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Groupoids and the Associative Law I. (Associative Triples)

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Associative triples of elements in grupoids are investigated.

Zkoumají se asociativní trojice prvků v grupoidech.

В статье изучаются ассоциативные тройки в групоидах.

This paper starts a series of (more or less) expository articles devoted to the same topic, namely, to the role of the associative law in groupoids. In this first part, sets of associative triples are investigated.

I.1. Associative triples - first concepts

1.1. Let G be a groupoid (i.e. a non empty set together with a binary operation). An ordered triple (a, b, c) of elements of G is said to be associative if $ab \,.\, c = (ab)\, c = a(bc) = a \,.\, bc$. The triple is said to be non-associative in the opposite case.

We denote by $\mathrm{As}(G)$ the set of associative triples of the groupoid G and by $\mathrm{Ns}(G)$ the set of non-associative triples. Thus $\mathrm{Ns}(G) = G^{(3)} - \mathrm{As}(G)$, $\mathrm{As}(G) = G(3) - \mathrm{Ns}(G)$, $\mathrm{As}(G) \cap \mathrm{Ns}(G) = \emptyset$ and $\mathrm{As}(G) \cup \mathrm{Ns}(G) = G^{(3)}$.

Further, we put $\mathrm{as}(G) = \mathrm{card}(\mathrm{As}(G))$ and $\mathrm{ns}(G) = \mathrm{card}(\mathrm{Ns}(G))$, so that $\mathrm{card}(G^{(3)}) = \mathrm{as}(G) + \mathrm{ns}(G)$. If G is finite and $\mathrm{card}(G) = n$, then $0 \leqq \mathrm{as}(G), \mathrm{ns}(G) \leqq n^3$ and $\mathrm{as}(G) + \mathrm{ns}(G) = n^3$. If G is infinite, then $0 \leqq \mathrm{as}(G)$, $\mathrm{ns}(G) \leqq \mathrm{card}(G)$ and at least one of the cardinal numbers $\mathrm{as}(G)$, $\mathrm{ns}(G)$ is equal to $\mathrm{card}(G)$.

1.2. A groupoid G is said to be associative (or a semigroup) if $\mathrm{As}(G) = G^{(3)}$ (or, equivalently, $\mathrm{Ns}(G) = \emptyset$).

A groupoid G is said to be antiassociative if $\mathrm{As}(G) = \emptyset$ (or, equivalently, $\mathrm{Ns}(G) = G^{(3)}$).

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1.3. Proposition. (*i*) *A groupoid G is associative (resp. antiassociative) iff* $\mathrm{ns}(G) = 0$ (*resp.* $\mathrm{as}(G) = 0$).
(ii) A finite groupoid G of order n is associative (resp. antiassociative) iff $\mathrm{as}(G) = n^3$ (resp. $\mathrm{ns}(G) = n^3$).

Proof. The assertion follow easily from the definitions.

1.4. Lemma. *Let G be a groupoid.*

(i) *If* $a, b \in G$ *are such that* $ab = ba$ *and* $a \,.\, ab = ab \,.\, a$, *then* $(a, b, a) \in \mathrm{As}(G)$.
(ii) *If* $a, b, c \in G$ *are such that* $ab = b = bc$, *then* $(a, b, c) \in \mathrm{As}(G)$.
(iii) *If* $a \in G$ *is an idempotent element (i.e.* $aa = a$*), then* $(a, a, a) \in \mathrm{As}(G)$.
(iv) *If* $e \in G$ *is a left neutral element (i.e.* $ex = x$*), then* $(e, x, y) \in \mathrm{As}(G)$ *for all* $x, y \in G$.
(v) *If* $e \in G$ *is a right neutral element (i.e.* $xe = x$*), then* $(x, y, e) \in \mathrm{As}(G)$ *for all* $x, y \in G$.
(vi) *If* $e \in G$ *is a neutral element, then* $(x, e, y) \in \mathrm{As}(G)$ *for all* $x, y \in G$.
(vii) *If* $z \in G$ *is a left dominant element (i.e.* $zx = z$*), then* $(z, x, y) \in \mathrm{As}(G)$ *for all* $x, y \in G$.
(viii) *If* $z \in G$ *is a right dominant element (i.e.* $xz = z$*), then* $(x, y, z) \in \mathrm{As}(G)$ *for all* $x, y \in G$.
(ix) *If* $z \in G$ *is a dominant element, then* $(x, z, y) \in G$ *for all* $x, y \in G$.

Proof. All these assertions are easy to check.

1.5. Proposition. *Let G be a finite groupoid of order n.*

(i) *If G is commutative, then* $n^2 \leqq \mathrm{as}(G)$ *and* $\mathrm{ns}(G) \leqq n^3 - n^2$.
(ii) *If G is idempotent, then* $n \leqq \mathrm{as}(G)$ *and* $\mathrm{ns}(G) \leqq n^3 - n$.
(iii) *If G contains at least one left (or right) neutral (or dominant) element, then* $n^2 \leqq \mathrm{as}(G)$ *and* $\mathrm{ns}(G) \leqq n^3 - n^2$.
(iv) *If G contains a neutral (dominant) element, then* $n^3 - (n - 1)^3 = 3n^2 - 3n + 1 \leqq \mathrm{as}(G)$ *and* $\mathrm{ns}(G) \leqq (n - 1)^3$.

Proof. Easy consequence of 1.4.

1.6. Proposition. *Let G be an infinite groupoid. Then* $\mathrm{as}(G) = \mathrm{card}(G)$ *in each of the following cases:*

(i) *G is commutative.*
(ii) *G is idempotent.*
(iii) *G contains at least one (left, right) neutral (dominant) element.*

Proof. Easy consequences of 1.4.

1.7. Lemma. *Let G, H be groupoids and* $K = G \times H$ *(the cartesian product). Then* $\mathrm{as}(K) = \mathrm{as}(G) \,.\, \mathrm{as}(H)$. *In particular, if at least one of the groupoids G, H is antiassociative, then K is so.*

Proof. Easy to check.

1.8. Construction. Let G be a set containing at least two elements and let $f \in \mathrm{T}(G)$ be such that $f(x) \neq x$ for each $x \in G$ (here, $\mathrm{T}(G)$ denotes the monoid of transformations of G). Define a multiplication on G by $xy = f(y)$ for all $x, y \in G$. If $a, b, c \in G$, then $a \,.\, bc = f(bc) = f^2(c) \neq f(c) = ab \,.\, c$ and we see that the groupoid G is antiassociative.

1.9. Construction. Let G be a set containing at least two elements and let $f \in \mathrm{T}(G)$. Put $\mathrm{k}(f) = \mathrm{card}(\{x \in G; f^2(x) = f(x)\})$ and define a multiplication on G by $xy = f(y)$ for all $x, y \in G$. Then G becomes a groupoid and $(x, y, z) \in \mathrm{As}(G)$ iff $f^2(z) = f(z)$. Thus $\mathrm{as}(G) = n^2 \,.\, \mathrm{k}(f)$ for G finite of order n, $\mathrm{as}(G) = \mathrm{card}(G)$ for G infinite and $\mathrm{k}(f) \neq \emptyset$ and $\mathrm{as}(G) = 0$ if $\mathrm{k}(f) = 0$.

1.10. Remark. Let S be a set with $\mathrm{card}(S) \geqq 2$. It is easy to check that for each $0 \leqq \alpha \leqq \mathrm{card}(S)$ one can find at least one $f \in \mathrm{T}(S)$ with $\mathrm{k}(f) = \alpha$.

1.11. Construction. Let K be a finite groupoid (or empty set), M a non-empty finite set disjoint with K and $G = K \cup M$; put $n = \mathrm{card}(K)$ and $m = \mathrm{card}(M)$. Further, let $f, g \in \mathrm{T}(M)$ and $\mathrm{k}(f) = \mathrm{card}(\{x \in M; f^2(x) = f(x)\})$, $\mathrm{k}(g) = \mathrm{card}(\{x \in M; g^2(x) = g(x)\})$, $\mathrm{l}(f, g) = \mathrm{card}(\{x \in M;\ f g(x) = g f(x)\})$. Now, define a multiplication on G in such a way that K is a subgroupoid of G (if $K \neq \emptyset$), $xy = f(y)$, $ax = f(x)$ and $xa = g(x)$ for all $x, y \in M$ and $a \in K$. Then G becomes a groupoid and, as one may check easily, $\mathrm{as}(G) = \mathrm{as}(K) + (n + m)^2\, \mathrm{k}(f) + n^2\, \mathrm{k}(g) + n(n + m)\, \mathrm{l}(f, g)$.

