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Groupoids and the Associative Law III. (Szász – Hájek Groupoids)

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This paper deals with groupoids possessing just one non-associative triple of elements. The triple is of the form (a, a, a).

Článek se zabývá grupoidy, které mají právě jednu neasociativní trojici prvků. Tato trojice je tvaru (a, a, a).

In this paper (which is a free continuation of [3] and [4]), Szász – Hájek groupoids (i.e., groupoids with just one non-associative triple) are studied in more detail.

III.1 Introduction

1.1 A groupoid G will be called an SH-groupoid (Szász – Hájek groupoid) if $\mathrm{ns}(G) = 1$, i.e., if G possesses just one non-associative triple (see I.1.1). If this is so and if (a, b, c) is that triple, then exactly one of the following five cases takes place:

$a = b = c$ (and then we shall say that G is an SH-groupoid of type (a, a, a));
$a = b \neq c$ (type (a, a, b));
$a \neq b = c$ (type (a, b, b) – this type is dual to (a, a, b));
$a = c \neq b$ (type (a, b, a));
$a \neq b \neq c \neq a$ (type (a, b, c)).

1.2 Proposition. *Let G be an SH-groupoid and let $a, b, c \in G$ be such that $a \,.\, bc \neq ab \,.\, c$. Then:*
(i) If $x, y \in G$ are such that $xy = a$ $(xy = b, xy = c)$, then either $x = a$ $(x = b, x = c)$ or $y = a$ $(y = b, y = c)$.
(ii) If A is a non-empty generator set of G, then $\{a, b, c\} \subseteq A$.

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(iii) If H is a subgroupoid of G, then either $\{a,b,c\} \subseteq H$ and H is an SH-*groupoid of the same type as G or $\{a,b,c\} \nsubseteq H$ and H is a semigroup.*
(iv) If r is a congruence of G, then either $(a\,.\,bc,\ ab\,.\,c) \notin r$ and G/r is an SH-*groupoid of the same type as G or $(a\,.\,bc,\ ab\,.\,c) \in r$ and G/r is a semigroup.*

Proof. (i) If $x \neq a \neq y$, then $a\,.\,bc = xy\,.\,bc = x(y\,.\,bc) = x(yb\,.\,c) = (x\,.\,yb)\,c = (xy\,.\,b)\,c = ab\,.\,c$, a contradiction. The other cases are similar.
(ii) Let W be an absolutely free groupoid with a free basis X such that there exists a bijection $f: X \to A$. This bijection can be uniquely extended to a projective homomorphism $g: W \to G$. Now, suppose that $a \notin A$ and take $t \in W$ such that the length $l(t)$ of t is minimal with respect to the property that $g(t) = a$. Since $a \notin A$, $t \notin X$ and $t = rs$ for some $r, s \in W$. We have $l(r) < l(t)$, $l(s) < l(t)$ and either $f(r) = a$ or $f(s) = a$ (see (i)), which is a contradiction. We have proved that $a \in A$. Quite similarly, $b, c \in A$.
(iii) and (iv). These two assertions are obvious.

1.3 An SH-groupoid G is said to be minimal if every proper subgroupoid of G is associative (i.e., if no proper subgroupoid of G is an SH-groupoid).

1.4 For a groupoid G, let $\sigma(G)$ denote the smallest cardinal number α such that there exists a generator set A of G with $\mathrm{card}(A) = \alpha$. We have $0 \le \sigma(G)$ and $\sigma(G) = 0$ iff G contains no proper subgroupoid. Groupoids with $\sigma(G) \le 1$ are sometimes called cyclic.

1.5 Proposition. *Let G be an* SH-*groupoid.*
(i) If G is of type (a, a, a), *then $\sigma(G) \ge 1$ and G is minimal iff $\sigma(G) = 1$.*
(ii) If G is of type (a, a, b) *(or* (a, b, b), (a, b, a)*), then $\sigma(G) \ge 2$ and G is minimal iff $\sigma(G) = 2$.*
(iii) If G is of type (a, b, c), *then $(G) \ge 3$ and G is minimal iff $\sigma(G) = 3$.*

Proof. (i) Let $a \in G$ be such that $a\,.\,aa \neq aa\,.\,a$. Put $b = aa$. Then $b \neq a$. Now, let A be a generator set of G. If $A = \emptyset$, then $\{b\}$ is also a generator set, and hence $a \in \{b\}$ by 1.1(ii) and $a = b$, a contradiction. Thus $A \neq \emptyset$, $a \in A$ and $\mathrm{card}(A) \ge 1$. This means that $\sigma(G) \ge 1$. If $\sigma(G) = 1$, then G possesses a one-element generator set, and therefore $\{a\}$ is a generator set of G (again, by 1.1(ii)). In this case, if H is a proper subgroupoid of G, then $a \notin H$, and so H is associative. We have proved that G is minimal. Conversely, if G is minimal, then G is generated by a, so that $\sigma(G) = 1$.
(ii) and (iii). We can proceed similarly as in the proof of (i).

1.6 Proposition. *Let G be an* SH-*groupoid, let $a, b, c \in G$ be such that $a\,.\,bc \neq ab\,.\,c$ and let H be the subgroupoid generated by $\{a,b,c\}$. Then H is a minimal* SH-*groupoid and H is of the same type as G.*

Proof. Obvious.

2.1 Throughout this section, let G be an SH-groupoid of type (a, a, a). Further, let $a \in G$ be such that $a . aa \neq aa . a$ and put $b = aa$, $c = ab$, $d = ba$, $e = ac$, $f = ad$.

2.2 Lemma. *(i) If $x \in G$, then $ax = a$ iff $xa = a$.*
(ii) If $x, y \in G$ are such that $a = xy$, then either $x = a$ and $ay = ya = a$ or $y = a$ and $ax = xa = a$.
(iii) If $x, y, z \in G$ are such that $a = ax$ (resp. $a = xa$) and $x = yz$, then $a = ax = xa = ay = ya = az = za$ and $x \neq a$, $y \neq a$, $z \neq a$.

