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Weak Convergence in Hoffmann-Jørgensen Sense

PETR LACHOUT

Praha*)

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This paper presents a general concept of the weak convergence in Hoffmann-Jørgensen's sense. The goal is in removing technical assumptions of the standard weak convergence which are not necessary. Just the limit has to possess "nice" properties; i.e. to be a finite Radon measure.

1 Introduction and Definitions

The weak convergence of probability measures is a helpful tool for mathematical statistics. But from the statisticians' point of view, the handling restricted on measures is just a technical difficulty. The only one important thing seems to be the quality of the limit. The limit is the base of each statistical test and allows investigation of estimators properties.

That is the reason for the concept due by Hoffmann-Jørgensen (1977). The proposed generalization was developed by Andersen & Dobrić (1989) for the space of all bounded functions equipped with supremal norm. They considered the limit concentrated in the set of all ϱ-continuous functions, where ϱ is a totally bounded pseudometric. General concept on Banach spaces is treated in Bickel, Klassen, Ritov and Wellner (1993). The present paper contributes with the definition of the weak convergence for general topological spaces.

In the sequel, we will use the following notation. E denotes the topological space, $\mathscr{G}(E)$ is the set of all open sets in E and $\mathscr{B}(E)$ is the set of all Borel sets in E. We will consider finite non-negative monotone supadditive set functions defined on $\mathscr{G}(E)$ (FNNMSA) instead of finite Borel measures.

Definition 1. *A set function μ is called FNNMSA on E if it is defined on $\mathscr{G}(E)$, $0 \leq \mu(G) \leq \mu(Q) \leq \mu(E) < +\infty$ if $G \subset Q$ are open sets and $\mu(G) + \mu(Q) \leq \mu(G \cup Q)$ for each couple of disjoint open sets G, Q.*

*) Institute of Information Theory and Automation, Czech Academy of Sciences, Pod vodárenskou věží 4, 182 08 Prague, Czech Republic

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FNNMSA's are reasonable generalization of probability measures. If X is a random mapping into E then its distribution μ_X, defined by $\mu_X(A) = \mathrm{Prob}_*(X \in A)$, is a FNNMSA. Moreover, μ_X is submodular and $\mu_X(E) = 1$, but we do not need that fact.

We connect FNNMSA's and measures by the following two constructions.

Definition 2. *Let μ be a FNNMSA on E. We define two set functions on $\mathscr{B}(E)$ related to μ:*

$$\tilde{\mu}(A) = \inf\{\mu(G) : A \subset G \textit{ open set}\} \textit{ is called the regularization of } \mu \tag{1}$$

and

$$\hat{\mu}(A) = \sup\left\{\inf\left\{\sum_{i=1}^{n}\mu(G_i) : K \subset \bigcup_{i=1}^{n} G_i;\, G_1, \ldots, G_n \textit{ open}\right\} : A \supset K \textit{ compact}\right\} \tag{2}$$

is called the Radonization of μ.

Proposition 1. *Let $E|_{\mathscr{G}(E)}$ be Hausdorff space and μ be a FNNMSA on E. Then its Radonization $\hat{\mu}$ is a finite Radon measure.*

Proof. Define the set function

$$v(G) = \inf\left\{\sum_{n=1}^{n}\mu(G_i) : G \subset \bigcup_{i=1}^{n} G_i;\, G_1, \ldots, G_n \text{ open}\right\}$$

for each open set G. This set function is monotone, subadditive and additive on $\mathscr{G}(E)$; i.e. $0 \leq v(G) \leq v(G \cup Q) \leq v(G) + v(Q)$ for each pair of open sets and $v(G \cup Q) = v(G) + v(Q)$ whenever G, Q are disjoint. Topsøe (1970) calls such function the content on $\mathscr{G}(E)$ and he has shown the existence of its inner Radon measure, see Topsøe (1970), theorem 6.2, p. 29. The inner Radon measure of v coincides with $\hat{\mu}$.

Q.E.D.

Let us recall the definition of Radon measures.

