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On Simulation of Multidimensional Random Sequences

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V článku je nejdříve stručně zrekapitulována metoda pro simulaci závislých náhodných veličin, které mají dané marginální rozdělení a danou kovarianční funkci. Pak je uvedena Kallova metoda pro řešení mnohorozměrného momentového problému. Tyto výsledky jsou aplikovány na problém simulace dvojrozměrného AR(1) procesu. Je odvozena nutná podmínka k tomu, aby existoval dvojrozměrný AR(1) proces s danými momenty a s danou maticí autoregresních koeficientů.

В работе упоминается метод для моделирования зависимых случайных величин с заданным частным распределением и с заданной ковариационной функцией. Приводится метод Калла для решения многомерной проблемы моментов. Результаты применены к моделированию двумерного AR(1) процесса. Выведено необходимое условие для того, чтобы существовал двумерный AR(1) процесс с заданными моментами и с заданной матрицей коэффициентов авторегрессии.

A method for simulating dependent random variables with given moments of their marginal distribution and with a given covariance function is briefly reviewed. The Kall's method for solution of multidimensional moment problem is introduced. The results are applied to simulations of two-dimensional AR(1) processes. A necessary condition for existence of an AR(1) process with given moments and with a given matrix of autoregressive coefficients is derived.

1. Introduction

Classical random number generators produce random samples from the rectangular distribution $R(0, 1)$. If one needs a sample from another distribution, it is possible to use some transformations or special tricks. Since these methods are well known and available in books and papers, we do not introduce further details.

A more complicated problem arises when it is necessary to produce identically distributed random variables $X_1, \dots, X_N$ with a given distribution function F and with a given covariance function. The history of this problem is described in Anděl

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(1989). Since the exact solution is known only in very few cases, the problem was modified in the following way. Denote $m_k = \mathsf{E}X_t^k$ $(k = 0, 1, \ldots, n)$ and try to find a sequence of variables $X_1, \ldots, X_N$ such that each of these variables has the moments $m_0, \ldots, m_n$ and that the sequence has a given covariance function. The results for AR(1) processes are described in Anděl (1989), for linear processes in Anděl (1987) and for some non-linear processes in Anděl and Garrido (1988). Numerical experiences and applications are introduced in Anděl and Zvára (1988). It must be stressed that there are cases when the problem has no solution.

The method mentioned above has three steps:

(i) For a given time series model, from $m_0, \ldots, m_n$ calculate possible moments $s_0, \ldots, s_n$ of the corresponding white noise.

(ii) Check if $s_0, \ldots, s_n$ is a sequence of moments (i.e. if there exists a random variable e_t such that $\mathsf{E}e_t^k = s_k$ $(k = 0, \ldots, n)$).

(iii) If $s_0, \ldots, s_n$ are moments, find a distribution such that $s_0, \ldots, s_n$ are its moments.

A generalization of this procedure to the multidimensional case is rather complicated. Although there exist some papers devoted to the multidimensional moment problem (e.g. Eskin 1960 and Natanson 1954), they do not allow to solve (ii) and (iii) for two or more dimensions. In our paper we adopt the procedure suggested by Kall (1987), which gives the possibility to recognize when the multidimensional problem has no solution.

2. Multidimensional moment problem on the unit-hypercube

Consider random variables $\xi_1, \ldots, \xi_K$ such that $a_{i0} \leqq \xi_i \leqq a_{i1}$, where a_{i0} and a_{i1} are real numbers $(i = 1, \ldots, K)$. The interval

$$I = \mathop{\mathsf{X}}_{i=1}^{K} [a_{i0}, a_{i1}] \tag{2.1}$$

has vertices $\boldsymbol{a}_j$ $(j = 1, \ldots, 2^K)$. In this section we assume that $a_{i0} = 0$ and $a_{i1} = 1$ for $i = 1, \ldots, K$, so that I is the unit-hypercube.

Let B be the set of all subsets of $\{1, \ldots, K\}$. For $A \in B$ define

$$m_A = \mathsf{E} \prod_{i \in A} \xi_i .$$

Further, for any vector $\boldsymbol{b} = (b_1, \ldots, b_K)'$ define

$$h_A(\boldsymbol{b}) = \prod_{i \in A} b_i , \qquad A \in B .$$

To a given $A \in B$ there exists such a vertex $\boldsymbol{a}_j = (\alpha_{j1}, \ldots, \alpha_{jK})'$ that $\alpha_{ji} = 1$ for $i \in A$ and $\alpha_{ji} = 0$ for $i \notin A$. This is a one-to-one mapping from B onto the set of all vertices of the unit-hypercube. For $K = 1, 2, \ldots$ we order the vertices of the K-dimensional unit-hypercube in the following way.