Proof. (i) Let $ax = a \neq xa$. Then $x \neq a$ (otherwise $aa = a$ and $a . aa = a = aa . a$) and $aa . a = (a . ax) a = (aa . x) a = aa . xa = a(a . xa) = a(ax . a) = a . aa$, a contradiction. Similarly, if $ax \neq a = xa$.
(ii) If $x \neq a \neq y$, then $aa . a = (a . xy) a = (ax . y) a = ax . ya = a(x . ya) = a(xy . a) = a . aa$, a contradiction.
(iii) By (i), $ax = xa = a$, and hence $x \neq a, b$ (otherwise $a . aa = aa . a$). This implies that either $y \neq a$ or $z \neq a$. If $z = a$, then $y \neq a$ and $yb = y . aa = ya . a = yz . a = xa = a$, a contradiction with (ii) (since $y \neq a \neq b$). Hence $z \neq a$ and, similarly, $y \neq a$. Further, $ay . z = a . yz = ax = a$ and $ay = a$ by (ii). Similarly, $za = a$. The rest is clear from (i).

2.3 Lemma. *(i) a, b, c, d are pair-wise different elements of G.*
(ii) $b = aa$, $c = ab = a . aa$, $d = ba = aa . a$.
(iii) $e = ac = da = bb = a(a . aa) = (aa . a) a = aa . aa$ and $e \neq a, b$.
(iv) $f = ca = ad = (a . aa) a = a(aa . a)$ and $f \neq a, b$.

Proof. (i) Since $c = ab = a . aa \neq aa . a = ba = d$, we have $c \neq d$ and also $a \neq b$. If $c = a$, then $d = a$ by 2.2(i), and so $c = d$, a contradiction. Thus $a \neq c$ and, similarly, $a \neq d$. If $bb = b$, then $c = ab = a . bb = a(aa . b) = a(a . ab) = a . ac = aa . c = bc = b . ab = ba . b = db = d . aa = da . a = (ba . a) a = (b . aa) a = bb . a = ba = d$, a contradiction. Hence $bb \neq b$ and, if $c = b$, then $b = ab = a . ab = aa . b = bb$, a contradiction. Thus $b \neq c$ and, similarly, $b \neq d$.
(ii) This is clear from the definition of b, c, d.
(iii) We have $e = ac = a . ab = aa . b = bb = b . aa = ba . a = da$. If $e = a$, then $bb = a$, a contradiction with 2.1(ii). The inequality $e = bb \neq b$ was already proved in (i).
(iv) We have $f = ca = ab . a = a . ba = ad$. If $f = a$, then $ca = a = ac$ by 2.2(i), a contradiction with (iii). If $f = b$, then $c = ab = af = a . ca = ac . a = ea = da . a = d . aa = db = ba . b = b . ab = bc = aa . c = a . ac = a(a . ab) = a(aa . b) = a . bb = ab . b = cb = ca . a = fa = ba = d$, a contradiction.

2.4 Lemma. *(i) $cx = dx$, $xc = xd$, $ex = fx$ and $xe = xf$ for every $x \in G$, such that $x \neq a \neq ax$.*

(ii) $bx = b = xb$, $cx = c = xc$, $dx = d = xd$, $ex = e = xe$ *and* $fx = f = xf$ *for every* $x \in G$ *such that* $a \neq ax$.
(iii) $ea = fa = ae = af$.

Proof. (i) We have $cx = ab . x = a . bx = a(aa . x) = a(a . ax) = aa . ax = b . ax = ba . x = dx$ and similarly, $xc = xd$. Further, $ex = ac . x = a . cx = a . dx = ad . x = fx$ and, similarly, $xe = xf$.
(ii) We have $x \neq a$ and the rest is clear.
(iii) We have $fa = ad . a = a . da = ae = a . bb = a(aa . b) = a(a . ab) = aa . ab = b . ab = ba . b = ba . aa = (ba . a)\, a = (b . aa)\, a = bb . a = ea = ac . a = a . ca = af$.

2.5 Lemma. *(i)* $c = e$ *iff* $c = f$.
(ii) $d = e$ *iff* $d = f$.

Proof. (i) If $c = e$, then $c = e = ac = ae = ea = ca = f$ (use 2.3(iii), (iv) and 2.4(iii)). Similarly, if $c = f$, then $c = f = ca = fa = af = ac = e$.
(ii) This is dual to (i).

2.6 Lemma. *(i)* $xc = xd = d$ *for every* $x \in G$ *such that* $xb = b$ *and* $ax \neq a$.
(ii) $cx = dx = c$ *for every* $x \in G$ *such that* $bx = b$ *and* $ax \neq a$.

Proof. (i) By 2.4(i), $xc = xd$. However, $xd = x . ba = xb . a = ba = d$.
(ii) This is dual to (i).

2.7 Lemma. *Suppose that there exists an element* $u \in G$ *such that* $ub = b$ $(bu = b)$ *and* $au \neq a$. *Then* $bx \neq b$ $(xb \neq b)$ *whenever* $x \in G$ *and* $ax \neq x$.

Proof. Let, on the contrary, $bv = b$ for some $v \in G$ such that $av \neq a$. Now, by 2.6, $c = dv = uc . v = u . cv = uc = d$, a contradiction.

2.8 Put $\mathrm{An}(G) = \{u \in G;\ au = a\} = \{u \in G;\ ua = a\}$ (see 2.2(i)), $\mathrm{Bn_l}(G) = \{u \in G;\ ub = b\}$ and $\mathrm{Bn_r}(G) = \{u \in G;\ bu = b\}$.

2.9 Proposition. *(i)* $\mathrm{An}(G)$ *(resp.* $\mathrm{Bn_l}(G)$, $\mathrm{Bn_r}(G)$*) is either empty or a subgroupoid of* G.
(ii) $\mathrm{An}(G) = \mathrm{Bn_l}(G) \cap \mathrm{Bn_r}(G)$.
(iii) If $\mathrm{Bn_l}(G) \neq \mathrm{An}(G)$, *then* $\mathrm{Bn_r}(G) = \mathrm{An}(G)$.
(iv) If $\mathrm{Bn_r}(G) \neq \mathrm{An}(G)$, *then* $\mathrm{Bn_l}(G) = \mathrm{An}(G)$.

Proof. (i) If $u, v \in \mathrm{An}(G)$, then $u \neq a \neq v$ and $uv . a = u . va = u . a = a$.
(ii), (iii) and (iv). Apply 2.4(ii) and 2.7.

