm{var}(s) - \mathrm{var}(r)$. If $s = x$, then $\mathrm{M}(r \approx s)$ is the trivial variety. In the opposite case we have $sx \notin \{x, x^2\}$ and $\mathrm{M}(r \approx s) \subseteq \mathrm{M}(rx \approx sx) \subseteq \mathscr{R}$ by (i). □

3.5 Lemma. *Let V be a variety of LD-semigroups. If $V \cap \mathscr{I} \subseteq \mathscr{I}_6$, then $V \subseteq \mathscr{T}$. If $V \cap \mathscr{I} \subseteq \mathscr{I}_5$, then $V \subseteq \mathscr{R}$.*

Proof. First, let $V \cap \mathscr{I} \subseteq \mathscr{I}_6$. Then $abc = bac$ for all $a, b, c \in \mathrm{Id}(S)$, for any $S \in V$. Conseqeuntly, $V \subseteq \mathrm{M}(x^2yz^2 \approx y^2xz^2) \subseteq \mathscr{T}$ by 3.3(i).
Now, let $V \cap \mathscr{I} \subseteq \mathscr{I}_5$. Then $V \subseteq \mathrm{M}(x^3 \approx x^2yx^2) \subseteq \mathscr{R}$ by 3.4(ii). □

3.6 Lemma. *The following are true:*
(i) *Let r, s be two words such that $o(r) \neq o(s)$ and $\mathrm{var}(r) \neq \mathrm{var}(s)$. Then $\mathrm{M}(r \approx s) \subseteq \mathscr{T} \cap \mathscr{R}$.*
(ii) *Let V be a variety of LD-semigroups such that $V \cap \mathscr{I} \subseteq \mathscr{I}_3$. Then $V \subseteq$ $\subseteq \mathscr{T} \cap \mathscr{R}$.*

Proof. Use 3.3(i), 3.4(ii) and 3.5. □

3.7 Lemma. *Let r, s be two words.*
(i) *If $r, s \in W_2$, then $\mathrm{M}(r \approx s) = S_{4,j}$ for some j.*
(ii) *If $r, s \in W_1$, then $\mathrm{M}(r \approx s) \cap \mathscr{T} = T_{3,j}$ for some j.*
(iii) *If $r \in W_1$, then either $\mathrm{M}(r \approx s) \cap \mathscr{T} \subseteq \mathscr{R}$ or $\mathrm{M}(r \approx s) \cap \mathscr{T} = T_{3,j}$ or $\mathrm{M}(r \approx s) \cap \mathscr{T} = T_{2,j}$ for some j.*

Proof. Put $V = \mathrm{M}(r \approx s)$ and let $V \cap \mathscr{I} = \mathscr{I}_j$. Then $V \subseteq S_{4,j}$ and $V \cap \mathscr{T} \subseteq T_{3,j}$.
(i) Let $S \in S_{4,j}$ and let f be a homomorphism of $\mathbf{F}$ into S. Then $f(W_2) \subseteq \mathrm{Id}(S)$ and hence $f(r) = f(s)$. Thus $S \in V$ and $V = S_{4,j}$.
(ii) Let $S \in T_{3,j}$ and let f be a homomorphism of $\mathbf{F}$ into S. Denote by g the substitution with $g(x) = x^3$ for all variables x. Put $h(a) = a^3$ for all $a \in S$, so that h is an endomorphism of S. We have $g(\mathbf{F}) = W_2$ and $h(S) = \mathrm{Id}(S)$. Moreover, $\mathrm{Id}(S) \in \mathscr{I}_j \subseteq V \cap \mathscr{T}$ and $fg(\mathbf{F}) \subseteq \mathrm{Id}(S)$. Consequently, $fg(r) = fg(s)$. On the other hand, it is easy to see that $fg = hf$. Therefore $hf(r) = hf(s)$. But both $f(r)$ and $f(s)$ belong to $\mathrm{Id}(S)$, and so $f(r) = f(s)$.
(iii) By the construction of free LD-semigroups given in II.1.1 we can assume that $s = x_1^i x_2 \dots x_n$ where $n \geq 1$, $x_1, \dots, x_n$ are pairwise different variables and $i \leq 2$. Put $U = \mathrm{M}(s \approx s^3)$. Clearly, $V \cap \mathscr{T} = U \cap \mathscr{T} \cap \mathrm{M}(r \approx s^3)$. Since the words r and s^3 belong to W_1, we have $\mathrm{M}(r \approx s^3) \cap \mathscr{T} = T_{3,k}$ for some k. If $n = 1$ and $i = 1$, then $U = \mathscr{I}$ and $V \cap \mathscr{T} = \mathscr{I}_k$. If $n = 1$ and $i = 2$, then $U = S_2$ and $V \cap \mathscr{T} = T_{2,k}$. Let $n \geq 2$. Then

$$U = \mathrm{M}(x_1^i x_2 \dots x_n \approx x_1^i x_2 \dots x_{n-1} x_n^2) \subseteq \mathscr{R}$$

by 3.4(i). □

3.8 Lemma. *Let x, y be two variables and r, s be two words with $x \notin \mathrm{var}(rs)$. Let $V = \mathrm{M}(xyr \approx xys)$. If either $V \subseteq \mathscr{R}$ or $xyr, xys \in W_1$, then either $V = S_{4,j}$ or $V = R_{6,j}$ for some j.*

Proof. Put $r = u_1 \dots u_n$ and $s = v_1 \dots v_m$ $(u_i, v_i \in X)$.