In particular, we have:

(i) $\mathrm{as}(G) = m^2\, \mathrm{k}(f)$ if $n = 0$;
(ii) $\mathrm{as}(G) = 1 + (m + 1)^2\, \mathrm{k}(f) + \mathrm{k}(g) + (m + 1)\, \mathrm{l}(f, g)$ if $n = 1$;
(iii) $\mathrm{as}(G) = \mathrm{as}(K) + (m + 2)^2\, \mathrm{k}(f) + 4\mathrm{k}(g) + 2(m + 2)\, \mathrm{l}(f, g)$ if $n = 2$;
(iv) $\mathrm{as}(G) = \mathrm{as}(K) + 3n^2 + 3n + 1$ if $m = 1$;
(v) $\mathrm{as}(G) = 9\mathrm{k}(f)$ if $n = 0$ and $m = 3$;
(vi) $\mathrm{as}(G) = 1 + 9\mathrm{k}(f) + \mathrm{k}(g) + 3.\mathrm{l}(f, g)$ if $n = 1$ and $m = 2$;
(vii) $\mathrm{as}(G) = \mathrm{as}(K) + 19$ if $n = 2$ and $m = 1$.

1.12. Construction. Let H be a finite groupoid of order n and let $e \notin H$, $G = H \cup \{e\}$. Define a multiplication on G in such a way that H is a subgroupoid of G and e is a netrual (resp. dominant) element of G. Then $\mathrm{as}(G) = \mathrm{as}(H) + 3n^2 + 3n + 1$.

1.13. Construction. Let H be a finite groupoid of order n and let $e \notin H$, $G = H \cup \{e\}$. Define a multiplication on G in such a way that H is a subgroupoid of G and e is a left neutral and a right dominant element of G. Then $\mathrm{as}(G) = \mathrm{as}(H) + 2n^2 + 3n + 1 + \mathrm{z}(H)$, $\mathrm{z}(H) = \mathrm{card}(\{x, y);\ x, y \in H,\ xy = y\})$.

1.14. Construction. Let K be a finite groupoid of order n, $e \notin K$ and $G = K \cup \{e\}$. Let $f, g \in \mathrm{T}(G)$ be such that $f(e) = g(e)$. Now, define a multiplication on G in such a way that K is a subgroupoid of G and $xe = f(x)$, $ex = g(x)$ for each $x \in G$. We obtain a groupoid G and it is easy to check that $\mathrm{as}(G) = \mathrm{as}(K) + \mathrm{i}_1 + \mathrm{i}_2 + \mathrm{i}_3 + \mathrm{i}_4 + \mathrm{i}_5 + \mathrm{i}_6 + \mathrm{i}_7$ where:

$\mathrm{i}_1 = \mathrm{card}(\{(x, y) \in K^{(2)};\ f(xy) = x\,f(y)\})$,
$\mathrm{i}_2 = \mathrm{card}(\{(x, y) \in K^{(2)};\ g(xy) = g(x)\,y\})$,
$\mathrm{i}_3 = \mathrm{card}(\{(x, y) \in K^{(2)};\ f(x)\,y = x\,g(y)\})$,
$\mathrm{i}_4 = \mathrm{card}(\{x \in K;\ f^2(x) = x\,f(e)\})$,
$\mathrm{i}_5 = \mathrm{card}(\{x \in K;\ g^2(x) = g(e)\,x\})$,
$\mathrm{i}_6 = \mathrm{card}(\{x \in K;\ f\,g(x) = g\,f(x)\})$,
$\mathrm{i}_7 = 1$ if $f^2(e) = g^2(e)$ and $\mathrm{i}_7 = 0$ if $f^2(e) \neq g^2(e)$.

I.2. Auxiliary results

2.1. Let S be a non-empty finite set. For $f \in \mathrm{T}(S)$, let $\mathrm{k}(f) = \mathrm{card}(\{x \in S;\ f^2(x) = f(x)\})$; for $f, g \in \mathrm{T}(S)$, let $\mathrm{l}(f, g) = \mathrm{card}(\{x \in S; f\,g(x) = g\,f(x)\})$ – see 1.11. Further, put $\mathrm{o}(f, g) = (\mathrm{k}(f), \mathrm{k}(g), \mathrm{l}(f, g))$.

If $\mathrm{card}(S) = 1$, then $\{\mathrm{o}(f, g)\} = \{(1, 1, 1)\}$.

2.2. Let $S = \{0, 1\}$.

(i) If $f = g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $\mathrm{o}(f, g) = (0, 0, 2)$.

(ii) If $f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $\mathrm{o}(f, g) = (0, 2, 0)$ and $\mathrm{o}(g, f) = (2, 0, 0)$.

(iii) If $f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, then $\mathrm{o}(f, g) = (0, 2, 2)$ and $\mathrm{o}(g, f) = (2, 0, 2)$.

(iv) If $f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, then $\mathrm{o}(f, g) = (2, 2, 0)$.

(v) If $f = g = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, then $\mathrm{o}(f, g) = (2, 2, 2)$.

2.3. Let S be a two-element set. It is easy to check that $\{\mathrm{o}(f, g)\} = \{(0, 0, 2), (0, 2, 0), (2, 0, 0), (0, 2, 2), (2, 0, 2), (2, 2, 0), (2, 2, 2)\}$.

2.4. Let $S = \{0, 1, 2\}$.

(i) If $f = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$, then $\mathrm{o}(f, g) = (0, 0, 0)$.

(ii) If $f = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$, then $\mathrm{o}(f, g) = (0, 0, 1)$.

(iii) If $f = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}$, then $\mathrm{o}(f, g) = (0, 0, 2)$.

(iv) If $f = g = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$, then $o(f, g) = (0, 0, 3)$.

(v) If $f = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$, then $o(f, g) = (0, 1, 0)$ and $o(g, f) = (1, 0, 0)$.

(vi) If $f = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$, then $o(f, g) = (0, 1, 2)$ and $o(g, f) = (1, 0, 2)$.

(vii) If $f = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix}$, then $o(f, g) = (0, 2, 0)$ and $o(g, f) = (2, 0, 0)$.

(viii) If $f = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$, then $o(f, g) = (0, 2, 1)$ and $o(g, f) = (2, 0, 1)$.

(ix) If $f = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, then $o(f, g) = (0, 3, 0)$ and $o(g, f) = (3, 0, 0)$.

(x) If $f = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$, then $o(f, g) = (0, 3, 1)$ and $o(g, f) = (3, 0\ 1)$.

(xi) If $f = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 0 \end{pmatrix}$, then $o(f, g) = (0, 3, 2)$ and $o(g, f) = (3, 0, 2)$.

(xii) If $f = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$, then $o(f, g) = (0, 3, 3)$ and $o(g, f) = (3, 0, 3)$.

(xiii) If $f = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}$, then $o(f, g) = (1, 1, 0)$.

(xiv) If $f = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$, then $o(f, g) = (1, 1, 3)$.

(xv) If $f = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix}$, then $o(f, g) = (1, 2, 0)$ and $o(g, f) = (2, 1, 0)$.

(xvi) If $f = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix}$, then $o(f, g) = (1, 2, 1)$ and $o(g, f) = (2, 1, 1)$.

(xvii) If $f = \begin{pmatrix}0 & 1 & 2\\ 1 & 0 & 2\end{pmatrix}$ and $g = \begin{pmatrix}0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix}$, then $o(f, g) = (1, 3, 0)$ and $o(g, f) = (3, 1, 0)$.

