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Tomáš Kepka; Milan Trch

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Groupoids and the Associative Law IV. (Szász–Hájek Groupoids of Type (A,B,A))

TOMÁŠ KEPKA*) and MILAN TRCH**)

MFF UK Praha

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The paper is concerned with groupoids possessing only one non-associative triple of elements and it is of the form (a,b,a).

Článek se týká grupoidů, které mají jen jednu neasociativní trojici prvků, a ta je tvaru (a,b,a).

This paper is an immediate continuation of [3]. Here Szász–Hájek groupoids of type (a,b,a) are considered.

IV.1 Basic arithmetic of SH-groupoids of type (a,b,a)

1.1 Throughout this section let G be an SH-groupoid of type (a,b,a). Let $a, b \in G$ be such that $a \cdot ba \neq ab \cdot a$ and put $c = ab$, $d = ba$, $e = ad = a \cdot ba$ and $f = ca = ab \cdot a$; clearly, $a \neq b$ and $e \neq f$.

1.2 Proposition. (i) *If $x, y \in G$ are such that $xy = a$ (resp. $xy = b$), then either $x = a$ (resp. $x = b$) or $y = a$ (resp. $y = b$).*

(ii) *If M is a generator set of G, then $a, b \in M$.*

(iii) *If H is a subgroupoid of G, then either $\{a, b\} \subseteq H$ and H is an* SH-*groupoid of type* (a,b,a) *or $\{a, b\} \nsubseteq H$ and H is a semigroup.*

(iv) *If r is a congruence of G, then either $(e, f) \in r$ and G/r is an* SH-*groupoid of type* (a,b,a) *or $(e, f) \notin r$ and G/r is a semigroup.*

Proof. See III.1.2.

1.3 Lemma. (i) *If $x \in G$ is such that $x \neq a$ and either $x \neq b$ or $a \neq d$, then $ax = a$ iff $xb = b$.*

(ii) *If $x \in G$ is such that $x \neq a$ and either $x \neq b$ or $a \neq c$, then $xa = a$ if $bx = b$.*

*) Department of Mathematics, Charles University, Sokolovská 83, 186 00 Praha 8, Czech Republic

**) Department of Pedagogy, Charles University, M. D. Rettigové 4, 116 39 Praha 1, Czech Republic

Proof. (i) If $ax = a$, then $a(xb \cdot a) = a(x \cdot ba) = ax \cdot ba = a \cdot ba = e \neq f = ab \cdot a = (ax \cdot b)\,a = (a \cdot xb)\,a$ and consequently $xb = b$. Similarly, if $xb = b$, then $ax \cdot ba = a(x \cdot ba) = a(xb \cdot a) = a \cdot ba = e \neq f = ab \cdot a = (a \cdot xb)\,a = (ax \cdot b)\,a$ and consequently $ax = a$.

(ii) Similar to (i).

1.4 Lemma. *Suppose that $a^2 = a$.*

(i) *Either $a \neq c$ or $b \neq d$.*

(ii) *Either $b \neq c$ or $a \neq d$.*

(iii) *$c \neq d$.*

(iv) *If $b \neq c$, then $af = f = fa$.*

(v) *If $b \neq d$, then $ae = e = ea$.*

Proof. (i) If $a = a$ and $b = d$, then $e = ad = ab = c = a = aa = ca = f$, a contradiction.

(ii) Similar to (i).

(iii) If $c = d$, then $e = ad = ac = a \cdot ab = aa \cdot b = ab = c = d = ba = b \cdot aa = ba \cdot a = da = ca = f$, a contradiction.

(iv) We have $af = a(ab \cdot a) = (a \cdot ab)\,a = (aa \cdot b) \cdot a = ab \cdot a = f = ab \cdot a = ab \cdot aa = (ab \cdot a) \cdot a = fa$.

(v) Similar to (iv).

1.5 Lemma. (i) *Either $a \neq c$ or $b \neq d$.*

(ii) *Either $a \neq d$ or $b \neq c$.*

Proof. (i) Let $a = c$ and $b = d$. Then $f = ca = aa \neq a$ by 1.4(i) and we have $a = c = ab = ad = a \cdot ba = a \cdot da = a(ba \cdot a) = a(b \cdot aa) = a \cdot bf = ab \cdot f = cf = af = a \cdot aa = aa \cdot a = ac \cdot a = (a \cdot ab)\,a = (aa \cdot b) \cdot a = fb \cdot a = f \cdot ba = fd = fb = aa \cdot b = a \cdot ab = ac = aa = f$, a contradiction.

(ii) This is dual to (i) (e.g., consider the opposite groupoid).

1.6 Lemma. (i) *Let $x \in G$ be such that $xa \neq a$ and either $x \neq a$ or $b \neq c$. Then $xe = xf$.*

(ii) *Let $x \in G$ be such that $ax \neq a$ and either $x \neq a$ or $b \neq d$. Then $ex = fx$.*

Proof. (i) We have $xe = x(a \cdot ba) = xa \cdot ba = (xa \cdot b)\,a = (x \cdot ab)\,a = x(ab \cdot a) = xf$.

(ii) Dual to (i).