Let $V \subseteq \mathscr{R}$. It is enough to show that a semigroup $S \in \mathscr{R}$ satisfies $xyr \approx xys$ if and only if $\mathrm{Id}(S)$ satisfies $xyr \approx xys$. The direct implication is clear. Let $\mathrm{Id}(S)$ satisfy $xyr \approx xys$. In S we have

$$\begin{aligned} xyr &= xy^2r = (xy)^2r = (xy)^2r^2 = (xy)^3y^3r^3 = (xy)^3y^3u_1^3 \dots u_n^3 \\ &= (xy)^3y^3v_1^3 \dots v_m^3 = xys. \end{aligned}$$

Let $xyr, xys \in W_1$. Then $V = \mathrm{M}(xyu_1^3 \dots u_n^3 \approx xyv_1^3 \dots v_m^3)$. If $x = y$, then the result follows from 3.7(i). Hence suppose that $x \neq y$ and put $\mathscr{I}_j = V \cap \mathscr{I}$. Then $\mathscr{I}_j$ satisfies $yu_1 \dots u_n \approx yv_1 \dots v_m$ and $V \subseteq S_{4,j}$. Conversely, let $S \in S_{4,j}$. Then S satisfies $y^3u_1^3 \dots u_n^3 \approx y^3v_1^3 \dots v_m^3$ and hence $S \in V$. □

3.9 Lemma. *Let $i, j \leq 2 \leq n$, let $x_1, \dots, x_n$ be pairwise different variables and let p be a permutation of $\{1, \dots, n\}$ such that $p(1) \neq 1$. Put*

$$r = x_1^i x_2 \dots x_n, \qquad s = x_{p(1)}^j x_{p(2)} \dots x_{p(n)}$$

and $V = \mathrm{M}(r \approx s)$. Then either $V \subseteq \mathscr{T} \cap \mathscr{R}$ or $V = T_{3,6}$.

Proof. By 3.3(i), $V \subseteq \mathscr{T}$. If $p(n) \neq n$, then $V \subseteq \mathscr{R}$ by 3.4(i). So, we can assume tat $p(n) = n$. Then $n \geq 3$, $\mathscr{I}_1 \nsubseteq V$, $V \cap \mathscr{I} = \mathscr{I}_6$ and we get $V \subseteq T_{3,6}$. Conversely, let $S \in T_{3,6}$ and $a_1, \dots, a_n \in S$. Then

$$a_1^3 \dots a_{n-1}^3 a_{n-1}^3 = a_{p(1)}^3 \dots a_{p(n-1)}^3 a_{n-1}^3$$

and

$$\begin{aligned} a_1 \dots a_n &= a_1^2 a_2 \dots a_n = a_1^3 a_2^3 \dots a_{n-1}^3 a_{n-1}^3 a_n = a_{p(1)}^3 \dots a_{p(n-1)}^3 a_{n-1}^3 a_n \\ &= a_{p(1)} \dots a_{p(n-1)} a_{n-1} a_n = a_{p(1)} \dots a_{p(n-1)} a_n. \end{aligned}$$ □

3.10 Lemma. *Let r, s be two words such that $o(r) \neq o(s)$ and let $V = \mathrm{M}(r \approx s)$. Then either $V \subseteq \mathscr{T} \cap \mathscr{R}$ or $V = T_{2,j}$ or $V = T_{3,j}$ for some j.*

Proof. By 3.3(i) we have $V \subseteq \mathscr{T}$ and by 3.6(i) we can assume that $\mathrm{var}(r) = \mathrm{var}(s)$. Taking into account 3.7(iii), we may restrict ourselves to the case $r, s \in F - W_1$. Then $r = x_1^i x_2 \dots x_n$ and $s = y_1^k y_2 \dots y_m$. We have $n = m$ and there is a permutation p of $\{1, \dots, n\}$ with $p(1) \neq 1$, such that $y_1 = x_{p(1)}, \dots, y_n = x_{p(n)}$. The result now follows from 3.9. □

3.11 Lemma. *Let $i \leq 2$, $3 \leq n$, let $x_1, \dots, x_n$ be pairwise distinct variables and let p be a permutation of $\{2, \dots, n\}$ such that $p(2) \neq 2$. Put $r = x_1 x_2 \dots x_n$, $s = x_1^i x_{p(2)} \dots x_{p(n)}$ and $V = \mathrm{M}(r \approx s)$. Then:*

(i) *$V \subseteq \mathscr{T}$.*
(ii) *If $p(n) \neq n$, then $V \subseteq \mathscr{T} \cap \mathscr{R}$.*
(iii) *If $p(n) = n$, then $V = T_{3,7}$.*

Proof. (i) Use 3.3(iii).
(ii) Use (i) and 3.4(i).

(iii) It is easy to see that $V \cap \mathscr{I} = \mathscr{I}_7$ and $V \subseteq T_{3,7}$. Conversely, let $S \in T_{3,7}$ and let $a_1, \ldots, a_n$ be elements of S. Then

$$\begin{aligned} a_1 \ldots a_n &= a_1^3 \ldots a_{n-1}^3 a_1^3 a_n = a_1^3 a_{p(2)}^3 \ldots a_{p(n-1)}^3 a_1^3 a_n \\ &= a_1^2 a_{p(2)} \ldots a_{p(n-1)} a_n . \quad \square \end{aligned}$$

3.12 Lemma. *Let $n \geq 3$, let $x_1, \ldots, x_n$ be pairwise different variables and let p be a non-identical permutation of $\{1, \ldots, n\}$ such that $p(1) = 1$. Put $V = \mathrm{M}(x_1^2 x_2 \ldots x_n \approx x_1^2 x_{p(2)} \ldots x_{p(n)})$. Then:*

(i) *If $p(n) \neq n$, then $V = R_{6,4}$.*
(ii) *If $p(n) = n$, then $V = S_{4,7}$.*

Proof. If $p(n) \neq n$, then $V \subseteq \mathscr{R}$ according to 3.4(i). The rest is similar to 3.11. $\square$

3.13 Lemma. *Let $i, k, q, t \leq 2 \leq n$, let $x_1, \ldots, x_n$ be pairwise distinct variables and let p be a permutation of $\{1, \ldots, n\}$. Put*

$$V = \mathrm{M}(x_1^i x_2 \ldots x_{n-1} x_n^k \approx x_{p(1)}^q x_{p(2)} \ldots x_{p(n-1)} x_{p(n)}^t) .$$

Then either $V \subseteq \mathscr{T} \cap \mathscr{R}$ or $V = S_{4,j}$ or $V = T_{m,j}$ or $V = R_{6,j}$ for some m and j.