Definition 3. *A measure μ on a topological space E is called Radon if $\mu(K) < +\infty$ for all compact and*

$$\mu(A) = \sup\{\mu(K) : A \supset K \textit{ compact in } E\} \textit{ for each Borel set } A\,. \tag{3}$$

2 The Weak convergence

The standard weak convergence can be generalized in the intention proposed by Hoffmann-Jørgensen (1977).

Definition 4. *Let μ_α be a net of FNNMSA's on E, μ be another FNNMSA on E and S be a Borel subset of E. We will say that μ_α converges weakly to μ in (S, E) in the Hoffmann-Jørgensen's sense, notation is*

$$\mu_\alpha \xrightarrow[\alpha]{\text{HJ-w}} \mu \textit{ in } (S, E), \tag{4}$$

if

$$\liminf_\alpha \mu_\alpha (G) \geq \mu(G) \textit{ holds for each open set } G \tag{5}$$

and

$$\lim_\alpha \mu_\alpha (G) = \mu(G) = \mu(E) \textit{ whenever the open set } G \textit{ contains the Borel set } S\,. \tag{6}$$

If μ_α, μ are finite Radon measures, $S = E$ is a Hausdorff space, then the introduced convergence coincides with the standard weak convergence, see Topsøe (1970) or Berg, Christensen & Ressel (1984). The convergence keeps properties familiar for the weak convergence.

Corollary 1. *Let $f: E \to E'$ be a continuous mapping between two topological spaces, S be a Borel subset of E and S' be a Borel subset of E', $f(S) \subset S'$. If*

$$\mu_\alpha \xrightarrow[\alpha]{\text{HJ-w}} \mu \textit{ in } (S, E) \textit{ then } \mu_\alpha \circ f^{-1} \xrightarrow[\alpha]{\text{HJ-w}} \mu \circ f^{-1} \textit{ in } (S', E') \tag{7}$$

The assertion is evident, because of continuity. The introduced convergence also fulfills the projection property.

Corollary 2. *Let S and E' be Borel subsets of a topological space E, $S \subset E' \subset E$, μ_α be a net of FNNMSA's on E and μ be another FNNMSA on E, fulfilling $\mu_\alpha(G) = \mu_\alpha(Q)$, $\mu(G) = \mu(Q)$ whenever $G \cap E' = Q \cap E'$. Then*

$$\mu_\alpha \xrightarrow[\alpha]{\text{HJ-w}} \mu \textit{ in } (S, E) \tag{8}$$

if and only if

$$\mu_\alpha|_{E'} \xrightarrow[\alpha]{\text{HJ-w}} \mu|_{E'} \textit{ in } (S, E')\,, \tag{9}$$

where $.|_{E'}$ denotes the restriction to the set E'; i.e. $\nu|_{E'}(G \cap E') = \nu(G)$.

Proof. The restrictions $\mu_\alpha|_{E'}$, $\mu|_{E'}$ are well defined and FNNMSA on E, E' respectively. We have

$$\liminf_\alpha \mu_\alpha(G) = \liminf_\alpha \mu_\alpha|_{E'}(G \cap E') \geq \mu(G) = \mu|_{E'}(G \cap E')$$

for each open set G. Remember, that $G \cap E'$ represents all open sets in E'.

Q.E.D.

The mass of the regularization of the limit FNNMSA is concentrated in the set S.

Corollary 3. *Let*

$$\mu_\alpha \xrightarrow[\alpha]{\text{HJ-w}} \mu \textit{ in } (S, E) \tag{10}$$

then $\bar{\mu}(S) = \bar{\mu}(E) = \mu(E)$.

Proof. Let $S \subset G$ be open set. Then we get

$$\lim_\alpha \mu_\alpha(G) = \mu(G) = \mu(E)\,.$$

Therefore, $\tilde{\mu}(S) = \tilde{\mu}(E) = \mu(E)$ because of the definition of the regularization.

Q.E.D.

Some limit FNNMSA exists almost always. But usually, it is not determined uniquely. We receive the whole interval of FNNMSA's.