2.10 Lemma. *Suppose that* G *is minimal. Then* $a \neq xy$ *for all* $x, y \in G$.

Proof. Let W be an absolutely free groupoid with a one-element free basis $\{w\}$ and let $f\colon W \to G$ be the projective homomorphism such that $f(w) = a$ (the groupoid G is generated by a). Suppose, on the contrary, that $a = xy$ for some

$x, y \in G$. In view of 2.2(iv) we can assume that $x = a$. We have $y = f(t)$ for some $t \in W$ and we can also assume that the length $l(t)$ is minimal with respect to $a = af(t)$. Since $a \neq b = aa$, $t \neq x$ and $t = rs$, $r, s \in W$. Then $a = a \, . \, uv$, $u = f(r)$, $v = f(s)$ and, by 2.2(iii), $a = au = av$, a contradiction with the minimality of $l(t)$.

2.11 We shall say that G is of subtype (α) (resp. (β)) if $e = f$ (resp. $e \neq f$). Hence, if G is of subtype (α), then G contains at least four different elements (namely a, b, c, d) and, if G is of subtype (β), then G contains at least six different elements (namely a, b, c, d, e, f).

2.12 Proposition. *Let* s_G *denote the least congruence of* G *such that the corresponding factor is associative.*
(i) If G *is of subtype* (α), *then* $s_G = \mathrm{id}_G \cup \{(c,d), (d,c)\}$.
(ii) If G *is of subtype* (β), *then* $s_G = \mathrm{id}_G \cup \{(c,d), (d,c), (e,f), (f,e)\}$.

Proof. Put $r = \mathrm{id}_G \cup \{(c,d), (d,c), (e,f), (f,e)\}$. It follows from the preceding results that r is a congruence of G. Clearly, G/r is associative, and hence $s_G \subseteq r$. On the other hand, $(c, d) = (a \, . \, aa, aa \, . \, a) \in s_G$ and $(e, f) \in s_G$. Thus $r = s_G$.

III.3 Construction of some SH-groupoids of type (a, a, a)

3.1 Let G be an SH-groupoid of type (a, a, a) and of subtype (α) and let a, b, c, d, e be as in 2.1 (we have $e = f$). Further, assume that the following condition is satisfied:

(SH1) If $x, y \in G$ are such that $xy = b$, then either $x = y = a$ or $y = b$ and $ax = a$.

Now, define a binary operation $*$ on G by $x * y = xy$ if $(x, y) \neq (b, a)$ and $b * a = c$. We are going to check that $G(*)$ is a semigroup. For, take $x, y, z \in G$ and consider the following cases:

(1) $(y, z) \neq (b, a)$ and $x \neq b$. Then $x * (y * z) = x \, . \, yz$ and $(x * y) * z = xy * z$. If $xy \neq b$, then $(x, y) \neq (a, a)$ and $x \, . \, yz = xy \, . \, z = xy * z$. If $xy = b$, then either $y = b$, $z \neq a$ and $x \, . \, yz = xy \, . \, z = xy * z$ or $x = y = a$. If $x = y = a$ and $z \neq a$, then $x \, . \, yz = xy \, . \, z = xy * z$. If $x = y = z = a$, then $x \, . \, yz = c = b * a = xy * z$.

(2) $(y, z) \neq (b, a)$ and $x = b$. Then $x * (y * z) = b * yz$ and $(x * y) * z = (b * y) * z$. If $yz = a = y$, then $b * yz = b * a = c = cz = c * z = (b * y) * z$. If $yz = a \neq y$, then $z = a$ and $b * yz = c = b * a = by * a = (b * y) * a = (b * y) * z$. If $yz \neq a = y$, then $b * yz = byz = baz = dz = cz = c * z = (b * a) * z = (b * y) * z$. If $yz \neq a \neq y$ and $by \neq b$, then $b * yz = byz = by * z = (b * y) * z$. If $yz \neq a \neq y$ and $by = b$, then $ay = a$, $z \neq a$ and $b * yz = byz = bz = b * z = (b * y) * z$.

(3) $(y, z) = (b, a)$. Then $x * (y * z) = x * c = xc$ and $(x * y) * z = (x * b) * a = = xb * a$. If $xb \neq b$ and $x \neq a$, then $xa \neq a$ and $xc = xd = xba = xb * a$. If $xb = b$, then $xa = a$ and $xc = c = b * a = xb * a$. If $x = a$, then $xc = ac = e = f = ca = c * a = ab * a = xb * a$.

We have proved that $G(*)$ is a semigroup. Clearly, $G = G(*)\,[b, a, d]$ (see II.2.1) and $\mathrm{sdist}(G) = 1$ (see II.1.1).

3.2 Let G be a semigroup containing two elements a, d such that the following conditions are satisfied:

(a) $a^2 \neq a \neq a^3$ and $a^2 \neq d \neq a^3$.
(b) If $x \in G$, then $ax = a$ iff $xa = a$.
(c) If $x, y \in G$ and $a = xy$, then either $x = a$ or $y = a$.
(d) If $x, y \in G$ and $xy = a^2$, then either $x = y = a$ or $x = a^2$ and $ax = a$.
(e) If $x \in G$ and $ax \neq a$, then $xd = xa^3$ and $dx = a^3x$.
(f) If $x \in G$ and $ax = a$, then $xd = dx = d$.

Now, put $G(\circledast) = G[a^2, a, d]$ (see II.2.1). Then $\mathrm{Ns}(G(\circledast)) = \{(a, a, a)\}$, and so $G(\circledast)$ is an SH-groupoid of type (a, a, a) (compare with 3.1). Clearly, $G(\circledast)$ is of subtype (α) and $\mathrm{sdist}(G(\circledast)) = 1$.

3.3 Let G be an SH-groupoid of the type (a, a, a) and of subtype (β) and let a, b, c, d, e, f be as in 2.1. Further, assume that the following two conditions satisfied:

(SH1) from 3.1
(SH2) If $x, y \in G$ are such that $xy = c$, then either $x = a$, $y = b$ or $x = c$ and $ay = a$ or $y = c$ and $ax = a$.

Now, define a binary operation $*$ on G by $x * y = xy$ if $(x, y) \neq (b, a), (c, a)$ and $b * a = c$, $c * a = b$. Then $G(*)$ is a semigroup (it requires just a tedious checking), and so $\mathrm{sdist}(G) \leq 2$. We show that $\mathrm{sdist}(G) = 2$, provided that $g = b$ whenever $g \in G$ and $gb = bg = e$.