(xviii) If $f = \begin{pmatrix}0 & 1 & 2\\ 0 & 2 & 1\end{pmatrix}$ and $g = \begin{pmatrix}0 & 1 & 2\\ 0 & 0 & 2\end{pmatrix}$, then $o(f, g) = (1, 3, 1)$ and $o(g, f) = (3, 1, 1)$.

(xix) If $f = \begin{pmatrix}0 & 1 & 2\\ 0 & 2 & 1\end{pmatrix}$ and $g = \begin{pmatrix}0 & 1 & 2\\ 2 & 1 & 2\end{pmatrix}$, then $o(f, g) = (1, 3, 2)$ and $o(g, f) = (3, 1, 2)$.

(xx) If $f = \begin{pmatrix}0 & 1 & 2\\ 0 & 2 & 1\end{pmatrix}$ and $g = \begin{pmatrix}0 & 1 & 2\\ 0 & 1 & 0\end{pmatrix}$, then $o(f, g) = (1, 3, 3)$ and $o(g, f) = (3, 1, 3)$.

(xxi) If $f = \begin{pmatrix}0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix}$ $\begin{pmatrix}0 & 1 & 2\\ 1 & 1 & 0\end{pmatrix}$, then $o(f, g) = (2, 2, 0)$.

(xxii) If $f = \begin{pmatrix}0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix}$ and $g = \begin{pmatrix}0 & 1 & 2\\ 0 & 2 & 0\end{pmatrix}$, then $o(f, g) = ((2, 2, 1)$.

(xxiii) If $f = \begin{pmatrix}0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix}$ and $g = \begin{pmatrix}0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix}$, then $o(f, g) = (2, 2, 3)$.

(xxiv) If $f = \begin{pmatrix}0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix}$ and $g = \begin{pmatrix}0 & 1 & 2\\ 1 & 1 & 1\end{pmatrix}$, then $o(f, g) = (2, 3, 0)$ and $o(g, f) = (3, 2, 0)$.

(xxv) If $f = \begin{pmatrix}0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix}$ and $g = \begin{pmatrix}0 & 1 & 2\\ 0 & 2 & 2\end{pmatrix}$, then $o(f, g) = (2, 3, 1)$ and $o(g, f) = (3, 2, 1)$.

(xxvi) If $f = \begin{pmatrix}0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix}$ and $g = \begin{pmatrix}0 & 1 & 2\\ 0 & 1 & 0\end{pmatrix}$, then $o(f, g) = (2, 3, 2)$ and $o(g, f) = (3, 2, 2)$.

(xxvii) If $f = \begin{pmatrix}0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix}$ and $g = \begin{pmatrix}0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix}$, then $o(f, g) = (2, 3, 3)$ and $o(g, f) = (3, 2, 3)$.

(xxviii) If $f = \begin{pmatrix}0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix}$ and $g = \begin{pmatrix}0 & 1 & 2\\ 2 & 1 & 2\end{pmatrix}$, then $o(f, g) = (3, 3, 0)$.

(xxix) If $f = \begin{pmatrix}0 & 1 & 2\\ 0 & 1 & 0\end{pmatrix}$ and $g = \begin{pmatrix}0 & 1 & 2\\ 2 & 1 & 2\end{pmatrix}$, then $o(f, g) = (3, 3, 1)$.

(xxx) If $f = \begin{pmatrix}0 & 1 & 2\\ 0 & 1 & 0\end{pmatrix}$ and $g = \begin{pmatrix}0 & 1 & 2\\ 0 & 1 & 1\end{pmatrix}$, then $o(f, g) = (3, 3, 2)$.

(xxxi) If $f = \begin{pmatrix}0 & 1 & 2\\ 0 & 1 & 2\end{pmatrix} = g$, then $o(f, g) = (3, 3, 3)$.

2.5. Let S be a three-element set. It is tedious but easy to check that $\{o(f, g)\} = \{(0,0,0), (0,0,1), (0,0,2), (0,0,3), (0,1,0), (0,1,2), (0,2,0), (0,2,1), (0,3,0), (0,3,1), (0,3,2), (0,3,3), (1,0,0), (1,0,2), (1,1,0), (1,1,3), (1,2,0), (1,2,1), (1,3,0), (1,3,1), (1,3,2), (1,3,3), (2,0,0), (2,0,1), (2,1,0), (2,1,1), (2,2,0), (2,2,1), (2,2,3), (2,3,0), (2,3,1), (2,3,2), (2,3,3), (3,0,0), (3,0,1), (3,0,2), (3,0,3), (3,1,0), (3,1,1), (3,1,2), (3,1,3), (3,2,0), (3,2,1), (3,2,2), (3,2,3), (3,3,0), (3,3,1), (3,3,2), (3,3,3)\}$.

I.3. Two - element groupoids

3.1. Consider the following ten two-element groupoids:

A_1	0	1
0	0	0
1	0	0

A_2	0	1
0	1	0
1	0	0

A_3	0	1
0	0	1
1	0	0

A_4	0	1
0	0	0
1	1	0

A_5	0	1
0	0	0
1	0	1

A_6	0	1
0	1	1
1	0	0

A_7	0	1
0	1	0
1	1	0

A_8	0	1
0	0	1
1	0	1

A_9	0	1
0	0	0
1	1	1

A_{10}	0	1
0	0	1
1	1	0

It is easy to check that these groupoids are pair-wise non-isomorphic and, up to isomorphism, they are the only two-element groupoids. Moreover, A_3 is anti-isomorphic to A_4, A_6 to A_7 and A_8 to A_9 (in fact, $A_4 = A_3^{op}$, $A_7 = A_6^{op}$ and $A_9 = A_8^{op}$).

The groupoids $A_1, A_5, A_8, A_9, A_{10}$ are associative and the groupoids A_2, A_3, A_4, A_6, A_7 are non-associative. More precisely: $As(A_2) = \{(0,0,0), (0,1,0), (1,0,1), (1,1,1)\}$ $as(A_2) = 4 = ns(A_2)$; $As(A_3) = As(A_4) = \{(0,0,0), (0,0,1), (0,1,0), (1,0,0), (0,1,1), (1,1,0)\}$ and $as(A_3) = 6 = as(A_4)$, $ns(A_3) = 2 = ns(A_4)$; $As(A_6) = As(A_7) = \emptyset$, $as(A_6) = 0 = as(A_7)$, $ns(A_6) = 8 = ns(A_7)$.

Put $z(A_i) = card(\{(x, y);\ x, y \in A_i,\ xy = y\})$. Then $z(A_1) = z(A_6) = z(A_9) = z(A_{10}) = 2$, $z(A_2) = z(A_4) = 1$, $z(A_3) = z(A_5) = 3$, $z(A_7) = 0$ and $z(A_8) = 4$.

I.4. Three - element groupoids

4.1. By Construction 1.12, where $H = A_1, A_2, A_3, A_6$, resp., and $e = 2$, we get the following four three-element groupoids:

B_{27}	0	1	2
0	0	0	0
1	0	0	1
2	0	1	2

B_{23}	0	1	2
0	1	0	0
1	0	0	1
2	0	1	2

B_{25}	0	1	2
0	0	1	0
1	0	0	1
2	0	1	2

B_{19}	0	1	2
0	1	1	0
1	0	0	1
2	0	1	2

Now, $as(B_i) = as(H) + 19$, and so $as(B_{27}) = 27$, $as(B_{23}) = 23$, $as(B_{25}) = 25$ and $as(B_{19}) = 19$ (see 3.1).

4.2. By Construction 1.13, where H = A_2, A_3, A_4, A_5, A_6, A_7, resp., and $e = 2$, we get the following six three-element groupoids:

B_{20}	0	1	2
0	1	0	2
1	0	0	2
2	0	1	2

B_{24}	0	1	2
0	0	1	2
1	0	0	2
2	0	1	2

B_{22}	0	1	2
0	0	0	2
1	1	0	2
2	0	1	2

B_{26}	0	1	2
0	0	0	2
1	0	1	2
2	0	1	2

B_{17}	0	1	2
0	1	1	2
1	0	0	2
2	0	1	2

B_{15}	0	1	2
0	1	0	2
1	1	0	2
2	0	1	2

Now, $\mathrm{as}(B_i) = \mathrm{as}(H) + 15 + \mathrm{z}(H)$, and so $\mathrm{as}(B_{20}) = 20$, $\mathrm{as}(B_{24}) = 24$, $\mathrm{as}(B_{22}) =$ $= 22$, $\mathrm{as}(B_{26}) = 26$, $\mathrm{as}(B_{17}) = 17$, $\mathrm{as}(B_{15}) = 15$ (see 3.1).

