On Hereditary Subdirectly Irreducible Graphs

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We give a characterization of classes of antireflexive graphs in which every full subgraph of a subdirectly irreducible graph is subdirectly irreducible as well.

V článku je podána charakterizace tříd antireflexivních grafů, v nichž je každý úplný podgraf subdirektně ireducibilního grafu opět subdirektně ireducibilní.

В работе дается характеризация классов графов без петель, в которых каждый порожденный подграф подпрямо неприводимого графа является опять подпрямо неприводимым.

0. Introduction

The concept of the subdirect irreducibility was introduced for algebras by G. Birkhoff. It can be defined more generally for categories, in particular for graphs: Let C be a class of (some) graphs. Then a C-graph A (i.e. a graph $A \in C$) is said to be subdirectly irreducible (SI) if, whenever an isomorphic copy A' of A is contained as a full subgraph in a product $\times_{i \in \alpha} B_i$ with $B_i \in C$ and $p_j(A') = B_j$ for all the projections, there is a j such that the restriction of p_j to A' is an isomorphism onto B_j. (This formulation is due to A. Pultr – see [3]).

Importance of investigation this topic is following: having a list of subdirectly irreducible C-graphs, one can construct any C-graph from subdirectly irreducibles using such simple operations as product and restrictions to full subgraphs. If subdirectly irreducibles are, in some sense, "simple" then this procedure may be useful for recording of graphs to the machine memory.

Characterization theorem for the subdirect irreducibility is given in [4]. It enables us to find a list of subdirectly irreducibles in various categories. This theorem, however, does not solve the problem when the list of subdirectly irreducibles is closed to subobjects. This question is particularly interesting for the case of systems of

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antireflexive graphs where the list of subdirectly irreducibles is often infinite and so such a hereditary (with respect to full subgraphs) can be useful for its description.

In [5], a characterization of systems of symmetric antireflexive graphs in which any full subgraph of a subdirectly irreducible one is again SI, is given. In the present note, we are going to generalize this characterization to the case of antireflexive graphs.

1. Notations and definitions

1.1. Notations. Denote $\boldsymbol{G}$ the class of all antireflexive graphs. For any ordinal n denote $K_n = (n, \{(i, j) \mid i, j \in n,\ i \neq j\})$ (i.e. the complete antireflexive graph with n vertices), $K'_n = (n, \{(i, j) \mid i, j \in n, \mid i \neq j,\ (i, j) \neq (0, 1)\})$ (i.e. the antireflexive graph on n vertices with just one edge missing),

$L_n = (n, \{(i, j) \mid i, j \in n,\ i < j\})$.
$L_n^+ = (n, \{(i, j) \mid i, j \in n,\ i < j\} \cup \{(1, 0)\})$,
$L_n^- = (n, \{(i, j) \mid i, j \in n,\ i > j\} \cup \{(0, 1)\})$,
$A_3 = (3, \{(0, 1), (1, 0), (0, 2), (2, 1)\})$,
$C_3 = (3, \{(0, 1), (1, 2), (2, 0)\})$,
$A_4 = (4, \{(0,1), (1, 0), (0, 2), (1, 2), (2, 1), (1, 3), (2, 3), (3, 2), (3, 0), (0, 3)\})$.

Further, put $\mathscr{K} = \{K_n \mid n \in Ord\}$, $\mathscr{K}' = \{K'_n \mid n \in Ord\}$, $\mathscr{L}^+ = \{L_n^+ \mid n \in Ord\}$, $\mathscr{L}^- = \{L_n^- \mid n \in Ord\}$, $\mathscr{S} = \{(X, \emptyset) \mid X \text{ is a set}\}$ (the class of sets = discrete graphs),

$\mathscr{T} = \{(X, R) \mid \forall x, y \in X,\ x \neq y \Rightarrow \mid \{(x, y),\ (y, x)\} \cap R \mid = 1\}$ (the class of all tournaments),

$\mathscr{U} = \{(n, R) \mid n \leqq 6,\ |R| = n + [n/2],\ x \neq y \Rightarrow |\{(x, y), (y, x)\} \cap R| \geqq 1$ and (n, R) contains neither K'_3 nor A_3 as a full subgraph$\}$,

$\mathscr{V} = \{(n, R) \mid n \leqq 4,\ x \neq y \Rightarrow |\{(x, y),\ (y, x)\} \cap R| \geqq 1,\ R \supset \{(0, 1),\ (1, 0), (2, 3), (3, 2)\} \cap n \times n$ and (n, R) does not contain K'_3 as a full subgraph$\}$,

$\mathscr{W} = \{A \in \boldsymbol{G} \mid$ any full subgraph of A with 3 vertices is isomorphic either to A_3 or to $L_3\}$.