Let, on the contrary, $G(\circ)$ be a semigroup such that $\mathrm{dist}(G, G(\circ)) = 1$. Then $u \circ v = w \neq uv$ for just one ordered pair (u, v). If $(u, v) \notin \{(a, a), (a, b), (b, a)\}$, then $a \,.\, aa = a(a \circ a) = a \circ (a \circ a) = (a \circ a) \circ a = (a \circ a)\, a = aa \,.\, a$, a contradiction. If $(u, v) = (a, a)$ and $g = a \circ a$, then $b \,.\, g = b \circ g = b \circ (a \circ a) = (b \circ a) \circ a = = (ba) \circ a = ba \,.\, a = e = bb = aa \,.\, b = a \,.\, ab = a(a \circ b) = a \circ (a \circ b) = = (a \circ a) \circ b = g \circ b = gb$. According to our hypothesis, $g = b$, and therefore $a \circ a = aa$, a contradiction. If $(u, v) = (a, b)$ and $a \circ b = g$, then $g = a \circ b = = a \circ (a \circ a) = (a \circ a) \circ a = b \circ a = ba = d$ and $e = bb = b \circ b = (a \circ a) \circ b = = a \circ (a \circ b) = a \circ g = a \circ d = ad = f$, a contradiction. Similarly, if $(u, v) = (b, a)$, then $g = b \circ a = (a \circ a) \circ a = a \circ (a \circ a) = a \circ b = ab = c$ and $e = ac = ag = = a \circ g = a \circ (b \circ a) = (a \circ b) \circ a = ab \,.\, a = ca = f$, a contradiction.

3.4 Let G be a semigroup containing three elements a, d, f such that the conditions (a), (b), (c), (d), (f) from 3.2 are satisfied and, moreover, the following are true:

(e′) If $x \in G$, $x \neq a$ and $ax \neq a$, then $xd = xa^3$ and $dx = a^3x$.
(g) $ad = f$ and $da = a^4$.
(h) $f \neq a^4$.
(i) If $x, y \in G$ and $xy = a^3$, then either $x = a$, $y = a^2$ or $x = a^2$, $y = a$ or $x = a^3$, $ay = a$ or $y = a^3$, $ax = a$.
(j) If $x \in G$ and $ax \neq a$, then $xf = xa^4$ and $fx = a^4x$.
(k) If $x \in G$ and $ax = a$, then $af = f = fa$.

Now, define a binary operation $\circledast$ on G by $x \circledast y = xy$ if $(x, y) \neq (a^2, a), (a^3, a)$ and $a^2 \circledast a = d$, $a^3 \circledast a = f$. Then $G(\circledast)$ is an SH-groupoid of the type (a, a, a) and subtype (β) (compare with 3.3).

III.4 A variety of "almost" associative groupoids

4.1 Denote by $\mathcal{R}_1$ the variety pf groupoids satisfying the following identities:

$$(xy \,.\, u)\, v \triangleq xy \,.\, uv, \quad x(y \,.\, uv) \triangleq xy \,.\, uv, \quad (x \,.\, yu)\, v \triangleq x(yu \,.\, v).$$

Clearly, $\mathcal{S} \subseteq \mathcal{R}_1$, where $\mathcal{S}$ denotes the variety of semigroups.

4.2 Throughout this section, let W be an absolutely free groupoid with a free basis X.

4.3 Lemma. *Let $t \in X$ be such that $\mathrm{l}(t) \geq 4$. Then there are $x \in X$ and $q \in X$ such that the identity $t \triangleq xq$ is satisfied in $\mathcal{R}_1$.*

Proof. We have $t = rs$ for some $r, s \in W$ and we can assume that $r \notin X$. Then $r = uv$, $u, v \in W$. If $u \in X$, then either $v = wz$ and $t = (u \,.\, wz)\, s \triangleq u(wz \,.\, s) = u \,.\, vs$ is satisfied in $\mathcal{R}_1$ or $v \in X$, $s = wz$ and $t = uv \,.\, wz \triangleq u(v \,.\, wz) = u \,.\, vs$ is satisfied in $\mathcal{R}_1$, too.

4.4. Lemma. *Let $r, s \in W$, $\mathrm{l}(r) \geq 5$. Then the identity $r \triangleq s$ is satisfied in $\mathcal{R}_1$ iff it is satisfied in $\mathcal{S}$.*

Proof. Assume that $r \triangleq s$ is true in $\mathcal{S}$. Then $\mathrm{l}(s) = \mathrm{l}(r) \geq 5$ and we shall proceed by induction on $\mathrm{l}(r)$. By 4.3, there are $x, x' \in X$ and $q, q' \in W$ such that the identities $r \triangleq xq$ and $s \triangleq x'q'$ are satisfied in $\mathcal{R}_1$. Then these identities are satisfied in $\mathcal{S}$, and hence $x = x'$ and $q \triangleq q'$ is satisfied in $\mathcal{S}$ (take into account that free semigroups are cancellative). If $\mathrm{l}(q) \geq 5$, then $q \triangleq q'$ is true in $\mathcal{R}_1$ by the induction hypothesis, and so $r \triangleq xq \triangleq xq' \triangleq s$ are satisfied in $\mathcal{R}_1$. Now, the remainig case is $\mathrm{l}(q) = \mathrm{l}(q') = 4$. Then there are $y, u, v, z \in X$ such that $q, q' \in \{y(u \,.\, vz), y(uv \,.\, z), yu \,.\, vz, (yu \,.\, v)\, z, (y \,.\, uv)\, z\}$ and $xq, xq' \in \{x(y(u \,.\, vz)), x(y(uv \,.\, z)), x(yz \,.\, vz), x((yu \,.\, v)\, z), x((y \,.\, uv)\, z)\}$. However, using the three identities from 4.1, it is easy to show that the following identities hold in $\mathcal{R}_1$: $x((yu \,.\, v)\, z) \triangleq x(yu \,.\, vz) \triangleq x(y(u \,.\, vz)) \triangleq (xy)\,(u \,.\, vz) \triangleq ((xy \,.\, u)\, v)\, z \triangleq (xy \,.\, uv)\, z \triangleq (xy)\,(uv \,.\, z) \triangleq x(y(uv \,.\, z)) \triangleq x((y \,.\, uv)\, z)$.