Proof. The result can be put together from the following nine cases.

(i) Let $p(1) \neq 1$. Then we can apply 3.10.

(ii) Let $p(1) = 1$, $k = t = 1$ and $i = q = 2$. This case is clear from 3.12.

(iii) Let $p(1) = 1$, $p(2) \neq 2$, $k = t = 1$ and $i + q \leq 3$. In this case we can use 3.11.

(iv) Let $p(1) = 1$, $p(2) = 2$, $k = t = 1$ and $i = q = 1$. If p is the identical permutation, then $V = \mathscr{L}$. Hence assume that p is non-identical. Then $n \geq 4$. *If* $p(n) \neq n$, then $V \subseteq \mathscr{R}$ y 3.4(i), $V \cap \mathscr{I} = \mathscr{I}_4$ and it is easy to see that $V = R_{6,4}$. Now, let $p(n) = n$. Then $V \cap \mathscr{I} = \mathscr{I}_7$ and $V \subseteq S_{4,7}$. Conversely, if $S \in S_{4,7}$ and if $a_1, \ldots, a_n$ are elements of S, then

$$\begin{aligned} a_1 \ldots a_n &= a_1 a_2^3 \ldots a_{n-1}^3 a_2^3 a_n^3 = a_1 a_2^3 a_{p(3)}^3 \ldots a_{p(n-1)}^3 a_2^3 a_n \\ &= a_1 a_2 a_{p(3)} \ldots a_{p(n-1)} a_n \end{aligned}$$

and $S \in V$.

(v) Let $p(1) = 1$, $p(2) = 2$, $k = t = 1$, $i = 1$ and $q = 2$. We have $V \subseteq \mathscr{T}$ by 3.3(ii). If $p(n) \neq n$, then $V \subseteq \mathscr{T} \cap \mathscr{R}$ follows from 3.4(i). Let $p(n) = n$ and $n \geq 3$. Then it is easy to see that $V = \mathscr{T} \cap \mathrm{M}(x_1^2 x_2 \ldots x_n \approx x_1^2 x_2 x_{p(3)} \ldots x_{p(n)})$. If p is non-identical, then $V = T_{3,7}$ by 3.12; if p is the identity, then $V = T_{3,9}$.

(vi) Let $p(1) = 1$, $k = t = 2$, $i = 2$ and $q = 1$. Then $V \subseteq \mathscr{T}$ by 3.3(ii) and we can use 3.7(ii).

(vii) Let $p(1) = 1$, $k = t = 2$ and $i = q = 1$. If $p(2) = 2$, then the result follows from 3.8. If $p(2) \neq 2$, then $n \geq 3$, $V \subseteq \mathscr{T}$ by 3.3(iii) and the result follows from 3.7(ii).

(viii) Let $p(1) = 1$, $k = t = 2$ and $i = q = 2$. In this case, it is possible to use 3.7(i).

(ix) Let $p(1) = 1$, $k = 2$ and $t = 1$. If $p(n) \neq n$, then $V \subseteq \mathscr{R}$ by 3.4(i). If $p(n) = n$, then the inclusion $V \subseteq \mathscr{R}$ is obvious. Hence we have

$$V = \mathscr{R} \cap \mathrm{M}(x_1^i x_2 \ldots x_{n-1} x_n^2 \approx x_1^q x_{p(2)} \ldots x_{p(n-1)} x_{p(n)}^2).$$

The result is now clear from (vi), (vii) and (viii). □

3.14 Lemma. *Let r, s be two words and let $V = \mathrm{M}(r \approx s)$. Then either $V \subseteq \subseteq \mathscr{T} \cap \mathscr{R}$ or $V = T_{i,j}$ for some i and j.*

Proof. According to 3.4(ii) and 3.7(iii), we can assume that $\mathrm{var}(r) = \mathrm{var}(s)$ and $r, s \in F - W_1$. However, then 3.13 can be applied. □

V.4 The lattice of subvarieties of $\mathscr{T}$

4.1 Lemma. *The following are true:*

(i) $T_{1,j} \cap \mathscr{A} = \mathscr{A}_1$, $T_{2,j} \cap \mathscr{A} = \mathscr{A}_4$, $T_{3,j} \cap \mathscr{A} = \mathscr{A}_5$ *and* $T_{1,j} \cap \mathscr{I} = T_{2,j} \cap \mathscr{I} = T_{3,j} \cap \mathscr{I} = \mathscr{I}_j$ *for every* $0 \leq j \leq 9$.

(ii) $T_{1,j} = \mathscr{A}_1 \vee \mathscr{I}_j$, $T_{2,j} = \mathscr{A}_4 \vee \mathscr{I}_j$ *and* $T_{3,j} = \mathscr{A}_5 \vee \mathscr{I}_j$ *for* $j \in \{0,1,3,5\}$.