Let $\boldsymbol{D}$ be a collection of graphs. Then $\boldsymbol{SP}(\boldsymbol{D})$ denotes (similarly as in [2]) the class of all the graphs which can be embedded as full subgraphs into products of graphs from $\boldsymbol{D}$.

1.2. Definition. A class $\boldsymbol{C}$ of graphs closed to categorical products $\boldsymbol{X}$ ($\underset{i \in I}{\boldsymbol{X}}(X_i, R_i) = (\underset{i \in I}{\boldsymbol{X}} X_i, R)$ where $((x_i)_I, (y_i)_I) \in R \Leftrightarrow (x_i, y_i) \in R_i$ for any $i \in I$) and to full subgraphs is said to be be hereditary with respect to subdirect irreducibility (HSI) if any full subgraph of a SI graph is again SI.

2. Main theorem

We are going to prove the following:

2.1. Theorem. Let $\boldsymbol{C} \subset \boldsymbol{G}$ be a productive hereditary class of graphs (i.e. a class closed to categorical products $\boldsymbol{X}$ and to full subgraphs). Then $\boldsymbol{C}$ is HSI iff either $\boldsymbol{C} = \mathscr{S}$, or $\boldsymbol{C} = \boldsymbol{SP}(\boldsymbol{D})$ where $\boldsymbol{D}$ satisfies one of the following conditions:

(i) $\boldsymbol{D} \subset \mathscr{K} \cup \mathscr{K}'$
(ii) $\boldsymbol{D} \subset \mathscr{K} \cup \{K_3', A_4\}$
(iii) $\boldsymbol{D} \subset \mathscr{K} \cup \mathscr{L}^+ \cup \mathscr{T}$
(iv) $\boldsymbol{D} \subset \mathscr{K} \cup \mathscr{L}^- \cup \mathscr{T}$
(v) $\boldsymbol{D} \subset \mathscr{K} \cup \mathscr{U}$
(vi) $\boldsymbol{D} \subset \mathscr{K} \cup \mathscr{V}$
(vii) $\boldsymbol{D} \subset \mathscr{K} \cup \mathscr{W}$

3. Subdirect irreducibility

Before proving Main Theorem, recall the characterization of subdirectly irreducibles:

3.1. Definition. a) A graph (X, R) is said to be meet-irreducible (in $\boldsymbol{C}$) iff, whenever $R = \bigcap_{i \in I} R_i$, $(X, R_i) \in \boldsymbol{C}$ then there exists $i_0 \in I$ such that $R_{i_0} = R$.
b) A graph (X, R) is said to be maximal (in $\boldsymbol{C}$) iff $R' \supset R$, $(X, R') \in \boldsymbol{C}$ implies that $R' = R$.
c) A monomorphic system is a system $(u_i : (X, R) \to (Y_i, R_i))_{i \in I}$ of homomorphisms such that if $u_i \alpha = u_i \beta$ for all $i \in I$ then $\alpha = \beta$.

3.2. Theorem. A $\boldsymbol{C}$-graph $A = (X, R)$ is SI iff either A is maximal in $\boldsymbol{C}$ and for any monomorphic system $(u_i : A \to B_i)_{i \in I}$ there exists an $i_0 \in I$ such that u_{i_0} is one-to-one, or A is not maximal, it is meet-irreducible in $\boldsymbol{C}$ and for any $\Phi : A \to B$ not one-to-one there exists $R' \supsetneq R$ such that Φ can be extended to a homomorphism $\bar{\Phi} : (X, R') \to B$.