Corollary 4. *Let μ_α be a net of FNNMSA's on E and S be a Borel subset of E. Then a limit FNNMSA exists if and only if the following limits exist and are equal*

$$\lim_\alpha \mu_\alpha(E) = \lim_\alpha \mu_\alpha(G) \textit{ for each open set } G \supset S\,. \tag{11}$$

Every limit FNNMSA μ fulfills $\underline{\mu} \leq \mu \leq \overline{\mu}$, where

$$\overline{\mu}(G) = \liminf_\alpha \mu_\alpha(G) \textit{ for each open set } G \tag{12}$$

and

$$\underline{\mu}(G) = \begin{cases} \lim_\alpha \mu_\alpha(E) & \textit{for each open set } G \supset S \\ 0 & \textit{otherwise} \end{cases} \tag{13}$$

Proof. $\underline{\mu}$ and $\overline{\mu}$ are FNNMSA's on E. The condition (11) is then necessary and sufficient for them being the required limit.

Q.E.D.

There is at most one finite Radon measure being the limit.

Proposition 2. *Let μ be a finite Radon measure on the topological space E, the factor space $E|_{\mathscr{G}(E)}$ be a Hausdorff space and ν be a FNNMSA on E. If $\mu(G) \leq \nu(G)$ for each open set G and $\mu(E) = \nu(E)$ then $\mu \equiv \hat{\nu}$.*

Proof. Evidently, $\mu \leq \hat{\nu}$ and $\hat{\nu}(E) = \mu(E)$. $\hat{\nu}$ is a finite Radon measure, according to Proposition 1. Therefore, $\mu \equiv \hat{\nu}$.

Q.E.D.

For more detailed study on Radon measures, we recommend the book of Topsøe (1970) or Berg & Christensen & Ressel (1984). We will just need that the Radon limit is really determined uniquely.

Theorem 1. *Let E be a topological space, S be a Borel subset of E and the factor space $E|_{\mathscr{G}(E)}$ be a Hausdorff space. Let μ_α, μ be FNNMSA's on E such that*

$$\mu_\alpha \xrightarrow[\alpha]{\text{HJ-w}} \mu \textit{ in } (S, E)$$

and μ be a finite Radon measure. Then μ is uniquelly determined and

$$\mu \equiv \hat{\nu}\,, \textit{ where } \nu(G) = \liminf_\alpha \mu_\alpha(G) \textit{ for each open set } G\,. \tag{14}$$

Moreover, $\mu(S) = \mu(E)$.

Proof. ν is a FNNMSA on E, $\nu(E) = \mu(E)$ and $\mu(G) \leq \nu(G)$ for all open sets. Thus, $\mu \equiv \hat{\nu}$ according to Proposition 2. μ is a finite Radon measure and then regular; i.e. $\tilde{\mu} \equiv \mu$. Hence, Corollary 3 given $\mu(S) = \nu(E)$.

Q.E.D.

If the quotient space is not Hausdorff the uniqueness fails.

Example 1. *Consider the space $E = \{a, b, c\}$ with topology $\mathscr{G} = \{\emptyset, \{a, b\}, \{a, c\}, \{a, b, c\}\}$ and the net $X_\alpha \equiv a$. Then every random mapping X taking values just in $\{b, c\}$ is the limit of this net, i.e.*

$$\mu_{X_\alpha} \xrightarrow[\alpha]{\text{HJ-w}} \mu_X \text{ in } (\{b,c\}, \{a,b,c\}).$$

Whenever X is measurable then μ_X is a Radon probability on $\{b, c\}$. Moreover, the subspace $\{b, c\}$ is a Hausdorff space but the space $\{a, b, c\}$ is not Hausdorff.

There is a connection between the HJ-weak convergence of FNNMSA's and the weak convergence of their Radonizations.

Definition 5. *Let μ_α be a net of FNNMSA's on a topological space E and S be a Borel subset of E. We will say that the net is eventually Radon compact in (S, E) if every subnet of μ_α has a futher subnet*

$$\nu_\beta \xrightarrow[\beta]{\text{HJ-w}} \nu \text{ in } (S, E) \text{ where } \nu \text{ is a finite Radon measure}. \tag{15}$$

Definition 6. *Let μ_α be a net of finite Radon measures on E. We say:*

- *the net is eventually compact if every subnet has a futher weakly convergent subnet;*
- *the net is eventually tight if*

$$\inf \left\{ \sup \left\{ \limsup_\alpha \mu_\alpha(E - G) : G \supset K \text{ open} \right\} : K \text{ compact} \right\} = 0. \tag{16}$$