If $K = 1$, then $a_1 = 0$, $a_2 = 1$. If the vertices $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{2^K}$ of the K-dimensional unit-hypercube are already ordered, then the vertices $\boldsymbol{a}_1^*, \ldots, \boldsymbol{a}_{2^{K+1}}^*$ of the $(K + 1)$-dimensional unit-hypercube are given by

$$\boldsymbol{a}_i^* = (\boldsymbol{a}_i', 0)' \qquad \text{for} \quad i = 1, \ldots, 2^K,$$
$$\boldsymbol{a}_i^* = (\boldsymbol{a}_{i-2^K}', 1)' \qquad \text{for} \quad i = 2^K + 1, \ldots, 2^{K+1}.$$

For a given K, define the $2^K \times 2^K$ matrix

$$\boldsymbol{H}_K = (h_\Lambda(\boldsymbol{a}_j)), \qquad \Lambda \in B\,;\ j = 1, \ldots, 2^K,$$

where $\boldsymbol{a}_j$ are ordered according to the above rule and the ordering of Λ corresponds to it in view of the introduced one-to-one mapping. It can be proved (see Kall 1987) that

$$\boldsymbol{H}_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \boldsymbol{H}_{K+1} = \begin{pmatrix} \boldsymbol{H}_K & \boldsymbol{H}_K \\ \boldsymbol{0} & \boldsymbol{H}_K \end{pmatrix}.$$

All the matrices $\boldsymbol{H}_K$ are regular and their inverses are

$$\boldsymbol{H}_1^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \qquad \boldsymbol{H}_{K+1}^{-1} = \begin{pmatrix} \boldsymbol{H}_K^{-1} & -\boldsymbol{H}_K^{-1} \\ \boldsymbol{0} & \boldsymbol{H}_K^{-1} \end{pmatrix}. \tag{2.2}$$

Denote $\boldsymbol{m} = (m_\Lambda, \Lambda \in B)'$ where the components of $\boldsymbol{m}$ are arranged in the same way as the elements $h_\Lambda(\boldsymbol{a}_j)$.

Theorem 2.1. *Let $\boldsymbol{m}$ be a given vector with 2^K components. Then there exists a probability distribution on the K-dimensional unit-hypercube with the moments $\boldsymbol{m}$ if and only if $\boldsymbol{H}_K^{-1}\boldsymbol{m} \geqq 0$.*

Proof. See Kall (1987), Proposition 3.1.

This criterion concerns the existence of a probability measure on the unit-hypercube. It is not very restrictive, since every moment problem with a finite number of moments can be reduced to this case. It follows from the next theorem.

Theorem 2.2. *Let $f_1, \ldots, f_L$ be real Borel measurable functions defined on $(\mathbb{R}_M, \mathscr{B}_M)$. Let μ be a probability measure on $(\mathbb{R}_M, \mathscr{B}_M)$ such that all the functions $f_1, \ldots, f_L$ are integrable with respect to μ. Then there exists a probability measure μ^* with finite support such that*

$$\int f_i \,\mathrm{d}\mu = \int f_i \,\mathrm{d}\mu^* \qquad (i = 1, \ldots, L).$$

It is possible to find μ^* such that its support has maximaly $L + 1$ points.

Proof. See Mulholland and Rogers (1958).

Thus to every M-dimensional distribution with some L moments there exists an M-dimensional distribution with finite support having the same L moments. Further, there exists a linear transformation such that the image of this finite support lies inside the unit-hypercube. Of course, before using Kall's criterion we must transform also the moments. The details can be found in Kall (1987).

Let us remark that if $\boldsymbol{m} = (m_\Lambda, \Lambda \in B)'$ is a vector of moments of a distribution with the support on a general interval (2.1), then there exists a discrete probability distribution concentrated on the vertices of the interval I with the same moments m_Λ, $\Lambda \in B$. An explicit formula for this discrete distribution is given on p. A 122 in Kall (1987).