3.3. Remark. The previous theorem is just a reformulation of Theorem 3.6 from [6].
Using Theorem 3.2 one can characterize subdirectly irreducible $\boldsymbol{G}$-graphs:

3.4. Proposition. A $\boldsymbol{G}$-graph A is SI in $\boldsymbol{G}$ iff $A \in \mathscr{K} \cup \mathscr{K}'$.

Proof. One can easily see that meet-irreducible $\boldsymbol{G}$-graphs are just elements of $\mathscr{K} \cup \mathscr{K}'$. Since any SI graph must be meet-irreducible, it has to be an element of $\mathscr{K} \cup \mathscr{K}'$.

Any element of $\mathscr{K}$ is maximal $\boldsymbol{G}$-graph which cannot be mapped to a $\boldsymbol{G}$-graph of a smaller cardinality, hence it is SI. Any element A of $\mathscr{K}'$ is non-maximal meet-irreducible graph. Moreover, every mapping $\Phi : A \to B$ is one-to-one. Hence, A is SI. □

4. Proof of the main theorem

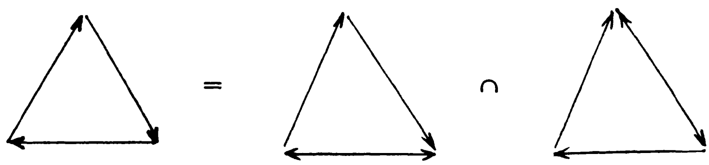
We are going to prove Theorem 2.1 by a series of lemmas:

4.1. Lemma. Let $\boldsymbol{C} \neq \mathscr{S}$ be a productive hereditary class of $\boldsymbol{G}$-graphs. If $\boldsymbol{C}$ is HSI then for every (X, R) a SI $\boldsymbol{C}$-graph and for any $x, y \in X$, $x \neq y$, there is $\{(x, y), (y, x)\} \cap R \neq \emptyset$.

Proof. Since $\boldsymbol{C} \neq \mathscr{S}$, $\boldsymbol{C}$ must contain a non-discrete graph. Since $\boldsymbol{C}$ is hereditary, it must contain a non-discrete graph A with two vertices. Hence, $(2, \emptyset) \cong A \times (1, \emptyset)$ is not SI and HSI of $\boldsymbol{C}$ implies the assertion of lemma. □

4.2. Lemma. Let $\boldsymbol{C}$ be a productive hereditary class of $\boldsymbol{G}$-graphs. If $A_3 \in \boldsymbol{C}$ then C_3 is not SI in $\boldsymbol{C}$.

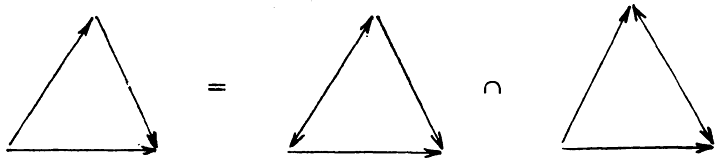
Proof.



Hence, C_3 is not meet-irreducible and according to 3.2 it is not SI. □

4.3. Lemma. Let $\boldsymbol{C}$ be a productive hereditary class of $\boldsymbol{G}$-graphs. If $L_3^+ \in \boldsymbol{C}$, $L_3^- \in \boldsymbol{C}$, then L_3 is not SI in $\boldsymbol{C}$.

Proof.

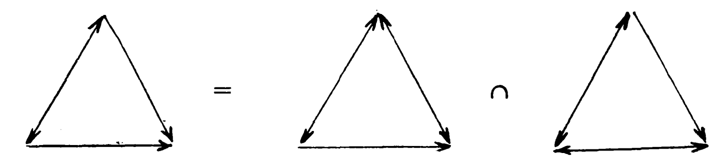


Hence, L_3 is not meet-irreducible and according to 3.2 it is not SI. □

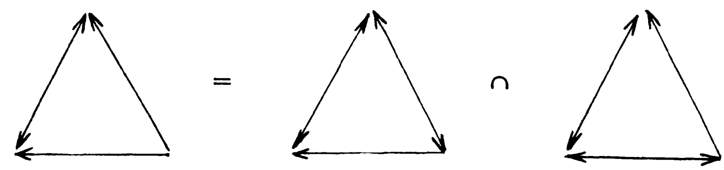
4.4. Lemma. Let C be a productive hereditary class of G-graphs. If $K_3' \in C$ then L_3^+, L_3^-, A_3 are not SI in C.