Theorem 2. *Let μ_α be a net of FNNMSA's on the topological space E and S be a Borel subset of E. If the net of the Radonizations $\hat{\mu}_\alpha$ is eventually compact and $\lim_\alpha \hat{\mu}_\alpha(G) = \lim_\alpha \mu_\alpha(E)$ for each open set $G \supset S$ then the net μ_α is eventually Radon compact in (S, E).*

Proof. Let ν_β be a subnet of μ_α. The net $\hat{\nu}_\beta$ is eventually compact and therefore has a futher subnet

$$\hat{\nu}_{\beta_\iota} \xrightarrow[\iota]{\text{w}} \nu \text{ in } E.$$

Then

$$\nu_{\beta_\iota} \xrightarrow[\iota]{\text{HJ-w}} \nu \text{ in } (S, E) \text{ since } \liminf_\iota \nu_{\beta_\iota}(G) \geq \liminf_\iota \hat{\nu}_{\beta_\iota}(G) \geq \nu(G)$$

for each open set G and

$$\lim_\iota \nu_{\beta_\iota}(G) = \lim_\iota \hat{\nu}_{\beta_\iota}(G) = \nu(G) = \nu(E)$$

if G contains the set S.

Q.E.D.

Corollary 5. *Let μ_α be a net of FNNMSA's on the topological space E fulfilling the following two conditions:*

$$\lim_\alpha \mu_\alpha(E) = \lim_\alpha \mu_\alpha(G) \textit{ for each open set } G \supset S\,. \tag{17}$$

and

$$\inf\left\{\sup\left\{\limsup_\alpha \sup\left\{\inf\left\{\sum_{i=1}^k \mu_\alpha(G_i) : G_1 \cup \ldots \cup G_k \supset L \textit{ open sets}\right\}:\right.\right.\right.\right.$$
$$\left.\left.\left.\left. L \subset E - G \textit{ compact}\right\} : G \supset K \textit{ open}\right\} : K \textit{ compact}\right\} = 0\,. \tag{18}$$

Then the net μ_α is eventually Radon compact in (S, E).

Proof. Under the assumptions, the net of finite Radon measures $\hat{\mu}_\alpha$ is eventually tight and therefore eventually compact, see Topsøe (1970), theorem 9.1, p. 43. The net of μ_α is eventually Radon compact in (S, E) according to Theorem 2.

Q.E.D.

Unfortunately, the reverse assertion fails.

Example 2. *Let $E = [0, 1]$ with usual topology. Consider the sequence of FNNMSA's given by*

$$\mu_n(G) = \begin{cases} \lambda(G) & \textit{if } \lambda(G) \geq \frac{1}{n} \\ 0 & \textit{if } \lambda(G) < \frac{1}{n} \end{cases}$$

for each open set G, where λ denotes Lebesgue measure on the interval $[0, 1]$. Evidently,

$$\mu_n \xrightarrow[n]{\text{HJ-w}} \lambda \textit{ in } (E, E)$$

but $\hat{\mu}_n \equiv 0$.

Definition 7. *Let μ_α be a net of FNNMSA's on a topological space E. We denote the set of all its cluster points, being finite Radon measure, by* $\lim_\alpha \mu_\alpha$.

The set of all Radon cluster points is always closed.

Lemma 1. *Let μ_α be a net of FNNMSA's on a topological space E, then the set of all Radon cluster points* $\lim_\alpha \mu_\alpha$ *is a closed subset of all finite Radon measures.*

Proof. The set of all cluster points of μ_α is a closed subset of the space of all FNNMSA's, see any monograph on topology; e.g. Kelly (1955).

The set $\lim_\alpha \mu_\alpha$ is intersection of all cluster points with the set of all finite Radon measures, therefore itself is closed as the subset of all finite Radon measures.

Q.E.D.

Assume finite Radon measures instead of FNNMSA's, we receive that any eventually Radon compact net possesses compact set of all Radon cluster points. But the space E must be regular spaces for that.