3. Two-dimensional AR(1) processes

Consider an AR(1) process $\boldsymbol{X}_t = (Y_t, Z_t)'$ given by

$$\boldsymbol{X}_t = \boldsymbol{A}\boldsymbol{X}_{t-1} + \boldsymbol{e}_t$$

where $\boldsymbol{e}_t = (\varepsilon_i, \eta_t)'$ is a two-dimensional strict white noise and

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

To ensure the stationarity of $\boldsymbol{X}_t$ we assume that both the roots of $\boldsymbol{A}$ lie inside the unit circle. Define

$$m_{ab} = \mathsf{E}Y_t^a Z_t^b, \qquad s_{ab} = \mathsf{E}\varepsilon_t^a \eta_t^b$$

for $a, b = 0, 1, \ldots$ Since

$$Y_t = a_{11}Y_{t-1} + a_{12}Z_{t-1} + \varepsilon_t,$$
$$Z_t = a_{21}Y_{t-1} + a_{22}Z_{t-1} + \eta_t,$$

we have

$$Y_t^a = \sum_{i=0}^{a} \sum_{j=0}^{i} \binom{a}{i} \binom{i}{j} a_{11}^j a_{12}^{i-j} Y_{t-1}^j Z_{t-1}^{i-j} \varepsilon_t^{a-i},$$

$$Z_t^b = \sum_{u=0}^{b} \sum_{v=0}^{u} \binom{b}{u} \binom{u}{v} a_{21}^v a_{22}^{u-v} Y_{t-1}^v Z_{t-1}^{u-v} \eta_t^{b-u},$$

and thus

$$m_{ab} = \sum_{i=0}^{a} \sum_{j=0}^{i} \sum_{u=0}^{b} \sum_{v=0}^{u} \binom{a}{i} \binom{i}{j} \binom{b}{u} \binom{u}{v} a_{11}^j a_{12}^{i-j} a_{21}^v a_{22}^{u-v} m_{j+v,\, i+u-j-v} s_{a-i,\, b-u}.$$

From here the values of s_{ab} can be computed recurrently.

Now, we show how to recognize that s_{ab} cannot be moments of a distribution with support inside the unit square. We use the results of Section 2. If we intend to use all the moments s_{ab} for $a \leqq 3$, $b \leqq 3$, we can apply the following theorem.

Theorem 3.1. *Define*

$$\boldsymbol{m} = \left(1, s_{10}, s_{20}, s_{30}, s_{01}, s_{11}, s_{21}, s_{31}, s_{02}, s_{12}, s_{22}, s_{32}, s_{03}, s_{13}, s_{23}, s_{33}\right)' .$$

If the relation $\boldsymbol{H}_4^{-1}\boldsymbol{m} \geqq 0$ does not hold, then there exists no two-dimensional distribution of $(\varepsilon_t, \eta_t)'$ with support inside the unit square such that the numbers s_{ab} $(a \leqq 3,\ b \leqq 3)$ are its moments.

Proof. The assertion is a consequence of the results introduced in Section 2. If $\boldsymbol{H}_4^{-1}\boldsymbol{m} \geqq 0$ does not hold, then there exists no vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3, \xi_4)'$ with moments $\boldsymbol{m}$ and with a distribution which has support inside the unit-hypercube. Thus it cannot exist a vector $\boldsymbol{\xi} = (\varepsilon_t, \varepsilon_t^2, \eta_t, \eta_t^2)'$ with moments $\boldsymbol{m}$ such that $0 \leqq \varepsilon_t \leqq 1$, $0 \leqq \eta_t \leqq 1$.

Using (2.2), we can easily derive that

$$\boldsymbol{H}_4^{-1} = \begin{pmatrix}
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\
 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \\
 & & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 \\
 & & & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\
 & & & & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\
 & & & & & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\
 & & & & & & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
 & & & & & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
 & & & & & & & & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
 & & & & & & & & & 1 & 0 & -1 & 0 & -1 & 0 & 1 \\
 & \mathbf{0} & & & & & & & & & 1 & -1 & 0 & 0 & -1 & 1 \\
 & & & & & & & & & & & 1 & 0 & 0 & 0 & -1 \\
 & & & & & & & & & & & & 1 & -1 & -1 & 1 \\
 & & & & & & & & & & & & & 1 & 0 & -1 \\
 & & & & & & & & & & & & & & 1 & -1 \\
 & & & & & & & & & & & & & & & 1
\end{pmatrix}.$$

If we want to include also higher moments, then we can use either $\boldsymbol{\xi} = (\varepsilon_t, \varepsilon_t^2, \varepsilon_t^3, \varepsilon_t^4, \eta_t, \eta_t^2, \eta_t^3, \eta_t^4)'$ or $\boldsymbol{\xi} = (\varepsilon_t, \varepsilon_t^2, \varepsilon_t^4, \eta_t, \eta_t^2, \eta_{4t})'$ and to apply Theorem 2.1 in a similar way.

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