Proof.

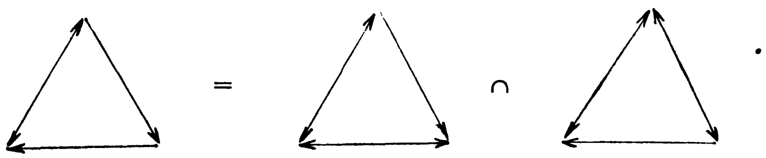
$L_3^+ \cong$



$L_3^- \cong$



$A_3 \cong$



Proposition 3.2 finishes the proof. □

4.5. Lemma. Let C be a productive hereditary class of G-graphs. If C is HSI, $K_3' \in C$, $K_4' \notin C$, then every SI C-graph with 4 vertices is either isomorphic to K_4 or isomorphic to A_4.

Proof. Let A be a SI C-graph with 4 vertices. Since C is HSI, A cannot contain a two-point discrete graph as a full subgraph. Lemmas 4.2–4.4 imply that any 3-point full subgraph of A has to be isomorphic either to K_3, or to K_3'. Since $K_4' \notin C$, there is either $A \cong K_4$ or $A \cong A_4$. □

4.6. Lemma. Let C be a productive hereditary class of G-graphs. If C is HSI, $K_3' \in C$, $K_4' \notin C$, then every SI C-graph with $n \geqq 5$ vertices is isomorphic to K_n.

Proof. Suppose that there exists SI C-graph with more than 4 vertices which is isomorphic to no K_n. Hereditary of C implies that there exists $A = (5, R) \not\cong K_5$ which is SI in C, Lemma 4.5 and the assumptions $K_3' \in C$, $K_4' \notin C$ imply that

$$A \restriction 4 = (4, R \cap 4 \times 4) \cong A_4 .$$

Suppose that $R \cap 4 \times 4 = \{(0,1),(1,0),(0,2),(1,2),(2,1),(1,3),(2,3),(3,2),(3,0),(0,3)\}$. Similarly, $A \upharpoonright \{1,2,3,4\} \cong A_4$. Hence, $R \cap \{1,2,3,4\} \times \{1,2,3,4\} = \{(1,2),(2,1),(1,3),(2,3),(3,2),(1,4),(4,1),(3,4),(4,3)\} \cup \{(i,j)\}$ where $(i,j) = (2,4)$ or $(i,j) = (4,2)$. Therefore, $B = A \upharpoonright \{0,2,4\} \not\cong K_3$, $B \not\cong K_3'$. Hence, B is not SI which contradicts the HSI property of C. □

4.7. Lemma. Let C be a productive hereditary class of G-graphs. If C is HSI, $K_3' \notin C$, $L_3^- \notin C$, $A_3 \notin C$, $L_3^+ \in C$, then every SI C-graph is an element of $\mathcal{K} \cup \mathcal{L}^+ \cup \mathcal{T}$.

Proof. Let A be a SI C-graph. If $A \notin \mathcal{K} \cup \mathcal{T}$ then A must contain three-point full subgraph B which is neither a tournament, nor an isomorphic copy of K_3. Since $K_3' \notin C$, $L_3^- \notin C$, $A_3 \notin C$, there is $B \cong L_3^+$ and $A \cong L_n^+$. □

4.8. Lemma. Let C be a productive hereditary class of G-graphs. If C is HSI, $K_3' \notin C$, $L_3^+ \notin C$, $A_3 \notin C$, $L_3^- \in C$, then every SI C-graph is an element of $\mathcal{K} \cup \mathcal{L}^- \cup \mathcal{T}$.