1.7 Lemma. (i) *If $x \in G$ and $xa = a$, then $xe = e$.*

(ii) *If $x \in G$, $xa = a$ and either $x \neq a$ or $b \neq c$, then $xf = f$.*

(iii) *If $x \in G$ and $ax = a$, then $fx = f$.*

(iv) *If $x \in G$, $ax = a$ and either $x \neq a$ or $b \neq d$, then $ex = e$.*

Proof. (i) We have $xe = x(a \cdot ba) = xa \cdot ba = a \cdot ba = e$.

(ii) We have $xf = x(ab \cdot a) = (x \cdot ab)\, a = (xa \cdot b)\, a = ab \cdot a = f$.

(iii) Dual to (i).

(iv) Dual to (ii).

1.8 Lemma. *Suppose that $c \neq b \neq d$.*

(i) *If $x \in G$ and $xa \neq a$, then $xe = xf$.*

(ii) *If $x \in G$ and $xa = a$, then $xe = e$ and $xf = f$.*

(iii) *If $x \in G$ and $xa \neq a$, then $ex = fx$.*

(iv) *If $x \in G$ and $xa = a$, then $ex = e$ and $fx = f$.*

Proof. See 1.6 and 1.7.

1.9 Lemma. *Suppose that $b = c = ab$.*

(i) *$d = f$, $e = ad = af$ and $b = c \neq d$.*

(ii) *If $a \neq a^2$, then $ae = d = f$ and $ea = da = fa$.*

(iii) *If $a = a^2$, then $ae = ad = af = e = ea$ and $fa = da = d = f$.*

Proof. (i) We have $f = ab \cdot a = ba = d$, $e = a \cdot ba = ad = af$. If $b = d$, then $f = d = b = ab = ad = e$, a contradiction.

(ii) We have $ae = a \cdot ad = a^2d = a^2 \cdot ba = a^2b \cdot a = (a \cdot ab)\, a = ab \cdot a = f = d$ and $ea = (a \cdot ba)\, a = a(ba \cdot a) = a \cdot ba^2 = ab \cdot a^2 = ba^2 = ba \cdot a = da = fa$.

(iii) First, $ae = e = ea$ by 1.4(v). Further, $af = ad = e$ by (i) and $fa = da = ba \cdot a = ba^2 = ba = d = f$.

1.10 Lemma. *Suppose that $b = c$.*

(i) *If $x \in G$ is such that $x \neq a$ and $xa \neq a$, then $xe = xf$.*

(ii) *If $x \in G$ is such that $x \neq a$ and $xa = a$, then $xe = e$ and $xf = f$.*

(iii) *If $a^2 \neq a$, then $ae = f$ and $af = a$.*

(iv) *If $a^2 = a$, then $ae = e = af$.*

(v) *If $x \in G$ and $ax \neq a$, then $ex = fx$.*

(vi) *If $x \in G$ and $ax = a$, then $ex = e$ and $fx = f$.*

Proof. Combine 1.6, 1.7 and 1.9.

1.11 Proposition. *The relation $s_G = \mathrm{id}_G \cup \{(e,f), (f,e)\}$ is just the least congruence of G such that the corresponding factor is associative.*

Proof. The result follows easily from 1.2(iv), 1.8, 1.10 and the dual of 1.10 (i.e., the case $b = d$).

1.12 It follows immediately from 1.2(ii), (iii) that every cyclic (i.e. one-generated) subgroupoid of G is associative. In other words, G is monoassociative (alias power associative). Further, $\sigma(G) \geqq 2$ and G is minimal (see III.1.3) iff $\sigma(G) = 2$. If this is so, then $\{a, b\}$ is a smallest generator set of G.

1.13 Lemma. *If $x \in G$, $n > 1$ and $x^n = a$ (resp. $x^n = b$), then $x = a$ (resp. $x = b$) and $a^2 = a$ (resp. $b^2 = b$).*

Proof. First, let $n^n = a$. By 1.2(i), $x = a$, $a^3 = a$ and $a^2 \neq b$. Assume $a^2 \neq a$. Then, by 1.3, $a \cdot ab = a^2b = b = ba^2 = ba \cdot a$ and $ab = b = ba$ by 1.2(i). However, then $e = a \cdot ba = ab = b = ba = ab \cdot a = f$, a contradiction.

Next, let $x^n = a$. Again, by 1.2(i), $x = b$, $b^3 = b$ and $b^2 \neq a$. Assume $b^2 \neq b$. We have $b \cdot b^2 = b = b^2 \cdot b$, and hence $ab^2 = a = b^2a$ by 1.3. Now, $ab \cdot b = a = {}$ $= ba \cdot a$, so that $ab = a = ba$ by 1.2(i) and $e = a \cdot ba = aa = ab \cdot a = f$, a contradiction.