Proof. Use 1.5 and 3.5. □

4.2 Lemma. *Let $1 \leq i,j \leq 3$ and $0 \leq p,q \leq 9$. Then $T_{i,p} \cap T_{j,q} = T_{r,s}$ for some r,s. Moreover, $T_{i,p} \subseteq T_{j,q}$ if and only if $i \leq j$ and $\mathscr{I}_p \subseteq \mathscr{I}_q$.*

Proof. It is easy. □

4.3 Lemma. *The varieties $T_{i,j}$ $(1 \leq i \leq 3,\ 0 \leq j \leq 9)$ are pairwise distinct.*

Proof. Use 4.2. □

4.4 Lemma. *Let V be a subvariety of $\mathscr{T}$. Then either V is contained in $\mathscr{T} \cap \mathscr{R}$ or $V = T_{i,j}$ for some i and j.*

Proof. If $V \subseteq \mathscr{R}$, then $V \subseteq \mathscr{T} \cap \mathscr{R}$. So, let $V \not\subseteq \mathscr{R}$. Then, by 3.14, V is the intersection of some varieties $T_{i,j}$, so that $V = T_{i,j}$ for some i,j by 4.2. □

4.5 Proposition. *The variety $\mathscr{T}$ has the following 62 subvarieties:*

$L_0, \ldots, L_{43}$,
$L_{44} = T_{1,2}$,
$L_{45} = T_{2,2}$,
$L_{46} = T_{3,2}$,
$L_{47} = T_{1,4}$,
$L_{48} = T_{2,4}$,
$L_{49} = T_{3,4}$,
$L_{50} = T_{1,6}$,

$L_{51} = T_{2,6}$,
$L_{52} = T_{3,6}$,
$L_{53} = T_{1,7}$,
$L_{54} = T_{2,7}$,
$L_{55} = T_{3,7}$,
$L_{56} = T_{1,8}$,
$L_{57} = T_{2,8}$,
$L_{58} = T_{3,8}$,
$L_{59} = T_{1,9}$,
$L_{60} = T_{2,9}$,
$L_{61} = T_{3,9} = \mathscr{T}$.

We have $L_{44}, \ldots, L_{61} \nsubseteq L_{43} = \mathscr{T} \cap \mathscr{R}$. We have $T_{i,p} \subseteq T_{j,q}$ if and only if $i \leq j$ and $\mathscr{I}_p \subseteq \mathscr{I}_q$. We have $\mathscr{A}_m \vee \mathscr{I}_n \subseteq T_{r,s}$ if and only if $\mathscr{I}_n \subseteq \mathscr{I}_s$ and either $r = 3$ or $r = 2$, $m \in \{0,1,2,4\}$ or $r = 1$, $m \in \{0,1\}$.

Proof. Let V be a subvariety of $\mathscr{T}$ such that $V \subseteq \mathscr{R}$. By 4.4 and 4.1(ii), $V = T_{i,j}$ where $i \in \{1,2,3\}$ and $j \in \{2,4,6,7,8,9\}$. Converserly, if i and j are such numbers, then $T_{1,2} \subseteq T_{i,j}$ and hence $T_{i,j} \nsubseteq \mathscr{R}$. The rest is easy. □

V.5 Auxiliary results

5.1 Lemma. *Let $i,j,k \leq 2$, $n \geq 0$, $x, x_1, \ldots, x_n$ be pairwise distinct variables and let p be a permutation of $\{1, \ldots, n\}$. Put*

$$V = \mathrm{M}(x^i x_1 \ldots x_{n-1} x_n^j \approx x^k x_{p(1)} \ldots x_{p(n)} x).$$

Then either $V \subseteq \mathscr{T}$ or $V = S_{r,s}$ or $V = R_{t,q}$ for some T and q.

Proof. We distinguish six cases.

(i) $n = 0$. Then either $S = \mathscr{L}$ or $V = S_{2,9}$ or $V = \mathscr{I}$.

(ii) $n \geq 1$ and $i = j = k = 2$. Then 3.7(i) can be applied.

(iii) $n \geq 1$, $i = k = 2$ and $j = 1$. By 3.4(i), $V \subseteq \mathscr{R}$ and then clearly $V = \mathscr{R} \cap U$ where

$$U = \mathrm{M}(x^i x_1 \ldots x_{n-1} x_n^2 \approx x^2 x_{p(1)} \ldots x_{p(n)} x).$$

But $U = S_{4,s}$ for some s and $V = R_{6,s}$.

(iv) $n \geq 1$ and $i + k = 3$. By 3.3(ii), $V \subseteq \mathscr{T}$.

(v) $n \geq 1$, $i = k = 1$ and $j = 2$. If $p(1) \neq 1$, then $V \subseteq \mathscr{T}$ due to 3.3(iii). Now we can assume that $p(1) = 1$. Consider first the case when p is the identity. Then it is easy to see that $V \subseteq S_{3,8}$. Conversely, if $S \in S_{3,8}$ and $a, b_1, \ldots, b_n \in S$, then

$$ab_1 \ldots b_n^2 = a(b_1 \ldots b_n)^2 = ab_1 \ldots b_n a$$

and $S \in V$. Now, let p be non-identical. Using similar arguments as in the last case, we see that $V = S_{3,4}$.

(vi) $n \geq 1$ and $i = j = k = 1$. Then $V \subseteq \mathscr{R}$,

$$V = \mathscr{R} \cap \mathrm{M}\left(xx_1 \dots x_{n-1}x_n^2 \approx xx_{p(1)} \dots x_{p(n)}x\right)$$

and either $V = R_{5,8}$ or $V = R_{5,4}$ by (v). $\square$

5.2 Lemma. *Let $i,j \leq 2$, $n \geq 0$, $x, x_1, \dots, x_n$ be pairwise distinct variables and let p be a permutation of $\{1, \dots, n\}$. Put*

$$V = \mathrm{M}\left(x^i x_1 \dots x_n x \approx x^j x_{p(1)} \dots x_{p(n)} x\right).$$

Then either $V \subseteq \mathscr{T}$ or $V = S_{4,9}$ or $V = S_{4,7}$.