4.3. In 1.11, choose $K = \{2\}$ and $M = \{0, 1\}$.
(i) Let $f(0) = 1, f(1) = 0, g(0) = 0, g(1) = 0$. Then $\mathrm{k}(f) = 0$, $\mathrm{k}(g) = 2$, $\mathrm{l}(f, g) = 0$ and so, by 1.11(vi), $\mathrm{as}(G) = 3$ for the corresponding groupoid $G = B_3$:

B_3	0	1	2
0	1	0	0
1	1	0	0
2	1	0	2

(ii) Let $f(0) = 1$, $f(1) = 0$ and $g = f$. Then $\mathrm{k}(f) = \mathrm{k}(g) = 0$, $\mathrm{l}(f, g) = 2$, and so $\mathrm{as}(G) = 7$ for the corresponding groupoid $G = B_7$:

B_7	0	1	2
0	1	0	1
1	1	0	0
2	1	0	2

(iii) Let $f(0) = 1, f(1) = 0, g(0) = 0, g(1) = 1$. Then $\mathrm{k}(f) = 0$, $\mathrm{k}(g) = 2 = \mathrm{l}(f, g)$, and so $\mathrm{as}(G) = 9$ for the corresponding groupoid $G = B_9$:

B_9	0	1	2
0	1	0	0
1	1	0	1
2	1	0	2

(iv) Let $f(0) = 0 = f(1)$, $g(0) = 1 = g(1)$. Then $\mathrm{k}(f) = 2 = \mathrm{k}(g)$, $\mathrm{l}(f, g) = 0$, and so $\mathrm{as}(G) = 21$ for the corresponding groupoid $G = B_{21}$:

B_{21}	0	1	2
0	0	0	1
1	0	0	1
2	0	0	2

4.4. In 1.11, choose $K = \emptyset$ and $M = \{0, 1, 2\}$.

(i) Let $f(0) = 1$, $f(1) = 0 = f(2)$. Then $k(f) = 0$, and so, by 1.11(v), $as(G) = 0$ for the corresponding groupoid $G = B_0$:

B_0	0	1	2
0	1	0	0
1	1	0	0
2	1	0	0

(ii) Let $f(0) = f(1) = 0$, $f(2) = 1$. Then $k(f) = 2$, and so $as(G) = 18$ for the corresponding groupoid $G = B_{18}$:

B_{18}	0	1	2
0	0	0	1
1	0	0	1
2	0	0	1

4.5. In 1.14, choose $K = A_1$, $e = 2$, so that $as(K) = 8$.

(i) Let $f(0) = 1$, $f(1) = 0$, $f(2) = 1$, $g(0) = 1$, $g(1) = 1$, $g(2) = 1$. Then $i_1 = i_2 = i_5 = i_6 = i_7 = 0$, $i_3 = 4$, $i_4 = 1$, and so $as(G) = 13$ for the corresponding groupoid $G = B_{13}$:

B_{13}	0	1	2
0	0	0	1
1	0	0	0
2	1	1	1

(ii) Let $f(0) = 1$, $f(1) = 0$, $f(2) = 0$, $g(0) = 1$, $g(1) = 1$, $g(2) = 0$; then $i_1 = i_2 = i_5 = i_6 = 0$, $i_3 = 4$, $i_4 = i_7 = 1$, and so $as(G) = 14$ for the corresponding groupoid $G = B_{14}$:

B_{14}	0	1	2
0	0	0	1
1	0	0	0
2	1	1	0

4.6. In 1.14, choose $K = A_6$, $e = 2$, so that $as(K) = 0$.

(i) Let $f(0) = 1$, $f(1) = 0$, $f(2) = 0$, $g(0) = 0$, $g(1) = 2$, $g(2) = 0$. Then $i_1 = i_3 = i_4 = i_5 = i_7 = 0$, $i_2 = i_6 = 1$, and so $as(G) = 2$ for the corresponding groupoid $G = B_2$:

B_2	0	1	2
0	1	1	1
1	0	0	0
2	0	2	0

$As(B_2) = \{(2, 1, 0), (2, 1, 2)\}$.

(ii) Let $f(0) = 1, f(1) = 0, f(2) = 2, g(0) = 0, g(1) = 2, g(2) = 2$. Then $i_1 = i_3 = i_4 = i_6 = 0, i_2 = i_7 = 1, i_5 = 2$, and so $as(G) = 4$ for the corresponding groupoid $G = B_4$:

B_4	0	1	2
0	1	1	1
1	0	0	0
2	0	2	2

(iii) Let $f(0) = 2, f(1) = 0, f(2) = 2, g(0) = 0, g(1) = 0, g(2) = 2$. Then $i_1 = i_2 = i_3 = 0, i_4 = i_6 = i_7 = 1, i_5 = 2$, and so $as(G) = 5$ for the corresponding groupoid $G = B_5$:

B_5	0	1	2
0	1	1	2
1	0	0	1
2	0	0	2

(iv) Let $f(0) = 1, f(1) = 2, f(2) = 2, g(0) = 0, g(1) = 0, g(2) = 2$. Then $i_1 = i_2 = i_6 = 0, i_5 = i_7 = 1, i_3 = i_4 = 2$, and so $as(G) = 6$ for the corresponding groupoid B_6:

B_6	0	1	2
0	1	1	1
1	0	0	2
2	0	0	2

(v) Let $f(0) = 0, f(1) = 0, f(2) = 0, g(0) = 0, g(1) = 0, g(2) = 0$. Then $i_2 = i_5 = 0, i_4 = i_7 = 1, i_1 = i_3 = i_6 = 2$, and so $as(G) = 8$ for the corresponding groupoid $G = B_8$:

B_8	0	1	2
0	1	1	0
1	0	0	0
2	0	0	0

(vi) Let $f(0) = 0, f(1) = 0, f(2) = 1, g(0) = 0, g(1) = 0, g(2) = 1$. Then $i_2 = 0, i_4 = i_7 = 1, i_1 = i_3 = i_5 = i_6 = 2$, and so $as(G) = 10$ for the corresponding groupoid $i_4 = i_7 = 1, i_1 = i_3 = i_5 = i_6 = 2$, and so $as(G) = 10$ for the corresponding groupoid $G = B_{10}$:

B_{10}	0	1	2
0	1	1	0
0	0	0	0
2	0	0	1

(vii) Let $f(0) = 0, f(1) = 0, f(2) = 2, g(0) = 0, g(1) = 0, g(2) = 2$. Then $i_2 = 0, i_7 = 1, i_1 = i_3 = i_4 = i_5 = i_6 = 2$, and so $as(G) = 11$ for the corresponding groupoid $G = B_{11}$:

B_{11}	0	1	2
0	1	1	0
1	0	0	0
2	0	0	2

(viii) Let $f(0) = 0$, $f(1) = 0$, $f(2) = 1$, $g(0) = 0$, $g(1) = 1$, $g(2) = 1$. Then $i_7 = 0$, $i_4 = i_5 = 1$, $i_1 = i_3 = i_6 = 2$, $i_2 = 4$, and so $as(G) = 12$ for the corresponding groupoid $G = B_{12}$:

B_{12}	0	1	2
0	1	1	0
1	0	0	0
2	0	1	1

(ix) Let $f(0) = 0$, $f(1) = 1$, $f(2) = 0$, $g(0) = 0$, $g(1) = 1$, $g(2) = 0$. Then $i_4 = 0$, $i_5 = i_7 = 1$, $i_6 = 2$, $i_1 = i_2 = i_3 = 4$, and so $as(G) = 16$ for the corresponding groupoid $G = B_{16}$:

B_{16}	0	1	2
0	1	1	0
1	0	0	1
2	0	1	0

4.7. Consider the following groupoid B_1:

B_1	0	1	2
0	1	2	1
1	0	0	0
2	2	1	0

It is easy to check that $As(B_1) = \{(0, 1, 1)\}$, and so $as(B_1) = 1$.