1.14 Lemma. *Suppose that G is minimal and such that $a \notin \{aa, ab, ba, a \cdot ba, ab \cdot a\}$. Then $a \neq xa$ for all $x, y \in G$.*

Proof. Suppose the contrary, denote by W an absolutely free groupoid with a two-element free basis $\{u, v\}$ and consider the projective homomorphism φ: $W \to G$ such that $\varphi(u) = a$ and $\varphi(v) = b$. In view of 1.2(i), we can assume that $t \in W$ is such that $a = a\varphi(t)$ and $\mathrm{l}(t)$ is minimal with the respect to $a = a\varphi(t)$ or $a = \varphi(t)\, a$. Now, $\mathrm{l}(t) \geqq 2$, and hence $t = rs$ and $a = a \cdot \varphi(r)\, \varphi(s)$. We have $(\varphi(r), \varphi(s)) \neq (a, b)$, and hence $a = a\varphi(r)\, \varphi(s)$. Since $\mathrm{l}(r) < \mathrm{l}(t)$, necessarily $a\varphi(r) \neq a$ and $\varphi(s) = a$. Again, $\mathrm{l}(s) < \mathrm{l}(t)$, and hence $\mathrm{l}(s) = 1$, $s = u$ and $a = a\varphi(r) \cdot a$. Further, $\mathrm{l}(r) \geqq 2$ (use also 1.13), $r = r_1r_2$ and $a = (a \cdot \varphi(r_1)\, \varphi(r_2))\, a$. If $(b, a) \neq (\varphi(r_1), \varphi(r_2))$, then $a = (a\varphi(r_1) \cdot \varphi(r_2))\, a = a\varphi(r_1) \cdot \varphi(r_2)\, a$, a contradiction with 1.2(i) and the minimality of $\mathrm{l}(t)$. Consequently, $\varphi(r_1) = b$, $\varphi(r_2) = a$ and $a = (a \cdot ba)\, a$. If $ba = b$, then $a = ab \cdot a$, which is not true, and therefore $ba \neq b$ and $a = (a \cdot ba)\, a = {}$ $= a(ba \cdot a) = a \cdot ba^2 = ab \cdot a^2$, a contradiction with $a \neq ab$ and $a \neq a^2$.

1.15 Lemma. *Suppose that G is minimal and that $b \notin \{ab, ba, bb, b \cdot ab\}$. Then $b \neq xy$ for all $x, y \in G$.*

Proof. Similar to that of 1.14.

1.16 Lemma. *If $a = ab = c$, then $b = b^2 \neq ba = d \neq a$ and $f = a^2$.*

Proof. By 1.5(i), $b \neq d = ba$. Further, if $a = a$, then $a = ba$ and $e = a \cdot ba = {}$ $= aa = ab \cdot a = f$, a contradiction. Hence $a \neq d$, and so $b^2 = b$ by 1.3(i). Finally, $f = ab \cdot a = a^2$ trivially.

1.17 Lemma. *If $a = ba = d$,* then *$b = b^2 \neq ab = c \neq a$ and $e = a^2$.*

Proof. Dual to that of 1.16.

1.18 Lemma. *If $a = a \cdot ba = e$, then $a = a^2 = ba$ and $b = b^2$.*

Proof. First, assume that $ba = a$. Then $a^2 = aa = a \cdot ba = a$. If $a = c = ab$, then $e = a \cdot ba = aa = ab \cdot a = f$, a contradiction. Hence $a \neq c$ and $b^2 = b$ by 1.3(ii).

Now, let $d = ba \neq a$. By 1.3(ii), $b = ba \cdot b = b \cdot ab$. If $c = ab = a$, then $b = b \cdot ab = ba = d$, a contradiction with 1.5(i). Thus $c = ab \neq a$, and so $a = {}$ $= ab \cdot a = f$ by 1.3(ii). However, then $e = a = f$, a contradiction.

1.19 Lemma. *If* $a = ab \cdot a = f$, *then* $a = a^2 = ab$ *and* $b = b^2$.

Proof. Dual to that of 1.18.

1.20 Lemma. (i) *If* $b = ab = c$, *then* $a = a^2$, $d = f$ *and* $e = af$.

(ii) *If* $b = ba = d$, *then* $a = a^2$, $c = e$ *and* $f = ca$.

(iii) *If* $b = b^2$, *then either* $a = ab = c$ *or* $a = ba = d$.

Proof. (i) Since $b = ab = c$, we have also $f = ab \cdot a = ba = d$ and $e = ad = af$. Further, if $a \neq a^2$, then $e = a \cdot ba = a(ab \cdot a) = a((a \cdot ab)\, a) = a((a^2 \cdot b)\, a) = a(a^2 \cdot ba) = a^3 \cdot ba = a^3 b \cdot a = (a^2 \cdot ab)\, a = a^2 b \cdot a = (a \cdot ab)\, a = ba = d = f$, a contradiction.

(ii) This is dual to (i).

(iii) Use 1.3.

1.21 Lemma. $b = b^2$ *iff* $a = c$ *or* $a = d$.

Proof. See 1.16, 1.17 and 1.20(iii).

1.22 Lemma. $a \neq b \cdot ab = ba \cdot b \neq b$.

Proof. Suppose that $b = b \cdot ab = ba \cdot b$. By 1.3, either $a = ab$ or $a = ab \cdot a$ and either $a = ba$ or $a = a \cdot ba$. Now, by 1.18 and 1.19, we get $ab = a = ba$ and $e = a^2 = f$, a contradiction.