Proof. It is similar to the proof of 5.1. $\square$

5.3 Lemma. *Let $i,j,k \leq 2 \leq n$, $1 \leq q < n$, $x, x_1, \dots, x_n$ be pairwise distinct variables and let p be a permutation of $\{1, \dots, n\}$. Put*

$$V = \mathrm{M}\left(x^i x_1 \dots x_{n-1} x_n^j \approx x^k x_{p(1)} \dots x_{p(n)} x_{p(q)}\right).$$

Then either $V \subseteq \mathscr{T}$ or $V = S_{4,r}$ or $V = R_{6,r}$ for some r.

Proof. We distinguish five cases.

(i) $i = j = k = 2$. In this case we can use 3.7(i).

(ii) $i = k = 2$ and $j = 1$. Clearly, $V \subseteq \mathscr{R}$ and we can use 3.8.

(iii) $i + k = 3$. Then $V \subseteq \mathscr{T}$.

(iv) $i = k = 1$ and $p(1) \neq 1$. Then $V \subseteq \mathscr{T}$ by 3.2.

(v) $i = k = 1$ and $p(1) = 1$. If $j = 2$, then we can use 3.8. If $j = 1$, then $V \subseteq \mathscr{R}$ and we can again use 3.8. $\square$

5.4 Lemma. *Let $i,j \leq 2 \leq n$, $1 \leq r,s < n$, $x, x_1, \dots, x_n$ be pairwise distinct variables and let p be a permutation of $\{1, \dots, n\}$. Put*

$$V = \mathrm{M}\left(x^i x_1 \dots x_n x_r \approx x^j x_{p(1)} \dots x_{p(n)} x_{p(s)}\right).$$

Then either $V \subseteq \mathscr{T}$ or $V = S_{4,q}$ or $V = S_{6,q}$ for some q.

Proof. It is similar to the proof of 5.3.

5.5 Lemma. *Let $i,j \leq 2 \leq n$, $1 \leq k < n$, $x, x_1, \dots, x_n$ be pairwise distinct variables and let p be a permutation of $\{1, \dots, n\}$. Put*

$$V = \mathrm{M}\left(x^i x_1 \dots x_n x \approx x^j x_{p(1)} \dots x_{p(n)} x_{p(k)}\right).$$

Then either $V \subseteq \mathscr{T}$ or $V = S_{r,s}$ for some r, s or $V = R_{t,s}$ for some t, s.

Proof. Clearly, $V \cap \mathscr{I} = \mathscr{I}_8$ and

$$V \subseteq \mathrm{M}\left(x_{p(k)}^3 \dots x_{p(n)}^3 x_{p(k)}^3 \approx x_{p(k)}^3 \dots x_{p(n)}^3\right).$$

Consequently, $V \subseteq U$ where

$$U = \mathrm{M}\left(x^i x_1 \dots x_n x \approx x^j x_{p(1)} \dots x_{p(n)}\right)$$

and $V = U \cap S_{4,8}$. The result now folows from 5.1. $\square$

5.6 Lemma. *Let r, s be two words such that* $\mathrm{var}(r) = \mathrm{var}(s)$ *and* $o(r) = o(s)$. *Put* $V = \mathrm{M}(r \approx s)$. Then either $V \subseteq \mathscr{T} \cap \mathscr{R}$ *or* $V = T_{i,j}$ *or* $V = R_{p,q}$ *or* $V = S_{n,m}$ *for some* i, j, p, q, n, m.

Proof. We can assume that $r, s \in F$. The result then follows from 3.13 and 5.1, ..., 5.5 □

5.7 Lemma. *Let r, s be two words such that* $\mathrm{var}(r) \neq \mathrm{var}(s)$ *and let* $V = \mathrm{M}(r \approx s)$. *Then either* $V = \mathscr{T} \cap \mathscr{R}$ *or* $V = R_{6,j}$ *or* $V = R_{4,j}$ *for some* j.

Proof. By 3.4(ii), $V \subseteq \mathscr{R}$ and we can assume that $o(r) = o(s)$; denote this variable by x. Recall that $o(w)$ is the first variable in a word w. The last variable in w will be denoted by $\bar{o}(w)$. We distinguish nine cases.

(i) $r = x^2p$ and $s = x^2q$ where p, q are two words with $o(p) \neq x \neq o(q)$. Then $V = R_{6,j}$ by 3.7(i).

(ii) $r = x^ip$ and $s = x^2q$ where p, q are two words with $o(p) \neq x \neq o(q)$ and $i + j = 3$. Then $V \subseteq \mathscr{T} \cap \mathscr{R}$ by 3.3(ii).

(iii) $r = xp$ and $s = xq$ where p, q are two words with $o(p) = o(q) \neq x$ and $\bar{o}(p) \neq x \neq \bar{o}(q)$. Then we can assume that $x \notin \mathrm{var}(pq)$ and the result follows from 3.8.

(iv) $r = xp$ and $s = xq$ where p, q are two words with $x \neq o(p) \neq o(q) \neq x$. Then $V \subseteq \mathscr{T} \cap \mathscr{R}$ by 3.3(iii).