4.8. Proposition. *Let* $0 \leqq m \leqq 27$. *Then there exist a three-element groupoid* G *such that* $as(G) = m$.

Proof. See 4.1, 4.2, 4.3, 4.4, 4.5, 4.6, and 4.7.

I.5. Four - element groupoids

5.1. Proposition. *Let* $0 \leqq m \leqq 64$. *Then there exists a four-element groupoid* G *such that* $as(G) = m$.

Proof. Denote by N the set of the numbers $as(G)$ where G runs through four-element groupoids. The rest of the proof is divided into several parts:

(i) By 1.8, $0 \in N$.

(ii) By 1.11 (where $n = 1$, $m = 3$), $1 + 16k(f) + k(g) + 4.1(f, g) \in N$ for all transformations f, g of a three-element set. Now, 2.4 implies

$1, \ldots, 5, 7, \ldots, 10, 12, 13, 16, \ldots, 20, 23, 24, 25, 28, 30, 32, \ldots, 40, 44, 47, \ldots, 64 \in N$.

(iii) By 1.11 (where $n = 2$, $m = 2$), $as(K) + 16k(f) + 4k(g) + 8.1(f, g) \in N$ for every two-element groupoid K and all transformations f, g of a two-elementset. Now, 3.1 and 2.2 imply that, in particular, $14, 22, 46 \in N$.

(iv) By 1.11 (where $n = 3$, $m = 1$), $\mathrm{as}(K) + 37 \in N$ for every three-element groupoid K. Now, 4.8 implies that, in particuler, $41, 42, 43, 45 \in N$.

(v) By 1.13, $\mathrm{as}(H) + \mathrm{z}(H) + 28 \in N$ for every three-element groupoid H. If $H = \mathbf{B}_1$ (see 4.7), then $\mathrm{as}(H) = 1$, $\mathrm{z}(H) = 2$, and so $31 \in N$.

(vi) In 1.14, choose $K = \mathbf{B}_0$ (see 4.4(i)) $e = 3$ and $f(0) = 1$, $f(1) = 0$, $f(2) = 0$, $f(3) = 0$, $g(0) = 2$, $g(1) = 0$, $g(2) = 0$, $g(3) = 0$. Then $\mathrm{i}_2 = \mathrm{i}_3 = \mathrm{i}_5 = \mathrm{i}_7 = 0$, $\mathrm{i}_6 = 1$, $\mathrm{i}_4 = 2$, $\mathrm{i}_1 = 3$, and so $\mathrm{as}(G) = 6$ for the corresponding four-element groupoid G (we have $\mathrm{as}(K) = 0$).

(vii) In 1.14, choose $K = \mathbf{B}_0$, $e = 3$ and $f(0) = 1$, $f(1) = 2$, $f(2) = 1$, $f(3) = 0$, $g(0) = 1$, $g(1) = 2$, $g(2) = 1$, $g(3) = 0$. Then $\mathrm{i}_1 = \mathrm{i}_2 = \mathrm{i}_5 = 0$, $\mathrm{i}_4 = \mathrm{i}_7 = 1$, $\mathrm{i}_6 = 3$, $\mathrm{i}_3 = 6$, and so $\mathrm{as}(G) = 11$ for the corresponding groupoid G.

(viii) In 1.14, choose $K = \mathbf{B}_0$, $e = 3$ and $f(0) = 1$, $f(1) = 0$, $f(2) = 1$, $f(3) = 0$, $g(0) = 1$, $g(1) = 0$, $g(2) = 1$, $g(3) = 0$. Then $\mathrm{i}_2 = 0$, $\mathrm{i}_4 = \mathrm{i}_5 = \mathrm{i}_7 = 1$, $\mathrm{i}_3 = \mathrm{i}_6 = 3$, $\mathrm{i}_1 = 6$, and so $\mathrm{as}(G) = 15$ for the corresponding groupoid G.

(ix) In 1.14, choose $K = \mathbf{B}_{18}$ (see 4.4(ii)), $e = 3$ and $f(0) = 1$, $f(1) = 1$, $f(2) = 0$, $f(3) = 3$, $g(0) = 2$, $g(1) = 2$, $g(2) = 1$, $g(3) = 3$. Then $\mathrm{i}_1 = \mathrm{i}_2 = \mathrm{i}_3 = \mathrm{i}_5 = \mathrm{i}_6 = 0$, $\mathrm{i}_7 = 1$, $\mathrm{i}_4 = 2$, and so $\mathrm{as}(G) = 21$ for the corresponding groupoid G (we have $\mathrm{as}(K) = 18$).

(x) In 1.14, choose $K = \mathbf{B}_{18}$, $e = 3$ and $f(0) = 1$, $f(1) = 0$, $f(2) = 1$, $f(3) = 3$, $g(0) = 1$, $g(1) = 2$, $g(2) = 1$, $g(3) = 3$. Then $\mathrm{i}_2 = \mathrm{i}_4 = \mathrm{i}_5 = 0$, $\mathrm{i}_6 = \mathrm{i}_7 = 1$, $\mathrm{i}_1 = \mathrm{i}_3 = 3$, and so $\mathrm{as}(G) = 26$ for the corresponding groupoid G.

(xi) In 1.14, choose $K = \mathbf{B}_{18}$, $e = 3$ and $f(0) = 1$, $f(1) = 2$, $f(2) = 2$, $f(3) = 0$, $g(0) = 2$, $g(1) = 2$, $g(2) = 2$, $g(3) = 0$. Then $\mathrm{i}_2 = \mathrm{i}_4 = \mathrm{i}_5 = \mathrm{i}_7 = 0$, $\mathrm{i}_1 = \mathrm{i}_3 = \mathrm{i}_6 = 3$, and so $\mathrm{as}(G) = 27$ for the corresponding groupoid G.

(xii) In 1.14, choose $K = \mathbf{B}_{18}$, $e = 3$ and $f(0) = 1$, $f(1) = 1$, $f(2) = 0$ $f(3) = 3$, $g(0) = 1$, $g(1) = 2$, $g(2) = 2$, $g(3) = 3$. Then $\mathrm{i}_1 = \mathrm{i}_2 = \mathrm{i}_7 = 0$, $\mathrm{i}_5 = \mathrm{i}_6 = 1$, $\mathrm{i}_4 = 3$, $\mathrm{i}_3 = 6$, and so $\mathrm{as}(G) = 29$ for corresponding groupoid G.

I.6. Five - element groupoids

6.1. Proposition. *Let* $0 \leqq m \leqq 125$. *Then there exists a five-element groupoid* G *such that* $\mathrm{as}(G) = m$.

Proof. Denote by N the set of the numbers $\mathrm{as}(G)$ where G runs through five-element groupoids. The rest of the proof is divided into several parts:

(i) By 1.8, $0 \in N$.

(ii) By 1.12 and 5.1, $61, \ldots, 125 \in N$.

(iii) By 1.11, $\mathrm{as}(K) + 25\mathrm{k}(f) + 9\mathrm{k}(g) + 15.\mathrm{l}(f, g) \in N$ for every three-element groupoid K and all transformations f, g of a two-element set. Now, 4.8 and 2.2 imply $18, \ldots, 45 \in N$ (for $\mathrm{o}(f, g) = (0, 2, 0)$), $30, \ldots, 57 \in N$ (for $\mathrm{o}(f, g) = (0, 0, 2)$), $48, \ldots, 75 \in N$ (for $\mathrm{o}(f, g) = (0, 2, 2)$). In particular, $18, \ldots, 60 \in N$.