4.5 (i) Let F with a free basis A be a free groupoid in $\mathscr{R}_1$. Denote by s_F the smallest congruence of F such that F/s_F is a semigroup and let $f: F \to F/s_F$ be a natural projection. Then F/s_F is a free semigroup, $f(A)$ is its free basis and $f \mid A$ is injective.

Let $a \in A$ and let g be the endomorphism of F such that $g(A) = \{a\}$. Then $g(F)$ is a free $\mathscr{R}_1$-groupoid of rank 1 and $s_F \cap \ker(g) = \mathrm{id}_F$. This implies that F can be imbedded into the cartesian product $g(F) \times F/s_F$.
(ii) Let F be a free $\mathscr{R}$-groupoid of rank 1. It follows from (i) that the variety $\mathscr{R}_1$ is generated by $\mathscr{S} \cup \{F\}$.

4.6 Consider pair-wise different elements $a, b, c, d, e, f, g_5, g_6, g_7, \ldots$ and define a grouipoid $R_1(\circ)$ by the following multiplication table:

$R_1(\circ)$	a	b	c	d	e	f	g_5	g_6	g_7	g_8	$\cdots$
a	b	c	e	f	g_5	g_5	g_6	g_7	g_8	g_9	$\cdots$
b	d	e	g_5	g_5	g_6	g_6	g_7	g_8	g_9	g_{10}	$\cdots$
c	f	g_5	g_6	g_6	g_7	g_7	g_8	g_9	g_{10}	g_{11}	$\cdots$
d	e	g_5	g_6	g_6	g_7	g_7	g_8	g_9	g_{10}	g_{11}	$\cdots$
e	g_5	g_6	g_7	g_7	g_8	g_8	g_9	g_{10}	g_{11}	g_{12}	$\cdots$
f	g_5	g_6	g_7	g_7	g_8	g_8	g_9	g_{10}	g_{11}	g_{12}	$\cdots$
g_5	g_6	g_7	g_8	g_8	g_9	g_9	g_{10}	g_{11}	g_{12}	g_{13}	$\cdots$
g_6	g_7	g_8	g_9	g_9	g_{10}	g_{10}	g_{11}	g_{12}	g_{13}	g_{14}	$\cdots$
g_7	g_8	g_9	g_{10}	g_{10}	g_{11}	g_{11}	g_{12}	g_{13}	g_{14}	g_{15}	$\cdots$
g_8	g_9	g_{10}	g_{11}	g_{11}	g_{12}	g_{12}	g_{13}	g_{14}	g_{15}	g_{16}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

It follows easily from 4.4 that $R_1(\circ)$ is a free $\mathscr{R}_1$-groupoid of rank 1; $\{a\}$is the only basis of $R_1(\circ)$.

4.7 Let S be a free semigroup with a free basis X. Put $F = \{(a, x); x \in X\} \cup \cup \{(b, xy); x, y \in X\} \cup \{(c, xyz), (d, xyz); x, y, z \in X\} \cup \{(e, xyuv), (f, xyuv); x, y, u, v \in X\} \cup \{(g_i, r); r \in S\}$, $l(r) = i \geq 5$. Then F is a subgroupoid of the cartesian product $R_1(\circ) \times S$, F is a free $\mathscr{R}_1$-groupoid and $\{(a, x); x \in X\}$ is its free basis.

4.8 Denote by $\mathscr{R}_2$ the subvariety of $\mathscr{R}_1$ determined in $\mathscr{R}_1$ by the identity $xy \,.\, uv \triangleq x(yu \,.\, v)$.

4.9 Lemma. *Let $r, s \in W$, $l(r) \geq 4$. Then the identity $r \triangleq s$ is satisfied in $\mathscr{R}_2$ iff it is satisfied in $\mathscr{S}$.*

Proof. Easy (use 4.4).

4.10 Consider the following groupoid $R_2(\circ)$:

$R_2(\circ)$	a	b	c	d	g_4	g_5	g_6	g_7	g_8	$\cdots$
a	b	c	g_4	g_4	g_5	g_6	g_7	g_8	g_9	$\cdots$
b	d	g_4	g_5	g_5	g_6	g_7	g_8	g_9	g_{10}	$\cdots$
c	g_4	g_5	g_6	g_6	g_7	g_8	g_9	g_{10}	g_{11}	$\cdots$
d	g_4	g_5	g_6	g_6	g_7	g_8	g_9	g_{10}	g_{11}	$\cdots$
g_4	g_5	g_6	g_7	g_7	g_8	g_9	g_{10}	g_{11}	g_{12}	$\cdots$
g_5	g_6	g_7	g_8	g_8	g_9	g_{10}	g_{11}	g_{12}	g_{13}	$\cdots$
g_6	g_7	g_8	g_9	g_9	g_{10}	g_{11}	g_{12}	g_{13}	g_{14}	$\cdots$
g_7	g_8	g_9	g_{10}	g_{10}	g_{11}	g_{12}	g_{13}	g_{14}	g_{15}	$\cdots$
g_8	g_9	g_{10}	g_{11}	g_{11}	g_{12}	g_{13}	g_{14}	g_{15}	g_{16}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Then $R_2(\circ)$ is a free $\mathscr{R}_2$-groupoid of rank 1.

4.11 Proposition. *Let G be a groupoid such that $\sigma(G) \leq 1$. The following conditions are equivalent:*
(i) G is an SH-*groupoid of type* (a, a, a) *(and then G is minimal).*
(ii) G is non-associative and $G \in \mathscr{R}_1$.

