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A New Simple Approach to Linear Dependence

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Ukazuje se, jak lze dokázat, že m daných vektorů z F^m je lineárně závislých a jak lze v tomto případě nalézt netriviální lineární kombinaci rovnou nulovému vektoru bez řešení soustavy rovnic.

On montre comment on peut prouver que m vecteurs donnés de F^m sont linéairement dépendants et comment on peut trouver une combinaison linéaire non-triviale égale au vecteur nul dans ce cas sans résoudre aucun système des equations.

We show how to prove that m given vectors of F^m are linearly dependent and how to find a non-trivial linear combination of these vectors giving the zero vector in this case without solving any system of equations.

Let

$$\mathbf{u}_1 = (a_{11}, a_{12}, \dots, a_{1m}), \mathbf{u}_2 = (a_{21}, a_{22}, \dots, a_{2m}), \mathbf{u}_m = (a_{m1}, a_{m2}, \dots, a_{mm})$$

be vectors of F^m (where F denotes a field), let $A = (a_{ij}) \in F^{m \times m}$ be the matrix formed by $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ and let Δ be the determinant of A so that
(i) $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ *are linearly dependent if and only if* $\Delta = 0$.

Lemma 1. *If* $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ *are the vectors of* F^m *specified above and if* A_{ij} *denotes the* (i, j)*-cofactor of* A*, then*

$$(1) \qquad \sum_{i=1}^{m} A_{i1}\mathbf{u}_i = (\Delta, 0, \dots, 0)$$

$$(2) \qquad \sum_{i=1}^{m} A_{i2}\mathbf{u}_i = (0, \Delta, \dots, 0)$$

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(m) $$\sum_{i=1}^{m} A_{im}\mathbf{u}_i = (0, 0, \ldots, \Delta).$$

Proof. Note that the i-th component of the row vector $\sum_{s=1}^{m} A_{st}\mathbf{u}_s$ is equal to $\sum_{s=1}^{m} a_{si}A_{st}$. Hence it is equal to 0 whenever $i \neq t$ and it is equal to Δ if $i = t$. □

Theorem 2. *Let* $\mathbf{o} = (0, 0, \ldots, 0)$ *be the zero vector of* F^m. *The following statements are equivalent:*

(1′) $$\sum_{i=1}^{m} A_{i1}\mathbf{u}_i = \mathbf{o};$$

(2′) $$\sum_{i=1}^{m} A_{i2}\mathbf{u}_i = \mathbf{o};$$

$$\cdots\cdots\cdots\cdots$$

(m′) $$\sum_{i=1}^{m} A_{im}\mathbf{u}_i = \mathbf{o}.$$

(m + 1) *the vectors* $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ *are linearly dependent.*

Remark 3. Form the matrix

$$A^* := \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1m} & u_1 \\ a_{21} & a_{22} & \ldots & a_{2m} & u_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \ldots & a_{mm} & u_m \end{pmatrix}$$

where $u_1, u_2, \ldots, u_m$ denote some elements of F. Notice that (1′), …, (m′) can be formally obtained by the expansion of the determinants

$$\begin{vmatrix} a_{12} & \ldots & a_{1m} & u_1 \\ a_{22} & \ldots & a_{2m} & u_2 \\ \cdots & \cdots & \cdots & \cdots \\ a_{m2} & \ldots & a_{mm} & u_m \end{vmatrix}, \ldots, \begin{vmatrix} a_{11} & \ldots & a_{1m-1} & u_1 \\ a_{21} & \ldots & a_{2m-1} & u_2 \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \ldots & a_{mm-1} & u_m \end{vmatrix}$$

by their last comumn.

Proof of Theorem 2. By (i), (m + 1) is true if and only if $\Delta = 0$. Using Lemma 1, we can see that (m + 1) is valid if and only if any relation of (1′), …, (m′) is satisfied. □

Example 4. Determine whether or not the vectors

$$\mathbf{u} = (1, 2, -9), \mathbf{v} = (2, 1, 3) \quad \text{and} \quad \mathbf{w} = (4, 3, -1)$$

of $\mathbb{R}^3$ are linearly dependent.

Solution. Let

$$A^* := \begin{pmatrix} 1 & 2 & -9 & u \\ 2 & 1 & 3 & v \\ 4 & 3 & -1 & w \end{pmatrix}$$

where u, v and w denote some elements of $\mathbb{R}$. By Remark 3, we delete the third column in A^* and consider the determinant

$$\begin{vmatrix} 1 & 2 & u \\ 2 & 1 & v \\ 4 & 3 & w \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} u - \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} v + \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} w = 2u + 5v - 3w .$$

Following Theorem 2, it is now sufficient to check whether $2\mathbf{u} + 5\mathbf{v} - 3\mathbf{w}$ is equal to $\mathbf{o}$ or not.

