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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 23 (1982), No. 1, 55--61

Persistent URL: http://dml.cz/dmlcz/142482

Embedding Of Presheaves Into Dual Unit Balls

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Received 19 November 1981

By means of embedding of a given presheaf $\mathscr{S} = \{X_\alpha \,|\varrho_{\alpha\beta}|\, \langle A \leqq \rangle\}$ into $\mathscr{T} = \{B_\alpha \mid \varrho^{**}_{\alpha\beta}| \,.\, \langle A \leqq \rangle\}$, where B_α is the dual unit ball of a normed space of continuous functions on X_α, we get a theorem on functional separation of inductive limit of $\mathscr{S}$; a theorem on representation of presheaves by sections follows from it.

Вложение предпучков в дуальные единичные шары. Вложением данного предпучка $\mathscr{S} = \{X_\alpha \,|\varrho_{\alpha\beta}|\, \langle A \leqq \rangle\}$ в $\mathscr{T} = \{B_\alpha \,|\varrho^{**}_{\alpha\beta}|\, \langle A \leqq \rangle\}$, где B_α есть дуальный единичный шар некоторого нормированного пространства непрерывных функций на X_α мы получим теорему о функциональной отделимости индуктивного предела $\mathscr{S}$; из этой теоремы следует теорема о представлении предпучка сечениями.

Vnoření předsvazku do duálních jednotkových koulí. Vnořením daného předsvazku $\mathscr{S} = \{X_\alpha \,|\varrho_{\alpha\beta}|\, \langle A \leqq \rangle\}$ do $\mathscr{T} = \{B_\alpha \,|\varrho^{**}_{\alpha\beta}|\, \langle A \leqq \rangle\}$, kde B_α je duální jednotková koule nějakého normovaného prostoru spojitých funkcí na X_α, dostaneme větu o funkcionální oddělitelnosti induktivní limity $\mathscr{S}$; z ní plyne věta o reprezentaci předsvazku řezy.

Introduction

There are two questions we have tried to solve in [2]–[5], and with which we also deal here. The one is when the topology t of the inductive limit of a presheaf $\mathscr{S} = \{(X_\alpha, t_\alpha) \,|\varrho_{\alpha\beta}|\, \langle A \leqq \rangle\}$ of topological spaces is Hausdorff, the other is when a presheaf of topological spaces over a topological space can be represented by sections in its covering space. The main means we have used there to get results in both mentioned directions is the embedding of $\mathscr{S}$ into a presheaf of compact spaces whose inductive limit resp. stalks are Hausdorff. We have dealt there with the embedding into cubes $Q^{F(X_\alpha)}$ (where Q is the compact unit interval and $F(X_\alpha)$ is a set of t_α-continuous functions on X_α with values in Q), and into the space of all continuous multiplicative linear functionals on a Banach algebra $\mathscr{A}(X_\alpha)$ of t_α-continuous functions on X_α. Both ways have their own traits and every theorem bears the marks of that by means of which it was gotten. The theorems proven by the second way are more general if we take in account the interplay of the algebras, but their drawback

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is the requirement that the $\mathscr{A}(X_\alpha)$'s be Banach algebras. On the other hand, those proven by the first way (of embedding in cubes) need not require so much from the sets $F(X_\alpha)$ themselves.

In this paper we employ another sort of embedding that has not been used for this end yet, namely the embedding into the unit ball of the dual of a normed space $N(X_\alpha)$ of t_α-continuous functions on X_α. This method keeps the advantages of the embedding into the functionals as to the requirements put on the interplay of the normed spaces $N(X_\alpha)$ on the one hand, and in a fair measure saves the freedom of choice of the sets F_α themselves which the embedding in cubes enjoys, because the F_α's may be any, under the only condition that there be a countable confinal subset in $\langle A \leqq \rangle$. Thus gotten Theorem 2.5 on separation of the topology of inductive limits, and Th. 2.6 on the representation extend the range of the theory.

1. Notation. Throughout the paper the category of topological spaces will be denoted by TOP. If $\mathscr{X}$ is a topological space then the set of all continuous (continuous and bounded) real functions on $\mathscr{X}$ will be denoted by $C(\mathscr{X} \to R)$ $(C^*(\mathscr{X} \to R))$. Where it is convenient we denote a topological space $\mathscr{X}$ by (X, t), where X is the underlying set of $\mathscr{X}$ (sometimes denoted also by $|\mathscr{X}|$) and t is its topology.