Proof is similar to the proof of Lemma 4.7. □

4.9. Lemma. Any tournament on 4 vertices contains L_3 as a full subgraph.

Proof is obvious. □

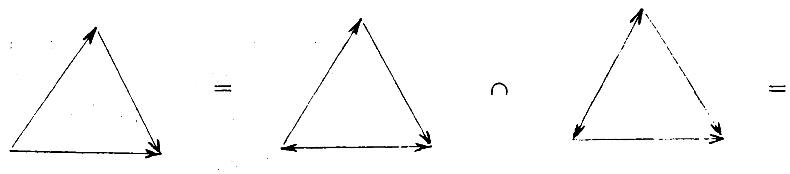
4.10. Lemma. Let C be a productive hereditary class of G-graphs. If C is HSI, L_3^+, L_3^-, are SI C-graphs, then every SI C-graph is an element of $\mathcal{K} \cup \mathcal{U}$.

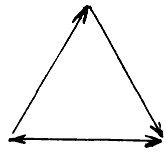
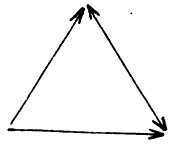
Proof. By Lemma 4.4, $K_3' \notin C$. Hence, $K_n' \notin C$ for any $n \geqq 3$. By Lemma 4.3, L_3 is not SI in C. Suppose there exists a SI C-graph A with $n \geqq 7$ vertices which is not isomorphic to K_n. Since $K_3' \notin C$, A contains a tournament on 4 vertices as a full subgraph. By Lemma 4.9 and HSI property of C, L_3 is SI in C which is a contradiction.

If B is a SI C-graph with $n \leqq 6$ vertices, $B \not\cong K_n$, then B does not contain K_3' as a full subgraph, on the other hand, B does not contain L_3 as a full subgraph as well. Hence, $B \in \mathcal{U}$. □

4.11. Lemma. Let C be a productive hereditary class of G-graphs which is HSI. If $A_3 \in C$, $L_3^+ \in C$ ($L_3^- \in C$, resp.) then no tournament with at least 3 vertices is SI.

Proof.



$=$  $\cap$  .

Hence, L_3 is not SI. By 4.2, C_3 is not SI. HSI property of $\boldsymbol{C}$ implies that no tournament with at least three verices is SI. □

4.12. Lemma. Let $\boldsymbol{C}$ be a productive hereditary class of $\boldsymbol{G}$-graphs. If $K_3' \notin \boldsymbol{C}$, $A_3 \in \boldsymbol{C}$, $L_3^+ \in \boldsymbol{C}$, ($L_3^- \in \boldsymbol{C}$, resp.) and $\boldsymbol{C}$ is HSI, then every SI $\boldsymbol{C}$-graph is an element of $\mathscr{K} \cup \mathscr{V}$.

Proof. By Lemma 4.11, any SI $\boldsymbol{C}$-graph A contains no tournament with at least 3 vertices. Hence, A is either complete, or an element of $\mathscr{V}$. □

4.13. Lemma. Let $\boldsymbol{C}$ be a productive hereditary class of $\boldsymbol{G}$-graphs. If $K_3' \notin \boldsymbol{C}$, $L_3^+ \notin \boldsymbol{C}$, $L_3^- \notin \boldsymbol{C}$, $A_3 \in \boldsymbol{C}$ and $\boldsymbol{C}$ is HSI then every SI $\boldsymbol{C}$-graph is an element of $\mathscr{K} \cup \mathscr{W}$.

Proof. Let A be a SI $\boldsymbol{C}$-graph, $A \notin \mathscr{K}$. By 4.2, A does not contain C_3 as a full subgraph. Hence, $A \in \mathscr{W}$. □

4.14. Lemma. Let $\boldsymbol{C}$ be a productive hereditary class of $\boldsymbol{G}$-graphs. If $K_4' \in \boldsymbol{C}$ and $\boldsymbol{C}$ is HSI, then every SI $\boldsymbol{C}$-graph is an element of $\mathscr{K} \cup \mathscr{K}'$.

Proof. Suppose there is a SI $\boldsymbol{C}$-graph $A \notin \mathscr{K} \cup \mathscr{K}'$. Then either, A has at most 3 vertices, or A contains a full subgraph B with four vertices such that $B \not\cong K_4$, $B \not\cong K_4'$.

In the first case, either there is $A \cong (2, \emptyset)$ which is a contradiction, or $A = (3, \boldsymbol{R})$, $A \not\cong K_3$, $A \not\cong K_3'$, Lemma 4.4 implies that A is not SI.