However,

$$\begin{aligned} 2\mathbf{u} \ \ldots \quad & (2, \quad 4, -18) \\ 5\mathbf{v} \ \ldots \quad & (10, \quad 5, \quad 15) \\ -3\mathbf{w} \ \ldots \quad & (-12, -9, \quad 3) \end{aligned}$$

Adding the vectors in the both columns we find that

$$2\mathbf{u} + 5\mathbf{v} - 3\mathbf{w} = (0, 0, 0) = \mathbf{o}$$

and the problem is solved: The given vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent.

It may happen that all the determinants of order $m - 1$ in (1′) are equal to 0. Then Theorem 2 says that $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ are linearly dependent. However, it is natural to ask whether we can guarantee a way to a non-trivial linear combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ giving $\mathbf{o}$ also in this case. If there exist $m - 1$ linearly independent vectors between $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$, then it can be shown that at least one linear combination of those given in Theorem 2 is non-trivial.

More precisely, we have the following result:

Lemma 5. *Let* $1 \leq t \leq m$. *If the vectors* $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{t-1}, \mathbf{u}_{t+1}, \ldots, \mathbf{u}_m$ *are linearly independent, then at least one of the cofactors* $A_{t1}, A_{t2}, \ldots, A_{tm}$ *of the t-th column in Theorem 2 is non-zero.*

Proof. Let $t = 1$ for simplicity of subsequent expressions and suppose that $\mathbf{u}_2, \ldots, \mathbf{u}_m$ are linearly independent. Assume to the contrary that

$$\det A_{11} = 0 \quad \& \quad \det A_{12} = 0 \quad \& \ldots \& \quad \det A_{1m} = 0 .$$

It follows that the rank r of the matrix

$$\begin{pmatrix} a_{21} & a_{22} & \cdots & a_{2m} \\ a_{31} & a_{32} & \cdots & a_{3m} \\ \cdots & \cdots & \cdots & . \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix}$$

satisfies the inequality $r < m - 1$. Since $\mathbf{u}_2, \ldots, \mathbf{u}_m$ are linearly independent, we have $m - 1 \leq r$, a contradiction. $\square$

Example 6. Determine whether the vectors

$$\mathbf{u} = (4, 2, 3),\ \mathbf{v} = (-1, 6, 9),\ \mathbf{w} = (5, 10, 15)$$

of $\mathbb{R}^3$ are linearly dependent.

Solution. Let

$$A^* := \begin{pmatrix} 4 & 2 & 3 & u \\ -1 & 6 & 9 & v \\ 5 & 10 & 15 & w \end{pmatrix} (u, v, w \in \mathbb{R}).$$

Delete the first column in A^* and consider the corresponding determinant:

$$\begin{vmatrix} 2 & 3 & u \\ 6 & 9 & v \\ 10 & 15 & w \end{vmatrix} = \begin{vmatrix} 6 & 9 \\ 10 & 15 \end{vmatrix} u - \begin{vmatrix} 2 & 3 \\ 10 & 15 \end{vmatrix} v + \begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} w = 0u - 0v + 0w.$$

Since $0\mathbf{u} - 0\mathbf{v} + 0\mathbf{w} = \mathbf{o}$ is evidently true, Theorem 2 implies that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent.

If we want to find a non-trivial linear combination of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ giving $\mathbf{o}$, we observe that

$$\begin{vmatrix} 4 & 2 \\ -1 & 6 \end{vmatrix} \neq 0.$$

According to Lemma 5 we delete the third column in A^* and write

$$\begin{vmatrix} 4 & 2 & u \\ -1 & 6 & v \\ 5 & 10 & w \end{vmatrix} = \begin{vmatrix} -1 & 6 \\ 5 & 10 \end{vmatrix} u - \begin{vmatrix} 4 & 2 \\ 5 & 10 \end{vmatrix} v + \begin{vmatrix} 4 & 2 \\ -1 & 6 \end{vmatrix} w = -2(20u + 15v - 13w).$$

Thus, by Theorem 2, we have $20\mathbf{u} + 15\mathbf{v} - 13\mathbf{w} = \mathbf{o}$.