Now, let $b \cdot ab = ba \cdot b = a$. Since $b \neq a$, we must have $ab = a = ba$, again a contradiction.

1.23 Corollary. *G contains at least three elements.*

1.24 Lemma. *If* $b = b^2$, *then* $c = ab \neq b \neq ba = d$.

Proof. Suppose, on the contrary, that $b = c$. Then $f = ca = ba = d = a$ (by 1.20(iii)) and $a = ab = c = b$ by 1.19, a contradiction. Thus $c \neq b$ and, similarly, $d \neq b$.

IV.2 Minimal SH-groupoids of the type (a,b,a)

2.1 In this section, let G be a minimal SH-groupoid of type (a,b,a). Let $a, b \in G$ be such that $a \cdot ba \neq ab \cdot a$ and put $c = ab$, $d = ba$, $e = ad$ and $f = ca$. We have $a \neq b$ and $e \neq f$.

2.2 Lemma. *If* $a \neq a^2$ *and* $b \notin \{ab, ba, b^2\}$, *then* $a \neq xy \neq b$ *for all* $x, y \in G$.

Proof. Combine 1.16, 1.17, 1.18, 1.19, 1.22 and 1.14, 1.15.

2.3 Lemma. *If* $a \notin \{b^2, ab, ba\}$, *then* $a \neq xy$ *for all* $x, y \in G$.

Proof. See 1.18, 1.19 and 1.14.

2.4 Lemma. *If* $b \notin \{b^2, ab, ba\}$, *then* $b \neq xy$ *for all* $x, y \in G$.

Proof. See 1.22 and 1.15.

2.5 Lemma. *If $ab \neq a \neq ba$, then $a \neq xy$ for all $x, y \in G$, $(x, y) \neq (a, a)$.*

Proof. With respect to 2.3, we can assume that $a = a^2$. Now, consider the projective homorphism φ: $W \to G$ such that: $\varphi(u) = a$, $\varphi(v) = b$ (see proof of 1.14) and suppose that $t \in W$ is such that $a = a\varphi(t)$ and $\mathrm{l}(t)$ is minimal with respect to $c \in \mathrm{var}(t)$ and $a = a\varphi(t)$ or $a = \varphi(t)\, a$.

Clearly, $\mathrm{l}(t) \geqq 2$, and so $t = rs$ for some $r, s \in W$. If $(\varphi(r), \varphi(s)) = (b, a)$, then $\varphi(t) = ba$ and $a = a \cdot ba$, a contradiction with 1.18. Thus $(\varphi(r), \varphi(s)) \neq (b, a)$, and hence $a = a \cdot \varphi(r)\, \varphi(s) = a\varphi(r) \cdot \varphi(s)$ and either $a = a\varphi(r)$ or $a = \varphi(s)$.

First, assume that $a = a\varphi(r)$. If $v \notin \mathrm{var}(r)$, then $\varphi(r) = a$, and therefore $a = a\varphi(r)$ and $a = a\varphi(s)$, $v \in \mathrm{var}(s)$, a contradiction with the minimality of $\mathrm{l}(t)$. Hence $v \in \mathrm{var}(r)$, again a contradiction with the minimality of $\mathrm{l}(t)$.

We have proved that $a\varphi(r) \neq a = \varphi(s)$. Consequently, $v \in \mathrm{var}(r)$ (otherwise $a\varphi(r) = a$) and $a = \varphi(ur)\, a$. Using the minimality of $\mathrm{l}(t)$, we get $\mathrm{l}(t) = \mathrm{l}(ur) = \mathrm{l}(r) + 1$, $\mathrm{l}(s) = 1$ and $s = u$. If $\mathrm{l}(r) = 1$, then $r = v$ and $a = a \cdot ba$, which is not true. Hence $\mathrm{l}(r) \geqq 2$ and $r = r_1 r_2$, $a = a(\varphi(r_1)\, \varphi(r_2) \cdot a)$. If $(\varphi(r_1), \varphi(r_2)) = (a, b)$, then $a = a(ab \cdot a)$ and either $ab = b$ and $a = a \cdot ba$, a contradiction with 1.18, or $ab \neq b$ and $a = a(ab \cdot a) = (a \cdot ab)\, a = (aa \cdot b)\, a = ab \cdot a$, a contradiction with 1.19. Thus $(\varphi(r_1), \varphi(r_2)) \neq (a, b)$, and hence $a = a(\varphi(r_1)\, \varphi(r_2) \cdot a) = a(\varphi(r_1) \cdot \varphi(r_2)\, a) = a\varphi(r_1) \cdot \varphi(r_2)\, a$ (clearly, $(\varphi(r_1), \varphi(r_2)\, a) \neq (b, a)$). Now, either $a\varphi(r_1) = a$ or $\varphi(r_2)\, a = a$. Suppose the former (the latter being similar). Then $v \notin \mathrm{var}(r_1)$ $v \in \mathrm{var}(r_2)$ and $a = a\varphi(r_2 u)$, a contradiction with the minimality of $\mathrm{l}(t)$.