(v) $r = xp$ and $s = xq$ where p, q are two words with $o(p) = o(q) \neq x$ and $\bar{o}(p) \neq x = \bar{o}(q)$. We can assume that $p = x_1 \dots x_n$, $x \notin \mathrm{var}(p)$, $q = y_1 \dots y_m x$, $x_1 = y_1$, $x \neq y_i$. Then $V \cap \mathscr{I} = \mathscr{I}_1$ and it is easy to see that $V = R_{6,1}$.

(vi) $r = xp$ and $s = xq$ where p, q are two words with $o(p) = o(q) \neq x = \bar{o}(p) = \bar{o}(q)$. We can assume that $p = x_1 \dots x_n x$, $q = y_1 \dots y_m x$, $x_1 = y_1$. Then $V \cap \mathscr{I} = \mathscr{I}_5$ and $V = R_{6,5}$.

(vii) $r = x$. Then $V \subseteq \mathscr{I}$.

(viii) $r = x^3$ and $s = x^iq$ where q is a word with $o(q) \neq x$. If $i = 1$, then $V \subseteq \mathscr{T} \cap \mathscr{R}$ by 3.3(ii). If $i = 2$, then 3.7(i) can be used.

(ix) $r = x^2$ and $s = x^iq$ where q is a word with $o(q) \neq x$. Then $V \subseteq S_2$ and $V = \mathrm{M}(x^3 \approx s) \cap S_2$. The result now follows from (viii). □

5.8 Proposition. *Let r, s be two words and let* $V = \mathrm{M}(r \approx s)$. *Then either* $V \subseteq \mathscr{R} \cap \mathscr{T}$ *or* $V = R_{i,j}$ *or* $V = T_{i,j}$ *or* $V = S_{i,j}$ *for some* i, j.

Proof. Apply 3.3, 5.6 and 5.7. □

V.6 The lattice of subvarieties of $\mathscr{R}$

6.1 Lemma. *The following are true:*

(i) $R_{1,j} \cap \mathscr{A} = R_{2,j} \cap \mathscr{A} = \mathscr{A}_1$, $R_{3,j} \cap \mathscr{A} = R_{4,j} \cap \mathscr{A} = \mathscr{A}_4$, $R_{5,j} \cap \mathscr{A} = R_{6,j} \cap \mathscr{A} = \mathscr{A}_5$, $R_{1,j} \cap \mathscr{I} = R_{3,j} \cap \mathscr{I} = R_{5,j} \cap \mathscr{I} = \mathscr{I}_j \cap \mathscr{I}_8$ *and* $R_{2,j} \cap \mathscr{I} = R_{4,j} \cap \mathscr{I} = \mathscr{R}_{6,j} \cap \mathscr{I} = \mathscr{I}_j$ *for every* $0 \leq j \leq 9$.

(ii) $R_{2,j} = \mathscr{A}_1 \vee \mathscr{I}_j$, $R_{4,j} = \mathscr{A}_4 \vee \mathscr{I}_j$, $R_{6,j} = \mathscr{A}_5 \vee \mathscr{I}_j$ *for every* $j \in \{0,2,3,6\}$.

(iii) $R_{1,0} = R_{1,3} = \mathscr{A}_1 \vee \mathscr{I}_0$, $R_{1,2} = R_{1,6} = \mathscr{A}_1 \vee \mathscr{I}_2$, $R_{3,0} = R_{3,3} = \mathscr{A}_4 \vee \mathscr{I}_0$, $R_{3,2} = R_{3,6} = \mathscr{A}_4 \vee \mathscr{I}_2$, $R_{5,0} = R_{5,3} = \mathscr{A}_5 \vee \mathscr{I}_0$ *and* $R_{5,2} = R_{5,6} = \mathscr{A}_5 \vee \mathscr{I}_2$.

(iv) $R_{1,j} = R_{2,j}$, $R_{3,j} = R_{4,j}$ *and* $R_{5,j} = R_{6,j}$ *for every* $j \in \{1,4,8\}$.

(v) $R_{i,k} = R_{i,j}$ *for* $i \in \{1,3,5\}$ *and* $(k,j) \in \{(1,5),(4,7),(8,9)\}$.

Proof. (i) is easy. In order to prove (ii), it is sufficient to show that $R_{6,6} \in \mathscr{T} \cap \mathscr{R}$. Let $S \in R_{6,6}$. We have $x^2y = x^2y^2$ and $efg = feg$ for all elements $x, y \in S$ and all idempotents $e, f, g \in S$. Hence $x^2y^2 = xx^3y^3y^3 = xy^3x^3y^3 = xy^2$.

(iii) follows from (ii). In order to prove (iv), it is sufficient to show that $R_{5,8} = R_{6,8}$. Let $S \in R_{6,8}$. We have $x^2y = x^2y^2$ and $efe = ef$ for all elements $x, y \in S$ and all idempotents $e, f \in S$. Hence $xyx = xy^3x^3 = xy^3x^3y^3 = xy^2$.

In order to prove (v), it is sufficient to show that $R_{5,8} = R_{5,9}$. Let $S \in R_{5,9}$. We have $x^2y = x^2y^2$ and $xy^2 = xyx$ for all elements $x, y \in S$. Then $efe = ef^2 = ef$ for all idempotents $e, f \in S$. □

6.2 Lemma. *Let* $1 \leq i,j \leq 6$ *and* $0 \leq r,\ s \leq 9$. *Then* $R_{i,r} \cap R_{j,s} = R_{p,q}$ *for some* p *and* q.

Proof. It is easy. □

6.3 Proposition. *We have the following inclusions between the varieties* $R_{i,j}$:

(i) $R_{i,j} \subseteq R_{p,q}$ *if* $R_i \subseteq R_p$ *and* $\mathscr{I}_j \subseteq \mathscr{I}_q$;

(ii) $R_{i,j} \subseteq R_{p,q}$ *if* $R_{i,j} = R_{p,q}$ *as described in 6.1.*

There are no other inclusions except those that follow by transitivity from these two cases.