(iv) By 1.11, $\mathrm{as}(K) + 25k(f) + 4\mathrm{k}(g) + 10.\mathrm{l}(f, g) \in N$ for every two-element groupoid K and all transformation f, g of a three-element set. Now, by 3.1 and 2.4 (choosing $\mathrm{o}(f, g) = (0, 0, 0), (0, 1, 0), (0, 0, 1))$, we get $4, 6, 8, 10, 12, 14, 16 \in N$.
(v) By 1.11 $\mathrm{q}(f, g) = 1 + 25\mathrm{k}(f) + \mathrm{k}(g) + 5.\mathrm{l}(f, g) \in N$ for all transformations f, g of a four-element set.

If $f = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \end{pmatrix}$, then $\mathrm{q}(f, g) = 1$.

If $f = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 0 & 0 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 0 & 1 & 3 \end{pmatrix}$, then $\mathrm{q}(f, g) = 2$.

If $f = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 2 \end{pmatrix}$, then $\mathrm{q}(f, g) = 3$.

If $f = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 2 & 1 \end{pmatrix}$, then $\mathrm{q}(f, g) = 7$.

If $f = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, then $\mathrm{q}(f, g) = 9$.

If $f = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 1 \end{pmatrix}$, then $\mathrm{q}(f, g) = 11$.

If $f = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 2 & 3 \end{pmatrix}$, then $\mathrm{q}(f, g) = 13$.

If $f = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 0 & 0 & 2 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 2 \end{pmatrix}$, then $\mathrm{q}(f, g) = 15$.

If $f = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ and $g = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 3 & 1 \end{pmatrix}$, then $\mathrm{q}(f, g) = 17$.

I.7. Spectrum of associativity

7.1. Let $\mathscr{A}$ be an abstract class of groupoids (i.e. $\mathscr{A}$ is closed under isomorphic images). For every $n \geqq 1$, we put $\mathrm{specass}(\mathscr{A}, n) = \{\mathrm{as}(G);\ G \in \mathscr{A},\ \mathrm{card}(G) = n\}$. We put also $\mathrm{specass}(n) = \mathrm{specass}(\mathscr{G}, n)$ where $\mathscr{G}$ is the class of all groupoids.

Clearly $\mathrm{specass}(1) = \{1\}$ and, by 2.1, $\mathrm{specass}(2) = \{0, 4, 6, 8\}$. Now, we are going to show that $\mathrm{specass}(n) = \{0, \ldots, n^3\}$ for each $n \geqq 3$.

7.2. Theorem. *Let* $3 \leqq n$ *and* $0 \leqq m \leqq n^3$. *Then there exists an n-element groupoid G such that* $\mathrm{as}(G) = m$.

Proof. It is divided into several parts.
(i) Denote by N the set of $n \geqq 3$ such that $\mathrm{specass}(n) = \{0, \ldots, n^3\}$. By 4.8, 5.1 and 6.1, we have $3, 4, 5 \in N$.
(ii) Let $n \in N$. By 1.11, $\{0, \ldots, n^3\} + (n + 2)^2\,\mathrm{k}(f) + n^2\,\mathrm{k}(g) + n(n + 2)\,\mathrm{l}(f, g) \subseteq$ $\subseteq \mathrm{specass}(n + 2)$ for all transformations f, g of a two-element set. Now, 2.2 implies

that $\{2n^2, \ldots, n^3 + 2n^2\}$, $\{2n^2 + 4n, \ldots, n^3 + 2n^2 + 4n\}$, $\{2n^2 + 8n + 8, \ldots, n^3 + 2n^2 + 8n + 8\}$, $\{4n^2 + 4n, \ldots, n^3 + 4n^2 + 4n\}$, $\{4n^2 + 8n + 8, \ldots, n^3 + 4n^2 + 8n + 8\}$, $\{4n^2 + 12n + 8, \ldots, n^3 + 4n^2 + 12n + 8\}$, $\{6n^2 + 12n + 8, \ldots, n^3 + 6n^2 + 12n + 8\}$ are all contained in specass$(n + 2)$. But $n^3 + 2n^2 \geqq 2n^2 + 4n$, $n^3 + 2n^2 + 4n \geqq 4n^2 + 4n$, $n^3 + 4n^2 + 4n \geqq 4n^2 + 8n + 8$, $n^3 + 4n^2 + 8n + 8 \geqq 4n^2 + 12n + 8$ and $n^3 + 6n^2 + 12n + 8 = (n + 2)^3$. Consequently, $\{2n^2, \ldots, (n + 2)^3\} \subseteq$ specass$(n + 2)$.

(iii) Let $n \in N$. By 1.11, $\{0, \ldots, n^3\} + (n + 3)^2\,\mathrm{k}(f) + n^2\,\mathrm{k}(g) + n(n + 3)\,\mathrm{l}(f, g) \in N$ for all transformations f, g of a three-element set. Now, 2.4 implies that $\{0, \ldots, n^3\}$ and $\{n^2, \ldots, n^3 + n^2\}$ are contained in specass$(n + 3)$. Consequently, $\{0, \ldots, n^3 + n^2\} \subseteq$ specass$(n + 3)$.

(iv) Let $n \in N$ be such that $n + 1 \in N$. We are going to show that $n + 3 \in N$. First, by (iii) $\{0, \ldots, n^3 + n^2\} \subseteq$ specass$(n + 3)$. Further, by (i) $\{2(n + 1)^2, \ldots, (n + 3)^3\} \subseteq$ specass$(n + 3)$. But $n^3 + n^2 = n^2(n + 1) \geqq 2(n + 1)^2$ and we see that $n + 3 \in N$.

(v) From (i) and (iv), it follows easily by induction, that $n \in N$ for each $n \geqq 3$.

I.8. Sets of associative triples - examples

8.1. Let S be a non-empty set. A subset T of $S^{(3)}$ will be called associatively admissible (or only admissible for short) if $T = \mathrm{As}(S(*))$ for a groupoid, say $S(*)$, defined on the set S.

8.2. Example. Let S be a set containing at least two elements. By 1.8 there exist at least one antiassociative groupoid defined on S and therefore $\emptyset$ is an associatively admissible subset of $S^{(3)}$.

8.3. Example. Let S be a non-empty set. We can define a structure of a semigroup on S (for instance with zero multiplication). Hence $S^{(3)}$ is an associatively admissible subset of $S^{(3)}$.

8.4. Remark. Let S be a finite set with card$(S) = n \geqq 3$. Let $0 \leqq m \leqq n^3$. By 7.2 there exists at least one admissible subset T of $S^{(3)}$ with card$(T) = m$.

8.5. Example. Let $S = \{0, 1\}$. It is easy to check (see 3.1) that only admissible subset of $S^{(3)}$ are the following five sets: $\emptyset$, $\{(0,0,0), (0,1,0), (1,0,1), (1,1,1)\}$, $\{(0,0,0), (0,1,0), (0,0,1), (0,1,1), (1,0,0), (1,1,0)\}$, $\{(1,1,1), (1,0,1), (1,1,0), (1,0,0), (0,1,1), (0,0,1)\}$ and $S^{(3)}$.

8.6. Example. Let S be a non-empty set and T an admissible subset of $S^{(3)}$. Then the set $\{(x, y, z); (z, y, x) \in T\}$ is also an admissible subset of $S^{(3)}$ (consider the opposite groupoid).

8.7. Example. Let S be a non-empty set, T an admissible subset of $S^{(3)}$ and f

a permutation on S. Then the set $\{(f(x), f(y), f(z));\ (x, y, z) \in T\}$ is again admissible (consider the isomorphic groupoid).

8.8. Example. Let S be a non-empty set and $T = S^{(3)} - \{(x, x, x);\ x \in S\}$. We are going to show that T is not an admissible subset of $S^{(3)}$.

Suppose, on the contrary, that $T = \mathrm{As}(S)$ and let $a \in S$ (the operation being denoted multiplicatively). Let $b = aa$. Since $a \,.\, aa \neq aa \,.\, a$, we have $a \neq b$. But we have also $b \,.\, bb = b(aa \,.\, b) = b(a \,.\, ab) = ba \,.\, ab = (ba \,.\, a)\, b = (b \,.\, aa)\, b = bb \,.\, b$, a contradiction.