2.6 Lemma. *If $a^2 \neq a = ab$, then $a \neq xy$ for all $x, y \in G$, $(x, y) \neq (a, b)$.*

Proof. Using 13, 1.16, 1.18 and 1.19, we can proceed similarly as in the proof of 2.5 (we consider $t \in W$ such that $\mathrm{l}(t)$ is minimal with the respect to $u \in \mathrm{var}\,(t)$ and either $a = \varphi(t)$ or $a = \varphi(t)a$).

2.7 Lemma. *If $a^2 \neq a = ab$, then $a \neq xy$ for all $x, y \in G$, $(x, y) \neq (b, a)$.*

Proof. Dual to that of 2.6.

2.8 Lemma. *If $b = b^2$, that $b \neq xy$ for all $x, y \in G$, $(x, y) \neq (b, b)$.*

Proof. Similar to that of 2.5 (use 1.22 and 1.24).

Lemma 2.9 *If $b = ab$ and $b = xy$ for some $x, y \in G$, then $y = b$ and $x = a^n$ for some $n \geqq 1$.*

Proof. Similar to that of 2.5 (use 1.22 and 1.24).

2.10 Lemma *If $b = ba$ and $b = xy$ for some $x, y \in G$, then $x = b$ and $y = a^n$ for some $n \geqq 1$.*

Proof. Dual to that of 2.9.

2.11 In the sequel, we shall say that G is of subtype

(α) if $a = a^2 = ab$.
(β) if $a = a^2 = ba$.
(γ) if $a = ab \neq a^2$.
(δ) if $a = ba \neq a^2$.
(ε) if $a = a^2 \neq ba$ and $b = ab$.
(φ) if $a = a^2 \neq ab$ and $b = ba$.
(ϱ) if $a = a^2 \neq ab, ba$ and $b \neq ab, ba$.
(η) if $a \neq a^2, ab, ba$ and $b \neq b^2, ab, ba$.

Proposition. *G is of just one of the preceding seven subtypes* (α), (β), ..., (η).

Proof. Suppose that G is not of subtype (η). Then $a \in \{a^2, ab, ba\}$ and $b \in \{b^2, ab, ba\}$. If $ab = b = ba$, then $e = a \cdot ba = ab = b = ba = ab \cdot a = f$, a contradiction. Now using 1.24, we see that exactly one of the three equalities $b = b^2$, $b = ab$, $b = ba$ takes place. Similarly, either $a \neq ab$ or $a \neq ba$. The rest is clear from 1.16, 1.17, 1.20, 1.21 and 1.24.

IV.3 Minimal SH-groupoids of type (a, b, a) and subtype (α)

3.1 Consider the following four-element groupoid $S_1(\circ)$:

$S_1(\circ)$	a	b	d	e
a	a	a	e	e
b	d	b	d	d
d	d	d	d	d
e	e	e	e	e

It is easy to check that $S_1(\circ)$ is a minimal SH-groupoid of type (a, b, c) and subtype (α). Clearly, sdist $(S_1(\circ)) = 1$ (put $a * a = e$).

3.2 Consider the following three-element groupouid $S_2(\circ)$:

$S_2(\circ)$	a	b	d
a	a	a	d
b	d	b	d
d	d	d	d

It is easy to check that $S_2(\circ)$ is a minimal SH-groupoid o type (a, b, c) and subtype (α). Clearly, sdist $(S_2(\circ)) = 1$ (put $a * d = a$ or $a * a = d$).

3.3 Proposition. *$S_1(\circ)$ and $S_2(\circ)$ are (up to isomorphism) the only minimal* SH-*groupoids of type* (a, b, a) *and subtype* (α).

Proof. Easy (use the preceding results).

4.1 Let G be a minimal SH-groupoid of type (a, b, a) and subtype (γ). Let $a, b \in G$, $a \cdot ba \neq ab \cdot a$. Put $d = ba$, $e = ad = a \cdot ba$. We have $a = ab$, $e \neq f = ab \cdot a = a^2, b = b^2, a^2 \neq a \neq d, b \neq d$ (otherwise $a = ab = a \cdot ba = e$), $a \neq e \neq b \neq a^2$. Furthermore, if $d = a^2$, then $e = ad = a^3 = da = ba \cdot a = ba^2 = bd = b \cdot ba = b^2a = ba = d = a^2 = ab \cdot a = f$, a contradiction. We have proved that $d \neq a$.

Now, consider the following four-element groupoid $S_3(\circ)$:

$S_3(\circ)$	a	b	a^2	d
a	a^2	a	d	d
b	d	b	d	d
a^2	d	a^2	d	d
d	d	d	d	d

It is easy to check that $S_3(\circ)$ is a minimal SH-groupoid of type (a, b, a) and subtype (γ). Moreover, $e = a \cdot ba = ad = d$, $f = ab \cdot a = a^2$ and $\text{sdist}(S_3(\circ)) = 1$ (put $a * a = d$).

If $d = e$ in G, then G is isomorphic to $S_3(\circ)$. If $d \neq e$ in G, then a, a^2, b, d, e are pair-wise different, and so G contains at least five elements.

4.2 Let G be a minimal SH-groupoid of type (a, b, a) and subtype (γ); let $a \cdot ba \neq ab \cdot a$, $d = ba$ and $e = a \cdot ba$. Now, define an operation $*$ on G by $x * y = xy$ iff $(x, y) \neq (a, a)$ and $a * a = e$.