In the second case, by the similar argument as in 4.4, B is an intersection of isomorphic copies of K_4', hence not SI. □

4.15. Proof of main theorem.

A. Suppose that $\boldsymbol{C}$ is HSI, $\boldsymbol{C} \neq \mathscr{S}$, $\boldsymbol{D} = \{A \in \boldsymbol{C};\ A \text{ is SI}\}$.
If $K_4' \in \boldsymbol{C}$ then $\boldsymbol{D} \subset \mathscr{K} \cup \mathscr{K}'$ according to 4.14.
If $K_4' \notin \boldsymbol{C}$, $K_3' \in \boldsymbol{C}$, then $\boldsymbol{D} \subset \mathscr{K} \cup \{K_3', A_4\}$ according to 4.6 and 4.4.
If $K_3' \notin \boldsymbol{C}$, $A_3 \notin \boldsymbol{C}$, then either $\boldsymbol{D} \subset \mathscr{K} \cup \mathscr{L}^+ \cup \mathscr{T}$($\boldsymbol{D} \subset \mathscr{K} \cup \mathscr{L}^- \cup \mathscr{T}$, resp.) by Lemmas 4.7 and 4.8, or $\boldsymbol{D} \subset \mathscr{K} \cup \mathscr{U}$ by Lemma 4.10.
If $K_3' \notin \boldsymbol{C}$, $A_3 \in \boldsymbol{C}$, $L_3^+ \in \boldsymbol{C}$($L_3^- \in \boldsymbol{C}$, resp.) then $\boldsymbol{D} \subset \mathscr{K} \cup \mathscr{V}$ according to Lemma 4.12.
If $K_3' \notin \boldsymbol{C}$, $L_3^+ \notin \boldsymbol{C}$, $L_3^- \notin \boldsymbol{C}$, $A_3 \in \boldsymbol{C}$, then $\boldsymbol{D} \subset \mathscr{K} \cup \mathscr{W}$ according to Lemma 4.13.
If $\boldsymbol{C} \cap \{K_3', A_3, L_3^+, L_3^-\} = \emptyset$ then $\boldsymbol{D} \subset \mathscr{K} \cup \mathscr{T}$.

B. One can check that each of systems $\mathscr{K} \cup \mathscr{K}'$, $\mathscr{K} \cup \{K_3', A_4\}$, $\mathscr{K} \cup \mathscr{L}^+ \cup \mathscr{T}$, $\mathscr{K} \cup \mathscr{L}^- \cup \mathscr{T}$, $\mathscr{K} \cup \mathscr{U}$, $\mathscr{K} \cup \mathscr{V}$, $\mathscr{K} \cup \mathscr{W}$ is hereditary. If $\boldsymbol{D}$ is its subsystem closed to full subgraphs, then $\boldsymbol{D}$ is a system of subdirectly irreducibles of $\boldsymbol{SP}(\boldsymbol{D})$ and $\boldsymbol{C} = \boldsymbol{SP}(\boldsymbol{D})$ is HSI. □

4.16. Concluding remark. In [5], types of dimensions of graphs are studied. Recall that a product dimension of a graph A in $\boldsymbol{C}$ is $p\text{-dim}_{\boldsymbol{C}} A = \min \{\alpha \mid A$ is a full subgraph of $\boldsymbol{X}_{i \in \alpha} A_i$ with A_i SI in $\boldsymbol{C}\}$, a subdirect dimension $s\text{-dim}_{\boldsymbol{C}} A = \min \{\alpha \mid A$ is a full subgraph of $\boldsymbol{X}_{i \in \alpha}$ with A_i SI and $p_i m$ onto$\}$ (p_i are projections, m is an embedding).

Theorem 2.1 implies that, if $\boldsymbol{C} \subset \boldsymbol{G}$ is a productive hereditary class of graphs then $p\text{-dim}_{\boldsymbol{C}} \equiv s\text{-dim}_{\boldsymbol{C}}$ iff $\boldsymbol{C} = \boldsymbol{SP}(\boldsymbol{D})$ where $\boldsymbol{D}$ satisfies one of the conditions (i)–(vii) from 2.1.

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