8.9. Example. Let S be a set containing at least two elements, let $a, b \in S$, $a \neq b$, $K = \{a, b\}$. Denote by R the set of all ordered triples $(x, y, z) \in S^{(3)}$ such that either $x = y = a$ or $x = a$, $y = b$ or $y = a$, $z = b$ or $x = y = z = b$. Further, let $T = S^{(3)} - \{(a, x, b);\ x \in S - \{a, b\}\} - \{(b, a, a)\}$. Clearly, if S is finite and $n = \mathrm{card}(S)$, then $\mathrm{card}(R) = 3n$ and $\mathrm{card}(T) = n^3 - n + 1$.

Now, let L be a subset of $S^{(3)}$ such that $R \subseteq L \subseteq T$. We are going to show that L is not an admissible subset of $S^{(3)}$. Suppose, on the contrary, that $L = \mathrm{As}(S)$ for some groupoid $S = S(\cdot)$. First, we show that K is a subgroupoid of S. Indeed, $a(aa \,.\, b) = a(a \,.\, ab) = aa \,.\, ab = (aa \,.\, a)\, b = (a \,.\, aa)\, b$. Then $(a, aa, b) \in L$ and $aa \in K$. Further, $a(ab \,.\, b) = a(a \,.\, bb) = aa \,.\, bb = (aa \,.\, b)\, b = (a \,.\, ab)\, b$; the third equality follows from the fact that either $aa = a$ or $aa = b$. Thus $(a, ab, b) \in L$ and $ab \in K$. Similarly, $a(bb \,.\, b) = a(b \,.\, bb) = ab \,.\, bb = (ab \,.\, b)\, b = (a \,.\, bb)\, b$, so that $bb \in K$, and finally $a(ba \,.\, b) = a(b \,.\, ab) = ab \,.\, ab = (ab \,.\, a)\, b = (a \,.\, ba)\, b$ and $ba \in K$. We have proved that K is a two-element subgroupoid of S. On the other hand, $\mathrm{As}(K) = \mathrm{As}(S) \cap K^{(3)} = L \cap K^{(3)}$, $R \cap K^{(3)} \subseteq L \cap K^{(3)}$, $(a, a, a), (b, b, b) \in R \cap K^{(3)}$. So $R \cap K^{(3)}$ contains six elements and $(b, a, a) \notin L \cap K^{(3)}$. However, such a situation is not possible by 8.5.

8.10. Example. Let S be a set containing at least three elements, let $a, b \in S$, $a \neq b$, and $R = S^{(3)} - \{(x, x, x);\ x \in S - \{a, b\}\}$. If S is finite and $\mathrm{card}(S) = n$, then $\mathrm{card}(R) = n^3 - n + 2$. We are going to show that R is not an admissible subset of $S^{(3)}$. Suppose, on the contrary, that $R = \mathrm{As}(S)$ (the operation on S denote multiplicatively). There is an element $c \in S$ with $a \neq c \neq b$. Put $d = cc$, $e = cd$, $f = dc$.

First, we show that $c \neq d, e, f$ abd $e \neq f$. If $c = d$, then $cc \,.\, c = c \,.\, cc$, so $(c, c, c) \in R$ and it is a contradiction. If $e = c \neq f$, then $cc \,.\, c = (c \,.\, cd) \,.\, c = (cc \,.\, d) \,.\, c = cc \,.\, dc = c \,.\, (c \,.\, dc) = c \,.\, (cd \,.\, c) = c \,.\, cc$, a contradiction.

Next, we show that $d \neq e, f$. If $d = e$, then $e = cd = ce = c \,.\, cd = c \,.\, ce = c(c \,.\, cd) = c(c \,.\, (c \,.\, cc)) = cc \,.\, (c \,.\, cc) = (cc \,.\, c) \,.\, cc = ((cc \,.\, c) \,.\, c) \,.\, c = (cc \,.\, cc) \,.\, c = (c \,.\, (c \,.\, cc)) \,.\, c = (c \,.\, cd) \,.\, c = ce \,.\, c = cd \,.\, c = ec = dc = f$, a contradiction. Similarly, $d \neq f$.

Further, we show that $d \in \{a, b\}$. Indeed, we have $dd \,.\, d = (d \,.\, cc) \,.\, d = (dc \,.\, c) \,.\, d = dc \,.\, cd = d \,.\, (c \,.\, cd) = d \,.\, (cc \,.\, d) = d \,.\, dd$. Similarly $ee \,.\, e =$

$= (e\,.\,cd)\,.\,e = (ec\,.\,d)\,.\,e = ec\,.\,de = e\,.\,(c\,.\,de) = e\,.\,(cd\,.\,e) = e\,.\,ee$, so $e \in \{a, b\}$ and $f \in \{a, b\}$.

We have proved that $d, e, f \in \{a, b\}$. But the elements d, e, f are pair-wise different and this is a contradiction.

8.11. Corollary. Let S be a groupoid of order $n \geqq 3$. Then for all k such that $3n \leqq k \leqq n^3 - n + 2$ there exists at least one subset T of $S^{(3)}$ such that $\mathrm{card}(T) = k$ and T is not an associatively admissible subset of $S^{(3)}$.

I.9. Small and large sets of associative triples

9.1. Let S be a non-empty set and R a subset of $S^{(3)}$. Put $\mathrm{V}(R) = \{(x, y);\ (x, y, z) \in R\}$, $\mathrm{W}(R) = \{x, y, z;\ (x, y, z) \in R\}$, $\mathrm{v}(R) = \mathrm{card}(\mathrm{V}(R))$ and $\mathrm{w}(R) = \mathrm{card}(\mathrm{W}(R))$.

9.2. Proposition. *Let S be a finite non-empty set and R a subset of $S^{(3)}$ such that* $\mathrm{v}(R) + \mathrm{w}(R) + 3 \leqq \mathrm{card}(S)$. *Then R is an admissible subset of $S^{(3)}$.*

Proof. Put $T = S - W$, $W = \mathrm{W}(R)$ and $V = \mathrm{V}(R)$. There are three different elements $a, b, c \in T$ and an injective mapping f of V into T such that $a, b, c \notin \mathrm{Im}(f)$. Define a transformation g of S by $g(a) = c$, $g(x) = g(b) = a$ for $x \in W$ and $g(y) = b$ for $y \in T - \{a, b\}$. Notice that $g(z) \in \{a, b, c\} \subseteq T$ for each $z \in S$ and that $g^2(z) \neq g(z)$. Now, define a multiplication on S as follows: $xy = f(x, y)$ for every $(x, y) \in V$; $f(x, y)\,.\,z = a$ for every $(x, y, z) \in R$ and $xy = g(x)$ in the remaining cases. Then the products of any two elements of S are in T. Therefore $yz \notin W$ and $x\,.\,yz = g(x)$ for all $x, y, z \in S$. Moreover, if $x, y, z \in S$ and $(x, y) \notin V$, then either $x = f(u, v)$ for some $(u, v, y) \in R$ or $x \neq f(u, v)$ for every (u, v, y). In the first case, $xy\,.\,z = a\,.\,z = c \neq b = g(x) = x\,.\,yz$. In the second case, $xy\,.\,z = g(x)\,.\,z = g^2(x) \neq g(x) = x\,.\,yz$; the second equality follows from the fact $g(x) = a, b$ or $c \notin W \cup \mathrm{Im}(f)$.

Finally, if $(x, y) \in V$, then either $(x, y, z) \notin R$, or $(x, y, z) \in R$. In the first case, $xy\,.\,z = f(x, y)\,.\,z = gf(x, y) = b \neq a = g(x) = x\,.\,yz$. In the second case, $xy\,.\,z = a = g(x) = x\,.\,yz$. So we have proved that $\mathrm{As}(S) = R$.

9.3. Remark. Let S be a finite non-empty set of order n and let $a, b, c \in S$, $R = \{(a, b, c)\}$ and $m = \mathrm{card}(\{a, b, c\})$. If $m = 1$ and $5 \leqq n$, then R is an admissible subset of $S^{(3)}$. If $m \leqq 2$ and $6 \leqq n$, then R is an admissible subset of $S^{(3)}$. If $7 \leqq n$, then R is an admissible subset of $S^{(3)}$.