2. Definition. Let $\mathscr{S} = \{\mathscr{X}_\alpha \,|\varrho_{\alpha\beta}|\, \langle A \leqq \rangle\}$ be a presheaf from TOP. A hull (weak hull) of $\mathscr{S}$ is a pair $(\mathscr{C}, \mathscr{Z})$ where $\mathscr{C} = \{\mathscr{C}_\alpha \,|\sigma_{\alpha\beta}|\, \langle A \leqq \rangle\}$ is a presheaf from TOP and $\mathscr{Z} = \{e_\alpha : \mathscr{X}_\alpha \to \mathscr{C}_\alpha \mid \alpha \in A\}$, where each e_α is a continuous 1-1 open (continuous 1-1) map of $\mathscr{X}_\alpha$ into $\mathscr{C}_\alpha$ such that the following diagram commutes for any $\alpha, \beta \in A$, $\alpha \leqq \beta$:

$$\begin{array}{ccc} \mathscr{X}_\alpha & \xrightarrow{\varrho_{\alpha\beta}} & \mathscr{X}_\beta \\ \big\downarrow{\scriptstyle e_\alpha} & & \big\downarrow{\scriptstyle e_\beta} \\ \mathscr{C}_\alpha & \xrightarrow{\sigma_{\alpha\beta}} & \mathscr{C}_\beta \end{array}$$

Where possible, we omit the family $\mathscr{Z}$ saying that $\mathscr{C}$ is the hull of $\mathscr{S}$. If moreover every $\mathscr{C}_\alpha$ is compact then $\mathscr{C}$ is called compact (weak compact) hull of $\mathscr{S}$.

3. Definition. A presheaf $\mathscr{S} = \{S_U \,|\varrho_{UV}|\, X\}$ of sets over a topological space X will be called sheaf if

a) Given an open set $U \subset X$ and $a, b \in S_U$ then the existence of an open cover $\mathscr{V}$ of U such that $\varrho_{UV}(a) = \varrho_{UV}(b)$ for all $V \in \mathscr{V}$ yields that $a = b$.

b) Given an open $U \subset X$, an open cover $\mathscr{V}$ of U and a family $\{a_V \in S_V \mid V \in \mathscr{V}\}$ such that $\varrho_{VV \cap W}(a_V) = \varrho_{WW \cap V}(a_W)$ whenever $V \cap W \neq \emptyset$ (such a family will be called smooth), then there is an $a \in S_U$ with $\varrho_{UV}(a) = a_V$ for all $V \in \mathscr{V}$.

In the following few lemmas we shall recall some well known properties of the things which shall occur later and order them for the further use. Then we shall be able to get the main theorems 8 and 9 following from it. The following lemma is well known:

4. *Lemma.* Given two topological linear spaces E, F and a continuous linear map $m : E \to F$, let us denote by (E^*, w_E^*), (F^*, w_F^*) their duals with the w^*-topology. Then the dual map $m^* : (F^*, w_F^*) \to (E^*, w_E^*)$ is continuous and linear; m^* is 1-1 if $m(E)$ is dense in F.

5. *Lemma. A.* Given a topological linear space X, let F_X be a normed space of some bounded continuous functions on X, endowed with the sup-norm. Let B_X be the unit ball of F_X^*. Then B_X is a w^*-compact subset of (F_X^*, w^*), and if we assign to any $x \in X$ a function $j_X(x)$ on F_X by $(j_X(x))(\varphi) = \varphi(x)$ for $\varphi \in F_X$ then for the map $j_X : X \to F_X^*$ (we call it a natural evaluation) we have

(a) j_X maps X into B_X,

(b) $j_X : X \to (B_X, w^*)$ is continuous; it is open if F_X separates points from closed sets of X,

(c) j_X is 1-1 if F_X separates points of X,

(d) for $\varphi \in F_X, f \in B_X$ set $\hat{\varphi}(f) = f(\varphi)$. Then $\hat{\varphi}$ is a continuous function on (B_X, w^*) and $\hat{F}_X = \{\hat{\varphi} \mid \varphi \in F_X\}$ separates points of B_X.

B. Given another topological space Y and a normed space F_Y of some bounded continuous functions on Y, with the sup-norm, and a continuous map $T: X \to Y$ such that T^* sends F_Y into F_X, then the dual map $T^{**} : F_X^* \to F_Y^*$ maps B_X into B_Y and $T^{**} : (B_X, w^*) \to (B_Y, w^*)$ is continuous. Further, the following diagram commutes:

$$\begin{array}{ccc} (B_X, w^*) & \xrightarrow{T^{**}} & (B_Y, w^*) \\ \uparrow{\scriptstyle j_X} & & \uparrow{\scriptstyle j_Y} \\ X & \xrightarrow{T} & Y \end{array}$$

Proof. The w^*-compactness of B_X is well known. The proof of A(b), (c), (d) is standard argument of embedding topological spaces into products. The proof of A(a) and B is straightforward.