Proof. (i) implies (ii). Let $a \in G$ be such that $a \,.\, aa \neq aa \,.\, a$. Then $xy, uv, yu \neq a$ for all $x, y, u, v \in G$ (see 2.10), and hence $xy \,.\, uv = (xy \,.\, u)\, v$, $xy \,.\, uv = x(y \,.\, uv)$, $x(yu \,.\, v) = (x \,.\, yu)\, v$. This means that $G \in \mathscr{R}_1$.
(ii) implies (i). There is an element $a \in G$ such that G is generated by $\{a\}$. Let $f\colon W \to G$ be the projective homomorphism such that $f(X) = \{a\}$. Now, take $u, v, w \in G$. There are $r, s, t \in W$ with $f(r) = u$, $f(s) = v$ and $f(t) = w$. If $\mathrm{l}(r) + \mathrm{l}(s) + \mathrm{l}(t) \geq 5$, then the identity $r \,.\, st \triangleq rs \,.\, t$ is satisfied in $\mathscr{R}_1$, and hence $u \,.\, vw = uv \,.\, w$. Assume that $n = \mathrm{l}(r) + \mathrm{l}(s) + \mathrm{l}(t) \leq 4$. Clearly, $3 \leq n$ and if $n = 3$, then $r, s, t \in X$ and $u = v = w = a$. Finally, assume that $n = 4$. If $\mathrm{l}(r) \geq 2$, then $\mathrm{l}(r) = 2$, $\mathrm{l}(s) = \mathrm{l}(t) = 1$, $u = aa$, $v = w = a$ and $u \,.\, vw = aa \,.\, aa = (aa \,.\, aa)\, a = uv \,.\, ww$, since $G \in \mathscr{R}_1$. The other two cases are similar and we have proved that $u \,.\, vw = uv \,.\, w$ except, possibly, for the case $u = v = w = a$. Since G is non-associative, $a \,.\, aa \neq aa \,.\, a$ and G is an SH-groupoid of type (a, a, a).

4.12. Proposition. *Let G be a groupoid such that $\sigma(G) \leq 1$. The following conditions are equivalent:*
(i) G is an SH-*groupoid of type* (a, a, a) *and of subtype* (α).
(ii) G is non-associative and $G \in \mathscr{R}_2$.

Proof. This follows easily from 4.9 and 4.11.

5.1 Proposition. *The following conditions are equivalent for a groupoid G:*
(i) G is a minimal SH-*groupoid of type* (a, a, a) *and subtype* (α).
(ii) G is non-associative and G is a homomorphic image of the groupoid $R_2(\circ)$ *(see 4.10).*

Proof. (i) implies (ii). We have $\sigma(G) = 1$ and $G \in \mathscr{R}_2$ by 4.12. However, $R_2(\circ)$ is free of rank 1 in $\mathscr{R}_2$, and so G is a homomorphic image of R_2.
(ii) implies (i). Clearly, $\sigma(G) \leq 1$ and $G \in \mathscr{R}_2$. Now, it suffices to use 4.12.

5.2 Lemma. *Let G be a minimal* SH-*groupoid of type* (a, a, a) *and of subtype* (α). *Let* $a' \in G$ *be such that* $a' . (a' . a') \neq (a' . a') . a'$. *Then* $x = y = a'$, *whenever* $x, y \in G$ *and* $xy = b' = a' . a'$.

Proof. Let $b' = a' . a'$, $c' = a' . b'$, $d' = b' . a'$ and let $\varphi : R_2(\circ) \to G$ be a projective homomorphism (see 5.1). The elements a', b', c', d' are pair-wise different and $\varphi(a) = a'$, $\varphi(b) = b'$, $\varphi(c) = c'$, $\varphi(d) = d'$. Further, there are $u, v \in R_2$ with $\varphi(u) = x$ and $\varphi(v) = y$. Then $\varphi(u \circ v) = \varphi(u)\,\varphi(v) = xy = b'$, and so $u \circ v \neq a, c, d$. If $u \circ v = g_i$ for some $i \geq 4$, then $a \circ (u \circ v) = g_{i+1} = (u \circ v) \circ a$, and therefore $c' = a' . b' = \varphi(a)\,\varphi(u \circ v) = \varphi(a \circ (u \circ v)) = \varphi(g_{i+1}) = \varphi((u \circ v) \circ a) = \varphi(u \circ v)\,\varphi(a) = b' . a' = d$, a contradiction. Thus $u \circ v = b$, $u = v = a$ and $x = y = a'$.

5.3 Let $3 \leq m \leq n$ and $R_{n,m} = \{a, b, c, d, g_4, \ldots, g_n\}$ ($n + 1$ pair-wise different elements). Define a structure of a semigroup on $R_{n,m}$ as follows: $b = a^2$, $c = a^3$, $g_i = a^i$ for $4 \leq i \leq n$, $a^{n+1} = a^m$ and $dx = a^3x$, $xd = xa^3$ for every $x \in R_{n,m}$. Clearly, $R_{n,m}$ becomes a semigroup and $R_{n,m}$ is not cyclic; every generator set of $R_{n,m}$ must contain the elements a and d. Moreover, the conditions (a), (b), (c), (d), (e) and (f) from 3.2 are satisfied. Now, put $R_{n,m}(\circledast) = R_{n,m}[b, a, d]$ (see 3.2), so that $R_{n,m}(\circledast)$ is a minimal $(n + 1)$-element SH-groupoid of type (a, a, a) and of subtype (α).

5.4 Let $4 \leq n$ and $R_n = \{a, b, c, d, g_4, \ldots, g_{n-1}\}$ (n pair-wise different elements). Define a structure of semigroup on R_n as follows: $b = a^2$, $c = a^3$, $g_i = a^i$ for $4 \leq i \leq n - 1$, $d = a^n$, $a^4 = a^{n+1}$. Clearly, R_n is cyclic semigroup generated by a and the condition (a), (b), (c), (d), (e), (f) from 3.2 are satisfied. Now, put $R_n(\circledast) = R_n[b, a, d]$ (see 3.2), so that $R_n(\circledast)$ is a minimal n-element SH-groupoid of type (a, a, a) and of subtype (α).

5.5 Theorem. *(i)* $R_2(\circ)$ *is (up to isomorphism) the only infinite minimal* SH-*groupoid of type* (a, a, a) *and of subtype* (α).
(ii) Let $n \geq 4$. *Then the* $n - 2$ *groupoids* $R_n(\circledast)$, $R_{n-1,m}(\circledast)$ $(3 \leq m, m \leq n - 1)$ *are pair-wise non-isomorphic and they are (up to isomorphism) the only n-element minimal* SH-*groupoids of type* (a, a, a) *and subtype* (α).