Proof. The other inclusions would imply incorrect inclusions between subvarieties of $\mathscr{T} \cap \mathscr{R}$ (intersect bot sides with $\mathscr{T}$). □

6.4 Proposition. *The variety* $\mathscr{R}$ *has the following 62 subvarieties:*

$L_0, \ldots, L_{43}$,
$L_{62} = R_{1,1} = R_{2,1} = R_{1,5}$,
$L_{63} = R_{3,1} = R_{4,1} = R_{3,5}$,
$L_{64} = R_{5,1} = R_{6,1} = R_{5,5}$,
$L_{65} = R_{1,4} = R_{2,4} = R_{1,7}$,
$L_{66} = R_{3,4} = R_{4,4} = R_{2,7}$,
$L_{67} = R_{5,4} = R_{6,4} = R_{5,7}$,
$L_{68} = R_{2,5}$,
$L_{69} = R_{4,5}$,
$L_{70} = R_{6,5}$,
$L_{71} = R_{2,7}$,
$L_{72} = R_{4,7}$,
$L_{73} = R_{6,7}$,

$L_{74} = R_{1,8} = R_{2,8} = R_{1,9}$,
$L_{75} = R_{3,8} = R_{4,8} = R_{3,9}$,
$L_{76} = R_{5,8} = R_{6,8} = R_{5,9}$,
$L_{77} = R_{2,9}$,
$L_{78} = R_{4,9}$,
$L_{79} = R_{6,9} = \mathscr{R}$.

Proof. Let V be a subvariety of $\mathscr{R}$ such that $V \not\subseteq \mathscr{T}$. It follows from 5.8 and 6.2 that $V = R_{i,j}$ for some $1 \leq i \leq 6$ and $0 \leq j \leq 9$. According to 6.1, V is one of the varieties $L_{62}, \ldots, L_{79}$. Example I.2.5 shows that $L_{62} \not\subseteq \mathscr{T}$. □

V.7 The lattice of subvarieties of $\mathscr{L}$

7.1 Lemma. *The following are true:*

(i) $S_{1,j} \cap \mathscr{A} = S_{2,j} \cap \mathscr{A} = \mathscr{A}_4$, $S_{3,j} \cap \mathscr{A} = S_{4,j} \cap \mathscr{A} = \mathscr{A}_5$, $S_{1,j} \cap \mathscr{I} = S_{3,j} \cap \cap \mathscr{I} = \mathscr{I}_j \cap \mathscr{I}_8$, $S_{2,j} \cap \mathscr{I} = S_{4,j} \cap \mathscr{I} = \mathscr{I}_j$ *for every* $0 \leq j \leq 9$.

(ii) $S_{1,0} = S_{2,0} = S_{1,3} = \mathscr{A}_4 \vee \mathscr{I}_0$, $S_{3,0} = S_{4,0} = S_{3,3} = \mathscr{A}_5 \vee \mathscr{I}_0$, $S_{2,3} = \mathscr{A}_4 \vee \vee \mathscr{I}_3$ *and* $S_{4,3} = \mathscr{A}_5 \vee \mathscr{I}_3$.

(iii) $S_3 \cap \mathscr{T} = T_{3,8}$.

(iv) $S_{1,2} = S_{2,2} = S_{1,6} = T_{2,2}$, $S_{3,2} = S_{4,2} = S_{3,6} = T_{3,2}$, $S_{2,6} = T_{2,6}$ *and* $S_{4,6} = T_{3,6}$.

(v) $S_{1,1} = S_{2,1} = R_{3,1}$, $S_{3,1} = S_{4,1} = R_{5,1}$, $S_{1,5} = R_{3,1}$, $S_{3,5} = R_{5,1}$, $S_{2,5} = R_{4,5}$ *and* $S_{4,5} = R_{6,5}$.

Proof. It is easy. □

7.2 Lemma. *Let* $0 \leq i \leq 9$ *and* $\mathscr{I}_j = \mathscr{I}_i \cap \mathscr{I}_8$. *Then* $S_{1,i} = S_{1,j}$ *and* $S_{3,i} = = S_{3,j}$.

Proof. It is easy. □

7.3 Lemma. *Let* $i \in \{0,1,2,4,8\}$. *Then* $S_{1,i} = S_{2,i}$ *and* $S_{3,i} = S_{4,i}$.

Proof. It is easy. □

7.4 Lemma. *Let* $1 \leq i,j \leq 4$ *and* $0 \leq r,s \leq 9$. *Then* $S_{i,r} \cap S_{j,s} = S_{p,q}$ *for some* p *and* q.

Proof. It is easy. □

7.5 Proposition. *We have the following inclusions between the varieties* $S_{i,j}$:

(i) $S_{i,j} \subseteq S_{p,q}$ *if* $S_i \subseteq S_p$ *and* $\mathscr{I}_j \subseteq \mathscr{I}_q$;

(ii) $S_{i,j} \subseteq S_{p,q}$ *if* $S_{i,j} = S_{p,q}$ *according to 7.1, 7.2 or 7.3.*

There are no other inclusions except those that follow by transitivity from these two cases.

Proof. It is easy. □

7.6 Theorem. *The variety $\mathscr{L}$ has the following 88 subvarieties:*
$L_0, \ldots, L_{79}$,
$L_{80} = S_{1,4}$,
$L_{81} = S_{3,4}$,
$L_{82} = S_{2,7}$,
$L_{83} = S_{4,7}$,
$L_{84} = S_{1,8}$,
$L_{85} = S_{3,8}$,
$L_{86} = S_{2,9}$,
$L_{87} = S_{4,9} = \mathscr{L}$.