Let $x, y, z \in G$. If $a \neq y \neq b$, then $x * (y * z) = x * yz = x \cdot yz = xy \cdot z = (x * y)z = (x * y) * z$ (we have $yz \neq a \neq xy$ by 2.6). Similarly, if $y = a$ and $x \neq a \neq z$ or if $y = b$ and either $x \neq a$ or $z \neq a$, then $x * (y * z) = (x * y) * z$.

If $x = y = a \neq z \neq b$, then $x * (y * z) = a * az = a \cdot az = a^2z = fz = ez = e * z = (a * a) * z = (x * y) * z$ ($fz = ez$ by 1.8(iii)).

If $x = y = a$, $z = b$, then $x * (y * z) = a * ab = a * a = e = a \cdot ba = a(b \cdot ab) = a(ba \cdot b) = (a \cdot ba)b = eb = e * b = (a * a) * b = (x * y) * z$.

If $x = y = z = a$, then $x * (y * z) = a * (a * a) = a * e = ae = a(a \cdot ba) = a^2 \cdot ba = a^2b \cdot a = (a \cdot ab)a = a^2a = a^3 = a \cdot a^2 = ab \cdot a^2 = a(b \cdot a^2) = a(ba \cdot a) = (a \cdot ba)a = ea = e * a = (a * a) = a = (x * y) * z$.

If $y = z = a \neq x$, then $xe = xf = xa^2$ by 1.8(i), and hence $x * (y * z) = x * (a * a) = x * e = xe = xa^2 = xa \cdot a = (x * a)a = (x * a) * a$.

If $y = b$ and either $x \neq a$ or $z \neq a$, then $x * (y * z) = x * (b * z) = x * bz = x \cdot bz = xb \cdot z = (x * b)z = (x * b) * z = (x * y) * z$.

Finally, if $y = b$ and $x = a = z$, then $x * (y * z) = a * (b * a) = a * ba = a \cdot ba = e = a * a = ab * a = (a * b) * a = (x * y) * z$.

We have checked that $G(*)$ is a semigroup. In particular, this implies that $\text{sdist}(G) = 1$.

4.3 Let S be a semigroup contains three (pair-wise different) elements a, b, f such that $a = ab \neq a^2 \neq f$, $a \neq xy$ if $(x, y) \neq (a, b)$, $b = b^2$, $b \neq xy$ if $(x, y) \neq (b, b)$, $fb = f$, $fx = a^2x$ for every $x \neq b$ and $yf = ya^2$ for every $y \in S$. Define an operation $\circ$ on S by $x \circ y = xy$ if $(x, y) \neq (a, a)$ and $a \circ a = f$. Then $S(\circ) = S(\circ, \gamma)$ becomes an SH-groupoid of type (a, b, a) $(a \circ (b \circ a) = a^2 \neq f = (a \circ b) \circ a)$ and subtype (γ). Moreover, $S(\circ)$ is minimal, provided that the semigroup S is generated by the elements a, b, f.

4.4 Proposition. *Every minimal* SH-*groupoid of type* (a, b, a) *and subtype* (α) *can be constructed in the way described in 4.3.*

Proof. See 4.2 and 4.3.

4.5 Example. Consider the following infinite groupoid $S_4(\circ)$:

$S_4(\circ)$	a	a^2	a^3	…	a^n	…	b	d_1	d_2	d_3	…	d_n	…	e
a	a^2	a^3	a^4	…	a^{n+1}	…	a	e	a^3	a^4	…	a^{n+1}	…	a^3
a^2	a^3	a^4	a^5	…	a^{n+2}	…	a^2	a^3	a^4	a^5	…	a^{n+2}	…	a^4
a^3	a^4	a^5	a^6	…	a^{n+3}	…	a^3	a^4	a^5	a^6	…	a^{n+3}	…	a^5
⋮	⋮	⋮	⋮	⋱	⋮	⋱	⋮	⋮	⋮	⋮	⋱	⋮	⋱	⋮
a^n	a^{n+1}	a^{n+2}	a^{n+3}	…	a^{2n}	…	a^n	a^{n+1}	a^{n+2}	a^{n+3}	…	a^{2n}	…	a^{n+2}
⋮	⋮	⋮	⋮	⋱	⋮	⋱	⋮	⋮	⋮	⋮	⋱	⋮	⋱	⋮
b	d_1	d_2	d_3	…	d_n	…	b	d_1	d_2	d_3	…	d_n	…	d_2
d_1	d_2	d_3	d_4	…	d_{n+1}	…	d_1	d_2	d_3	d_4	…	d_{n+1}	…	d_3
d_2	d_3	d_4	d_5	…	d_{n+2}	…	d_2	d_3	d_4	d_5	…	d_{n+2}	…	d_4
d_3	d_4	d_5	d_6	…	d_{n+3}	…	d_3	d_4	d_5	d_6	…	d_{n+3}	…	d_5
⋮	⋮	⋮	⋮	⋱	⋮	⋱	⋮	⋮	⋮	⋮	⋱	⋮	⋱	⋮
d_n	d_{n+1}	d_{n+2}	d_{n+3}	…	d_{2n}	…	d_n	d_{n+1}	d_{n+2}	d_{n+3}	…	d_{2n}	…	d_{n+2}
⋮	⋮	⋮	⋮	⋱	⋮	⋱	⋮	⋮	⋮	⋮	⋱	⋮	⋱	⋮
e	a^3	a^4	a^5	…	a^{n+2}	…	e	a^3	a^4	a^5	…	a^{n+2}	…	a^4

Then S_4 is (up to isomorphism) the only infinite SH-groupoid of type (a, b, a) and subtype (γ).