9.4. Corollary. *Let S be a finite non-empty set, $n = \mathrm{card}(S)$ and let R be a subset of $S^{(3)}$ such that* $\mathrm{card}(R) \leqq (n - 3)/4$. *Then R is an admissible subset of $S^{(3)}$.*

9.5. Corollary. *Let S be a finite non-empty set and R a subset of $S^{(3)}$. Then, for every finite set T such that $T \cap S = \emptyset$ and* $\mathrm{card}(T) \geqq 4\,.\,\mathrm{card}(R) + 3 - \mathrm{card}(S)$, *$R$ is an admissible subset of $(T \cup S)^{(3)}$.*

9.6. Corollary. *Let S be a non-empty subset of finite set T such that* $\mathrm{card}(S)^3 + 3 \leqq \mathrm{card}(T)$. *Then every subset of $S^{(3)}$ is an admissible subset of $T^{(3)}$.*

9.7. Proposition. *Let S be an infinite set and R a subset of $S^{(3)}$ such that* $\mathrm{card}(S - \mathrm{W}(R)) = \mathrm{card}(S)$. *Then R is an admissible subset of $S^{(3)}$.*

Proof. Similar to that of 9.2.

9.8. Corollary. *Let S be an infinite set and R a subset of $S^{(3)}$ such that* $\mathrm{card}(R) < \mathrm{card}(S)$. *Then R is an admissible subset of $S^{(3)}$.*

9.9. Proposition. *Let S be a finite non-empty set, T a subset of $S^{(3)}$, $R = S^{(3)} - T$ and suppose that* $\mathrm{v}(R) + \mathrm{w}(R) + 2 \leqq \mathrm{card}(S)$. *Then T is an admissible subset of $S^{(3)}$.*

Proof. Put $Z = S - \mathrm{W}(R)$ and $V = \mathrm{V}(R)$. There are two different elements $a, b \in Z$ and an injective mapping f of V into Z such that $a, b \notin \mathrm{Im}(f)$. Define a multiplication on S as follows: $xy = f(x, y)$ for every $(x, y) \in V$, $f(x, y) . z = a$ for every $(x, y, z) \in R$ and $xy = b$ in the remaining cases. It is easy to see that $x . yz = b$ for all $x, y, z \in S$. Moreover, if $x, y, z \in S$ and $(x, y) \notin V$, then $xy \in \{a, b\}$ and $xy . z = b = x . yz$. Finally, if $x, y, z \in S$ and $(x, y) \in V$, then either $(x, y, z) \notin R$, and we have $xy . z = f(x, y) . z = b = x . yz$, or $(x, y, z) \in R$, and we have $xy . z = f(x, y) . z = a \neq b = x . yz$. We have proved that $\mathrm{As}(S) = T$.

9.10. Remark. Let S be a finite non-empty set of order n and let $a, b, c \in S$, $R = S^{(3)} - \{(a, b, c)\}$, and $m = \mathrm{card}(\{a, b, c\})$.
If $m = 1$ and $4 \leqq n$, then R is admissible.
If $m \leqq 2$ and $5 \leqq n$, then R is admissible.
If $6 \leqq n$, then R is admissible.

9.11. Corollary. *Let S be a finite non-empty set, $n = \mathrm{card}(S)$, and let R be a subset of $S^{(3)}$ such that* $n^3 - n/4 + 1/2 \leqq \mathrm{card}(R)$. *Then R is admissible subset of $S^{(3)}$.*

9.12. Proposition. *Let S be an infinite set and R a subset of $S^{(3)}$ such that* $\mathrm{card}(S - \mathrm{W}(S^{(3)} - R)) = \mathrm{card}(S)$. *Then R is an admissible subset of $S^{(3)}$.*

Proof. Similar to that of 9.9.

9.13. Corollary. *Let S be an infinite set and R a subset of $S^{(3)}$ such that* $\mathrm{card}(S^{(3)} - R) < \mathrm{card}(S)$. *Then R is an admissible subset of $S^{(3)}$.*

9.14. Let S be a finite set with $n = \mathrm{card}(S) > 3$. Then:
(i) For every $0 \leqq m \leqq n^3$ there exists at least one admissible subset R of $S^{(3)}$ such that $\mathrm{card}(R) = m$.
(ii) If R is a subset of $S^{(3)}$ such that either $\mathrm{card}(R) \leqq (n - 3)/4$ or $n^3 - n/4 + 1/2 \leqq \mathrm{card}(R)$, then R is an admissible subset of $S^{(3)}$.
(iii) For every $3n \leqq m \leqq n^3 - n + 2$ there exists at least one non-admissible subset T of $S^{(3)}$ such that $\mathrm{card}(T) = m$.

Proof. See 7.2 and 8.9, 8.10, 9.4, 9.11.

9.15. Theorem. *Let S be an infinite set. Then:*
(i) *If R is a subset of $S^{(3)}$ such that either* $\mathrm{card}(R) < \mathrm{card}(S)$ *or* $\mathrm{card}(S^{(3)} - R) < \mathrm{card}(S)$, *then R is an admissible subset of $S^{(3)}$.*
(ii) *There exists at least one non-admissible subset of $S^{(3)}$.*

Proof. See 8.8, 9.8 and 9.13.

I.10. Comments and open problems

10.1. Theorem 7.2 and some other related results are proved in [1] and [2], while the results contain in I.8 and I.9 are adapted from [3].

10.2. The class of antiassociative groupoids is closed under subgroupoids and filtered product. On the other hand, every absolutely free groupoid is antiassociative, and hence the class is not closed under homomorphic images.

10.3. Let $\mathscr{N}$ designate the class of groupoids with neutral element. By 1.12, $\mathrm{specass}(\mathscr{N}, n) = \{3n^2 - 3n + 1, \ldots, n^3\}$ for every $n \geqq 4$. Further, $\mathrm{specass}(\mathscr{N}, 2) = \{8\}$ and $\mathrm{specass}(\mathscr{N}, 3) = \{19, 20, 21, 22, 23, 24, 25, 27\}$.

10.4. Find $\mathrm{specass}(\mathscr{A}, n)$ for the following classes $\mathscr{A}$ of groupoids: Commutative groupoids, idempotent groupoids, commutative groupoids, unipotent groupoids, commutative unipotent groupoids.

10.5. Let $\mathscr{T}$ designate the class of groupoids such that $\mathrm{card}(GG) = 2$ for each $G \in \mathscr{T}$. Then $\mathrm{specass}(\mathscr{T}, 2) = \{0, 4, 6, 8\}$. Further, $\mathrm{specass}(\mathscr{T}, 3) = \{3, 5, 7, \ldots, 22, 24, \ldots, 27\}$ (hence 1, 2, 4, 6 and 23 are missing). Find $\mathrm{specass}(\mathscr{T}, n)$ for $n \geqq 4$. In particular, does there exist a number $n \geqq 4$ with $\mathrm{specass}(\mathscr{T}, n) = \{0, 1, 2, \ldots, n^3\}$?

10.6. Let $n \geqq 2$. We can define two numbers $\varrho(n)$ and $\sigma(n)$ by $\varrho(n) = \min \mathrm{card}(R)$ and $\sigma(n) = \max \mathrm{card}(R)$, where R is running through all non-admissible subsets of $S^{(3)}$, $\mathrm{card}(S) = n$. Then $\varrho(2) = 1$ and $\sigma(2) = 7$, $\varrho(3) = 1$ and $\sigma(3) = 26$, $(n - 3)/4 < \varrho(n) \leqq 3n$ and $n^3 - n + 2 \leqq \sigma(n) < n^3 - n/4 + 1/2$ for every $n \geqq 4$.
(i) Find all admissible subset of $S^{(3)}$ for a three-element set.
(ii) Find $\varrho(n)$ and $\sigma(n)$ for "small" natural numbers n.
(iii) Improve the above estimates of $\varrho(n)$ and $\sigma(n)$.

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