Proof. Apply 5.8 and 7.1, ..., 7.5. □

The lattice of varieties of LD-semigroups is pictured in Fig. 3. An element labeled i in the picture represents the variety L_i $(i = 0, \ldots, 87)$.

V.8 Comments and open problems

The main result of this chapter (Theorem 7.6), i.e., description of the lattice of varieties of LD-semigroups, is adopted from [Kep,81]. Now, given a property defined for a semigroup variety, an open problem may be to determine which of the varieties L_i $(i = 0, \ldots, 87)$ enjoy this property.

List of symbols

$a(n)$	II.2.1
$a(n,m)$	II.2.1
$\mathscr{A}$	III.2.1
$\mathscr{A}_0, \ldots, \mathscr{A}_5$	III.2.1
$b(n)$	II.2.1
$f_1, \ldots, f_{16}$	III.3.1 – III.3.5
F	II.1.1
$\mathbf{F}$	II.1.1
$\mathscr{I}$	IV.2.1
$\mathscr{I}_0, \ldots, \mathscr{I}_9$	IV.2.1
$L_0, \ldots, L_{43}$	V.1.6
$L_{44}, \ldots, L_{61}$	V.4.5
$L_{62}, \ldots, L_{79}$	V.6.4
$L_{80}, \ldots, L_{87}$	V.7.6
$\mathrm{LA}(S)$	IV.3.8
$\mathrm{M}(u_1 \approx v_1, \ldots)$	V.2.1
R_i	V.2.1
$R_{i,j}$	V.2.1

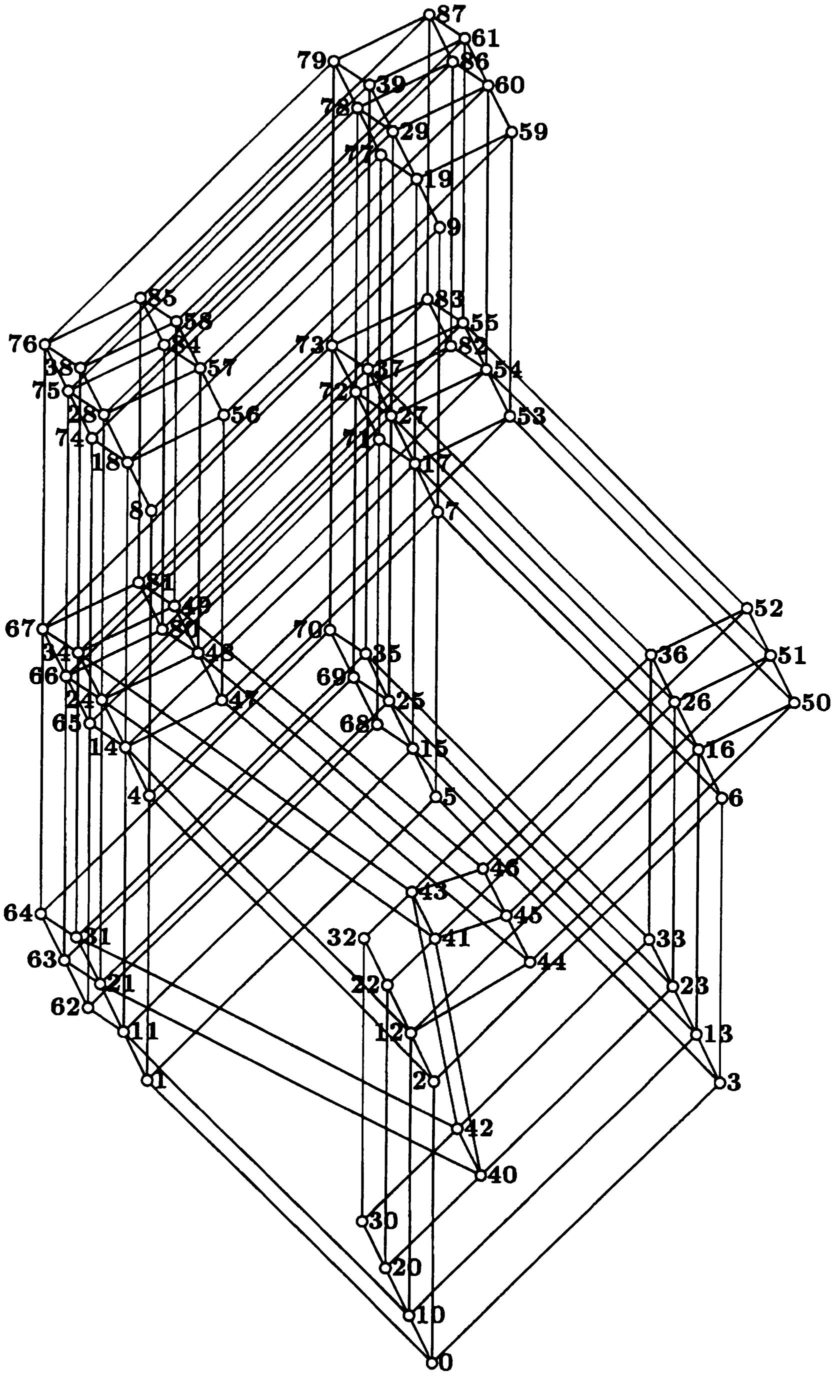
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Fig. 3

$\mathscr{R}$	V.1.1
S_i	V.2.1
$S_{i,j}$	V.2.1
T_i	V.2.1
$T_{i,j}$	V.2.1
$\mathscr{T}$	V.1.1
W_1, W_2	V.3.1

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