IV.5 Minimal SH-groupoids of type (a, b, a) and subtype (ε)

5.1 Let G be a minimal SH-groupoid of type (a, b, a) and subtype (ε). Let $a, b \in G$ be such that $a \cdot ba \neq ab \cdot a$ and put $d = ba$ and $e = ad = a \cdot ba$. We have $a^2 = a \neq d$, $a \neq e$, $ab = b \neq d = f$, $b \neq a$, $b \neq e$, $b \neq b^2 \neq a$, $d \neq e$. If $b^2 = d$, then $b^2 = ab^2 = ad = e$, and hence $f = d = e$, a contradiction. Consequently, $b^2 \neq d$ and the elements a, b, b^2, d are pair-wise different.

Now, consider the following four-element groupoid $S_5(\circ)$:

$S_5(\circ)$	a	b	b^2	d
a	a	b	b^2	b^2
b	d	b^2	b^2	b^2
b^2	b^2	b^2	b^2	b^2
d	d	b^2	b^2	b^2

Then $S_5(\circ)$ is a minimal SH-groupoid of type (a, b, a) and subtype (ε). Moreover, $e = a \cdot ba = ad = b^2, f = d$ and sdist $(S_5(\circ)) = 1$ (put $b * a = b^2$).

If $b^2 = e$ in G, then G is isomorphic to $S_5(\circ)$. If $b^2 \neq e$, then a, b, b^2, d, e are pair-wise different and G contains at least five elements.

5.2 Let G be a minimal SH-groupoid of type (a, b, a) and subtype (ε). Let $a \cdot ba \neq ab \cdot a, d = ba$ and $e = ad$. Define a binary operation $*$ on G by $x * y = xy$ if $(x, y) \neq (b, a)$ and $b * a = e$. Then $G(*)$ becomes a semigroup. In particular, sdist $(G) = 1$.

5.3 Let S be a semigroup containing three (pair-wise different) elements a, b, f such that $a = a^2$, $a \neq xy$ if $(x, y) \neq (a, a)$, $b = ab$, $b \neq xy$ if $(x, y) \neq (a, b)$, $xf = xba$ and $fx = bax$ for every $x \in S$, $x \neq a$, $af = ba$ and $fa = f$. Define an operation $\circ$ on S by $x \circ y = xy$ if $(x, y) \neq (b, a)$ and $b \circ a = f$. Then $S(\circ) = S(\circ, \varepsilon)$ is an SH-groupoid of type (a, b, a) and subtype (ε). Moreover, $S(\circ)$ is minimal, provided that S is generated by a, b, f.

5.4 Proposition. *Every minimal* SH-*groupoid of type* (a, b, a) *and subtype* (ε) *can be constructed in the way described in 5.3.*

Proof. See 5.2 and 5.3.

5.5 Example. Consider the following infinite groupoid $S_6(\circ)$:

$S_6(\circ)$	a	b	b^2	b^3	…	b^n	…	d_1	d_2	d_3	…	d_n	…	e
a	a	b	b^2	b^3	…	b^n	…	e	d_2	d_3	…	d_n	…	e
b	d_1	b^2	b^3	b^4	…	b^{n+1}	…	d_2	d_3	d_4	…	d_{n+1}	…	d_2
b^2	d_2	b^3	b^4	b^5	…	b^{n+2}	…	d_3	d_4	d_5	…	d_{n+2}	…	d_3
b^3	d_3	b^4	b^5	b^6	…	b^{n+3}	…	d_4	d_5	d_6	…	d_{n+3}	…	d_4
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
b^n	d_n	b^{n+1}	b^{n+2}	b^{n+3}	…	b^{2n}	…	d_2	d_3	d_4	…	d_{n+1}	…	d_2
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
d_1	d_1	b^2	b^3	b^4	…	b^{n+1}	…	d_2	d_3	d_4	…	d_{n+1}	…	d_2
d_2	d_2	b^3	b^4	b^5	…	b^{n+2}	…	d_3	d_4	d_5	…	d_{n+2}	…	d_3
d_3	d_3	b^4	b^5	b^6	…	b^{n+3}	…	d_4	d_5	d_6	…	d_{n+3}	…	d_4
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
d_n	d_n	b^{n+1}	b^{n+2}	b^{n+3}	…	b^{2n}	…	d_{n+1}	d_{n+2}	d_{n+3}	…	d_{2n}	…	d_{n+1}
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
e	e	b^2	b^3	b^4	…	b^{n+1}	…	d_2	d_3	d_4	…	d_{n+1}	…	d_2