Proposition 3. *Let S be a Borel subset of a regular topological space E. If μ_α is a net of finite Radon measures on E which is eventually Radon compact in (S, E) then the set $\lim_\alpha \mu_\alpha$ is a compact set.*

Proof. Let ν_β be a net of limits of the given net μ_α.
Define new indices $(\alpha, G) \in I$ if and only if G is an open set of finite Radon measures on E, $\mu_\alpha \in G$ and there is $\hat{\beta}$ such that $\nu_\beta \in G$ for any $\beta \geq \hat{\beta}$.
Set $\varrho_{\alpha, G} = \mu_\alpha$ whenever $(\alpha, G) \in I$.
The introduced net is actually a subnet of the net μ_α. Thus there exists a convergent subnet

$$\varrho_{\alpha_\iota, G_\iota} \xrightarrow[\iota]{\text{HJ-w}} \varrho \text{ in } (S, E) \text{ and } \varrho \text{ is a finite Radon measure.}$$

The space of all finite Radon measures on E is regular, since the space E is regular. Therefore, we may conclude that there is a subnet ν_{β_γ} weakly converging to ϱ. Consequently, the set of all finite Radon limits is a compact set.

Q.E.D.

3 Example

This section presents a simple example on the empirical distribution functions. We denote $D_-(I)$ the set of all function defined on the interval I which are left continuous and has right limit at each point of I. These functions are called cádlàg, too. $C(I)$ denotes the set of all continuous functions defined on the interval I and $C(I, \varrho)$ denotes the set of all ϱ-continuous functions defined on the interval I, where ϱ is a pseudometric on I. If $F : I \to \mathscr{R}_+$ is monotone we define the pseudometric $\varrho_F(t, s) = |F(t) - F(s)|$.

Let $\xi_1, \xi_2, \ldots$ be i.i.d. random variables with the distribution function F. Let us denote the empirical distribution function by $F_n(x) = \frac{1}{n}\sum_{i=1}^n I[\xi_i < x]$. Then we have the following observation.

Lemma 2.

$$\sqrt{n}(F_n - F) \xrightarrow[n]{\text{HJ-w}} B \circ F \ \textit{in} \ (C(\mathscr{R}, \varrho_F), (D_-(\mathscr{R}), \|.\|)), \tag{19}$$

where B is the standard Brownian bridge on $[0, 1]$, i.e. Gaussian process with continuous sample paths, zero mean and the covariance function $R(t, s) = t \wedge s(1 - t \vee s)$.

Proof. There is a classical result on uniformly distributed random variables, see Billingsley (1968). Let $\eta_1, \eta_2, \ldots$ are i.i.d. random variables with uniform distribution on the interval $[0, 1]$. Setting $G_n(x) = \frac{1}{n}\sum_{i=1}^{n} I[\eta_i < x]$ we receive the convergence

$$\sqrt{n}(G_n - Id_{[0,1]}) \xrightarrow{d} B \text{ in } (D_-([0, 1]), \text{Skorohod topology}).$$

Since the limit process possesses continuous sample paths, we have in the same time the convergence

$$\sqrt{n}(G_n - Id_{[0,1]}) \xrightarrow[n]{\text{HJ-w}} B \text{ in } (C([0, 1]), (D_-([0, 1]), \|.\|)),$$

see Andersen & Dobrić (1987) or the book of Bickel, Klassen, Ritov & Wellner (1993). Let us consider the transformation $\mathscr{F}(f) = f \circ F$. That transformation maps $(D_-([0, 1]), \|.\|)$ into $(D_-(\mathscr{R}), \|.\|)$ and is trivially continuous. Thus the weak convergence is preserved by $\mathscr{F}$ and we have

$$\sqrt{n}(G_n \circ F - F) \xrightarrow[n]{\text{HJ-w}} B \circ F \text{ in } (C(\mathscr{R}, \varrho_F), (D_-(\mathscr{R}), \|.\|)).$$

The processes $\sqrt{n}(G_n \circ F - F)$ and $\sqrt{n}(F_n - F)$ represents the same distribution on $(D_-(\mathscr{R}), \|.\|)$. Therefore the lemma is proved.

Q.E.D.

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