Proof. (i) Let G be an infinite minimal SH-groupoid of type (a, a, a) and of subtype (α). Let $a \in G$ be such that $a.aa \neq aa.a$ and let $b = aa$, $c = ab$, $d = ba$. The groupoid G satisfies the condition (SH1) from 3.1 (see 5.2), and so we have the corresponding semigroup $G(*)$. Proceeding similarly as in the proof of 5.2 and using the fact that G is infinite, we can show that $xy \neq d$ if $(x, y) \neq (b, a)$. This shows that $H(*)$ is a cyclic semigroup generated by a, where $H = G - \{d\}$. The rest is clear.
(ii) Let G be an n-element minimal SH-groupoid of type (a, a, a) and of subtype (α). Again, G satisfies (SH1) and we have the semigroup $G(*)$ from 3.1. If $d \neq x.y$ for all $x, y \in G$, then $G(*)$ is not cyclic, $H(*)$ is cyclic ($H = G - \{d\}$)and G is isomorphic to $R_{n-1,m}(\circledast)$ for some $3 \leq m \leq n-1$. Now, assume that $d = x * y$ for some $x, y \in G$, i.e., $d = uv$ for some $u, v \in G$ such that $(u, v) \neq (b, a)$. Then, $G(*)$ is a cyclic semigroup generated by a and we have $d = a * \ldots * a$ (k-times). From this $a * a * a * a = a * c = a * d = a * \ldots * a$ ($k + 1$-times) and, since G possesses just n elements, necessarily $k = n$. Consequently, G is isomorphic to $R_n(\circledast)$.

5.6 Corollary. *Let G be a minimal* SH-*groupoid of type* (a, a, a) *and of subtype* (α). *Then* $\text{sdist}(G) = 1$.

5.7 Example.

$R_4(\circ)$	a	b	c	d
a	b	c	d	d
b	d	d	d	d
c	d	d	d	d
d	d	d	d	d

$R_{3,3}(\circ)$	a	b	c	d
a	b	c	c	c
b	d	c	c	c
c	c	c	c	c
d	c	c	c	c

$R_5(\circ)$	a	b	c	d	g_4
a	b	c	g_4	g_4	d
b	d	g_4	d	d	g_4
c	g_4	d	g_4	g_4	d
d	g_4	d	g_4	g_4	d
g_4	d	g_4	d	d	g_4

$R_{4,3}(\circ)$	a	b	c	d	g_4
a	b	c	g_4	g_4	c
b	d	g_4	c	c	g_4
c	g_4	c	g_4	g_4	c
d	g_4	c	g_4	g_4	c
g_4	c	g_4	c	c	g_4

$R_{4,4}(\circ)$	a	b	c	d	g_4
a	b	c	g_4	g_4	g_4
b	d	g_4	g_4	g_4	g_4
c	g_4	g_4	g_4	g_4	g_4
d	g_4	g_4	g_4	g_4	g_4
g_4	g_4	g_4	g_4	g_4	g_4

6.1 Proposition. *The following conditions are equivalent for a groupoid G:*
(i) G is a minimal SH*-groupoid of type* (a, a, a).
(ii) G is non-associative and G is a homomorphic image of the groupoid $R_1(\circ)$ *(see 4.6).*

Proof. This is an easy consequence of 4.11 (see the proof of 5.1).

6.2 Lemma. *Let G be a minimal* SH*-groupoid of type* (a, a, a) *and or subtype* (β). *Let* $a' \in G$ *be such that* $a' \,.\, a'a' \neq a'a' \,.\, a'$. *Then:*
(i) $x = y = a'$, *whenever* $x, y \in G$ *and* $xy = b' = a'a'$.
(ii) $x = a'$ *and* $y = b'$, *whenever* $x, y \in G$ *and* $xy = c' = a'b'$.
(iii) $x = b'$ *and* $y = a'$, *whenever* $x, y \in G$ *and* $xy = d' = b'a'$.

Proof. Let $b' = a'a'$, $c' = a'b'$, $d' = b'a'$, $e = b'b' = a'c' = d'a'$, $f' = c'a' = a'd'$ and let $\varphi\colon R_1(\circ) \to G$ be a projective homomorphism (see 6.1). Then a', b', c', d', e', f' are pair-wise different and $\varphi(a) = a'$, $\varphi(b) = b'$, $\varphi(c) = c'$, $\varphi(d) = d'$, $\varphi(e) = e'$, $\varphi(f) = f'$. Further, let $x, y \in G$, $u, v \in R_1$ and $\varphi(u) = x$, $\varphi(v) = y$.
(i) Let $xy = b'$. Proceeding similarly as in the proof of 5.2, we can show that $x = y = a'$.
(ii) Let $xy = c'$. Then $\varphi(u \circ v) = c'$, and so $u \circ v \neq a, b, c, d, e, f$. If $u \circ v = g_i$ for some $i \geq 5$, then $a \circ (u \circ v) = g_{i+1} = (u \circ v) \circ a$ and this implies that $e' = a'c' = c'a' = f'$, a contradiction. Thus $u \circ v = c$, $u = a$, $v = b$ and $x = a'$, $y = b'$.
(iii) This is dual to (ii).

6.3 Let $n \geq 4$, $4 \leq m \leq n$ and $R'_{n,m} = \{a, b, c, d, e, f, g_5, \ldots, g_n\}$ ($n + 2$ pair-wise different elements). Define a structure of a semigroup on $R'_{n,m}$ as follows: $b = a^2$, $c = a^3$, $e = a^4$, $g_i = a^i$ for $5 \leq i \leq n$, $a^{n+1} = a^m$, $ad = f$, $dx = a^3x$, $fx = a^4x$, $xf = xa^4$ for every $x \in R'_{n,m}$, $yd = ya^3$ for every $y \in R'_{n,m}$, $y \neq a$. It is easy to check that $R'_{n,m}$ becomes a semigroup and that $R'_{n,m}$ is not cyclic; every generator set of $R'_{n,m}$ must contain the elements a, d. Moreover, the conditions (a), (b), (c), (d), (f) from 3.2 and the conditions (e'), (g), (h), (i), (j), (k) from 3.4 are satisfied. Now, define a binary operation $\circ$ on $R'_{n,m}$ by $x \circ y = xy$ if $(x, y) \neq (b, a), (c, a)$, and $b \circ a = d$, $c \circ a = f$ (see 3.4). Then $R'_{n,m}(\circ)$ is a minimal $(n + 2)$-element SH-groupoid of type (a, a, a) and of subtype (β). Clearly, $1 \leq \mathrm{sdist}(R'_{n,m}(\circ)) \leq 2$. If $5 \leq m$, then $e \neq b \circ g$ for every $g \in R'_{n,m}$, $g \neq b$, and it is easy to check that $\mathrm{sdist}(R'_{n,m}(\circ)) = 2$ (see 3.3). Finally, if $m = 4$, then $\mathrm{sdist}(R'_{n,m}(\circ)) = 1$ ($R'_{n,4}(\circ)\,[a, a, g_{n-1}]$ for $n \geq 6$ and $R'_{n,4}(\circ)\,[a, a, e]$ for $n = 4{,}5$ are semigroups).