IV.6 Minimal SH-groupoids of type (a, b, a) and subtype (ϱ)

6.1 Remark. Let G be a minimal SH-groupoid of type (a, b, a) and subtype (ϱ). Let $a, b \in G$ be such that $a \cdot ba \neq ab \cdot a$. Put $c = ab$, $d = ba$, $e = a \cdot ba$ and $f = ab \cdot a$. It is tedious, but easy to check that sdist$(G) = 1$ iff at least one of the following six conditions is satisfied:
(1) There exists an element $c' \in G$ such that $b \neq c' \neq e$, $cx = c'x$ and $xc = xc'$ for every $x \in G$, $x \neq a$, and $ac' = c'$, $c'a = e$ (this implies that $c' \neq a$);
(2) $d = ba = e$ and $bx = cx$, $xb = xc$ for every $x \in G$, $x \neq a$;
(3) There exists an element $d' \in G$ such that $b \neq d' \neq d$, $dx = dx'$ and $xd = xd'$ for every $x \in G$, $x \neq a$, and $d'a = d'$, $ad' = f$ (this implies that $d' \neq a$);
(4) $c = ab = f$ and $bx = dx$, $xb = xd$ for every $x \in G$, $x \neq a$;
(5) If $x, y \in G$ are such that $xy = c$, then $(x, y) \in \{\{a,b), (a, c), (c, a)\}$;
(6) If $x, y \in G$ are such that $xy = d$, then $(x, y) \in \{(b, a), (a, d), (d, a)\}$.

6.2 Let G be a minimal SH-groupoid of type (a, b, a) and subtype (ϱ). Then a, b, c, d are pair-wise different elements and $c \neq e$, $d \neq f$, $b \neq b^2$. If $b^2 = c$, then G is isomorphic to one of the following four (pair-wise non-isomorphic) groupoids $S_7(\circ)$, $S_8(\circ)$, $S_9(\circ)$, $S_{10}(\circ)$:

$S_7(\circ)$	a	b	c	d	e	f
a	a	c	c	e	e	f
b	d	c	c	f	f	f
c	f	c	c	f	f	f
d	d	c	c	f	f	f
e	e	c	c	f	f	f
f	f	c	c	f	f	f

$S_8(\circ)$	a	b	c	d	e
a	a	c	c	e	e
b	d	c	c	c	c
c	c	c	c	c	c
d	d	c	c	c	c
e	e	c	c	c	c

$S_9(\circ)$	a	b	c	d	f
a	a	c	c	d	f
b	d	c	c	f	f
c	f	c	c	f	f
d	d	c	c	f	f
f	f	c	c	f	f

$S_{10}(\circ)$	a	b	c	d
a	a	c	c	d
b	d	c	c	c
c	c	c	c	c
d	d	c	c	c

6.3 Example.

$S_{11}(\circ)$	a	b	c	d	g
a	a	c	c	d	g
b	d	g	g	g	g
c	c	g	g	g	g
d	d	g	g	g	g
g	g	g	g	g	g

IV.7 Minimal SH-groupoids of type (a, b, a) and subtype (η)

7.1 Remark. Let G be a minimal SH-groupoid of type (a, b, a) and subtype (η). Let $a, b \in G$ be such that $a \cdot ba \neq ab \cdot a$. Put $c = ab$, $d = ba$, $e = a \cdot ba$ and $f = ab \cdot a$. Then $\mathrm{sdist}(G) = 1$ iff at least one of the following four conditions is satisfied:

(1) There exists an element $c' \in G$ such that $c' \notin \{a, b, c\}, xc' = xc$ for every $x \in G$, $c'y = cy$ for every $y \in G$, $y \neq a$, and $c'a = e$;

(2) There exists an element $d' \in G$ such that $d' \notin \{a, b, d\}, d'x = dx$ for every $x \in G$, $yd' = yd$ for every $y \in G$, $y \neq a$, and $ad' = f$;

(3) If $x, y \in G$ are such that $xy = c$, then $(x, y) \in \{(a, b), (a, c), (c, a)\}$;

(4) If $x, y \in G$ are such that $xy = d$, then $(x, y) \in \{(b, a), (a, d), (d, a)\}$.

7.2 Example.

$S_{12}(\circ)$	a	b	c	d	e	f
a	e	c	e	e	e	e
b	d	e	e	e	e	e
c	f	e	e	e	e	e
d	e	e	e	e	e	e
e	e	e	e	e	e	e
f	e	e	e	e	e	e ;

$S_{13}(\circ)$	a	b	c	d	f
a	d	a	d	d	d
b	d	d	d	d	d
c	f	d	d	d	d
d	d	d	d	d	d
f	d	d	d	d	d ;

$S_{14}(\circ)$	a	b	c	d	e
a	e	c	e	e	e
b	d	e	e	e	e
c	d	e	e	e	e
d	e	e	e	e	e
e	e	e	e	e	e .

IV.8 Comments and open problems

8.1 Find the number $\mathrm{sdist}(G)$ for SH-groupoids of type (a, b, a). In particular, is $\mathrm{sdist}(G) = 1$ for every minimal SH-groupoid of type (a, b, a)?

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