6.4 Let $5 \leq n$ and $R'_n = \{a, b, c, d, e, f, g_5, \ldots, g_{n-1}\}$ ($n + 1$ pair-wise different elements). Define a structure of a semigroup on R'_n as follows: $b = a^2$, $c = a^3$, $e = a^4$, $g_i = a^i$ for $5 \leq i \leq n - 1$, $a^n = f$, $ad = f$, $dx = a^3x$, $fx = a^4x$ for every

$x \in R'_n$ and $yd = ya^3$ for every $y \in R'_n$, $y \neq a$. It is easy to check that R'_n becomes a semigroup (which is not cyclic) and that the conditions (a), (b), (c), (d), (f) from 3.2 and the contitions (e'), (g), (h), (i), (j), (k) from 3.4 are satisfied. Now, define a binary operation $\circ$ on R'_n by $x \circ y = xy$ if $(x, y) \neq (b, a), (c, a)$ and $b \circ a = d$, $c \circ a = f$ (see 3.4). Then $R'_n(\circ)$ is a minimal $(n + 1)$-element SH-groupoid of type (a, a, a) and of subtype (β). Moreover, $\mathrm{sdist}(R'_n(\circ)) = 2$ (see 3.2).

6.5 Theorem. *(i)* $R_1(\circ)$ *is (up to isomorphism) the only infinite minimal* SH-*groupoid of type* (a, a, a) *and of subtype* (β).
(ii) Let $n \geq 6$. *Then the* $n - 4$ *groupoids* $R'_{n-1}(\circ)$, $R'_{n-2,m}(\circ)$ $(4 \leq m \leq n - 2)$ *are pair-wise non-isomorphic and they are (up to isomorphism) the only n-element minimal* SH-*groupoid of type* (a, a, a) *and subtype* (β).

Proof. (i) Let G be an infinite minimal SH-groupoid of type (a, a, a) and of subtype (β). Let $a \in G$ be such that $a \,.\, aa \neq aa \,.\, a$ and let $b = aa$, $c = ab$, $d = ba$, $e = ac$, $f = ad$. The groupoid G satisfies the condition (SH1) from 3.1 and the condition (SH2) from 3.3 (see 6.2), and so we can consider the corresponding semigroup $G(*)$. Proceeding similarly as in the proof of 6.2 and using the fact that G is infinite, we can show that $xy \neq f$ if $(x, y) \neq (a, d), (c, a)$. This (together with 6.2(iii)) shows that $H(*)$ is a cyclic semigroup, where $H = G - \{d, f\}$. The rest is clear.
(ii) Let G be an n-element minimal SH-groupoid of type (a, a, a) and of subtype (β). By 6.2, G satisfies both (SH1) and (SH2) and we have the semigroup $G(*)$ from 3.3. By 6.2(iii), $d \neq x * y$ for all $x, y \in G$. If $f \neq x * y$, then $H(*)$ is cyclic $(H = G - \{d, f\})$ and G is ismorphic to $R'_{n-2,m}(\circ)$. Now, assume that $f = x * y$ for some $x, y \in G$, i.e., $f = uv$ for some $u, v \in G$ such that $(u, v) \neq (a, d), (c, a)$. Then $f = a * \ldots * a$ (k-times), which means that $a * a * a * a * a = a * f = a * \ldots * a$ ($k + 1$-times) and, since G possesses just n elements, necessarily $k = n - 1$. Consequently, G is ismorphic to $R'_{n-1}(\circ)$.

6.6 Corollary. Let G be a minimal SH-groupoid of type (a, a, a) and subtype (β). Then $\mathrm{sdist}(G) = 2$ except for the case when G is ismorphic to $R'_{n,4}$ for some $n \geq 4$ and then $\mathrm{sdist}(G) = 1$.

6.7 Example.

$R'_{4,4}(\circ)$	a	b	c	d	e	f
a	b	c	e	f	e	e
b	d	e	e	e	e	e
c	f	e	e	e	e	e
d	e	e	e	e	e	e
e	e	e	e	e	e	e
f	e	e	e	e	e	e

$R'_5(\circ)$	a	b	c	d	e	f
a	b	c	e	f	f	f
b	d	e	f	f	f	f
f	f	f	f	f	f	f
d	e	f	f	f	f	f
e	f	f	f	f	f	f
f	f	f	f	f	f	f

$R'_{5,4}(\circ)$	a	b	c	d	e	f	g_5
a	b	c	e	f	g_5	g_5	e
b	d	e	g_5	g_5	e	e	g_5
c	f	g_5	e	e	g_5	g_5	e
d	e	g_5	e	e	g_5	g_5	e
e	g_5	e	g_5	g_5	e	e	g_5
f	g_5	e	g_5	g_5	e	e	g_5
g_5	e	g_5	e	e	g_5	g_5	e

$R'_{5,5}(\circ)$	a	b	c	d	e	f	g_5
a	b	c	e	f	g_5	g_5	f
b	d	e	g_5	g_5	f	f	g_5
c	f	g_5	f	f	g_5	g_5	f
d	e	g_5	f	f	g_5	g_5	f
e	g_5	f	g_5	g_5	f	f	g_5
f	g_5	f	g_5	g_5	f	f	g_5
g_5	f	g_5	f	f	g_5	g_5	f

$R'_{6}(\circ)$	a	b	c	d	e	f	g_5
a	b	c	e	f	g_5	g_5	g_5
b	d	e	g_5	g_5	g_5	g_5	g_5
c	f	g_5	g_5	g_5	g_5	g_5	g_5
d	e	g_5	g_5	g_5	g_5	g_5	g_5
e	g_5	g_5	g_5	g_5	g_5	g_5	g_5
f	g_5	g_5	g_5	g_5	g_5	g_5	g_5
g_5	g_5	g_5	g_5	g_5	g_5	g_5	g_5

III.7 Comments and open problems

7.1 In this part, some results from [1] are reformulated. Besides, the semigroup distance of minimal SH-groupoids of the type (a, a, a) is found.

7.2 Find the numbers sdist(G) for SH-groupoids of the type (a, a, a). In particular, are these numbers bounded?

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