6. *Lemma.* Let $\mathscr{S} = \{\mathscr{X}_\alpha \,|\varrho_{\alpha\beta}|\, \langle A \leqq \rangle\}$ be a presheaf from TOP. Suppose that for every $\alpha \in A$ we have a normed space $F_\alpha \subset C^*(\mathscr{X}_\alpha \to R)$ with the sup-norm which separates points) (points, and points from closed sets) of $\mathscr{X}_\alpha$, and such that $\varrho_{\alpha\beta}^*$ maps F_β into F_α for all $\alpha, \beta \in A$, $\alpha \leqq \beta$. For each $\alpha \in A$ let B_α be the unit ball of the dual space F_α^* with the w^*-topology (denote it by w_α^*), and let $j_\alpha : \mathscr{X}_\alpha \to (B_\alpha, w_\alpha^*)$ be the natural evaluations (Lemma 5A). We set $\mathscr{E} = \{F_\alpha \mid \alpha \in A\}$, $\mathscr{Z} = \{j_\alpha \mid \alpha \in A\}$, $\mathscr{T} =$ $= \{(B_\alpha, w_\alpha^*) \,|\varrho_{\alpha\beta}^{**}|\, \langle A \leqq \rangle\}$. Then $(\mathscr{T}, \mathscr{Z})$ is a weak compact (compact) hull of $\mathscr{S}$ (see 2; it is called the $\mathscr{E}$-dual hull of $\mathscr{S}$), and $\xi = \{\hat{F}_\alpha \mid \alpha \in A\}$ is a separting family for $\mathscr{T}$ (see 5A(d)).

Proof. Straightforward from the lemma 4 and 5.

7. Notation. Let $\mathscr{S} = \{\mathscr{X}_\alpha \,|\varrho_{\alpha\beta}|\, \langle A \leqq \rangle\}$ be a presheaf from TOP. For $\alpha \in A$ we set $A[\alpha] = \{\beta \in A \mid \beta < \alpha\}$ and $\mathscr{S}_{A[\alpha]} = \{\mathscr{X}_\alpha \,|\varrho_{\alpha\beta}|\, \langle A[\alpha] \leqq \rangle\}$. We denote by $\mathscr{L}(\mathscr{S})$ the set of all $\alpha \in A$ for which $\langle A[\alpha] \leqq \rangle$ is right directed and $\langle \mathscr{X}_\alpha \mid \{\varrho_{\alpha\beta} \mid \beta \in A[\alpha]\}\rangle = \underrightarrow{\lim}\, \mathscr{S}_{A[\alpha]}$.

Now we have made things ready so that we can get the results on functional separation of inductive limits and representation of sheaves by sections.

8. Theorem. Let $\mathscr{S} = \{\mathscr{X}_\alpha \,|\varrho_{\alpha\beta}|\, \langle A \leqq \rangle\}$ be a presheaf from the category of topological spaces. Suppose that there is a set $B \subset A$ such that

(1) For every $\alpha \in B$ we have a separating set $\tilde{F}_\alpha \subset C^*(\mathscr{X}_\alpha \to R)$ such that if we consider the sup-norm in the sets $\tilde{F}_\alpha$ then $\varrho^*_{\alpha\beta}$ maps $\tilde{F}_\beta$ onto a norm-dense subset of $\tilde{F}_\alpha$ for all $\alpha, \beta \in B$, $\alpha \leqq \beta$.

(2) Either B is cofinal in $\langle A \leqq \rangle$, or $\langle A \leqq \rangle$ is ordered, $\langle A - B \leqq \rangle$ well ordered and $A - B \subset \mathscr{L}(\mathscr{S})$.

(3) There is a countable cofinal subset of $\langle B \leqq \rangle$.

For $\alpha \in A$ let $F_\alpha = \operatorname{sp} \tilde{F}_\alpha$ (it is the smallest module containing $\tilde{F}_\alpha$) and let us endow F_α with the sup-norm. Let $\mathscr{E} = \{F_\alpha \mid \alpha \in B\}$, let $\mathscr{T}$ be the $\mathscr{E}$-dual hull of $\mathscr{S}_B = \{\mathscr{X}_\alpha \,|\varrho_{\alpha\beta}|\, \langle B \leqq \rangle\}$ (see 6).

Then $\mathscr{K} = \underrightarrow{\lim}\, \mathscr{T}$ is functionally separated (f.s.) meaning that for any $p, q \in |\mathscr{K}|$, $p \neq q$ there is a continuous function f on $\mathscr{K}$ with $f(p) \neq f(q)$. $\mathscr{J} = \underrightarrow{\lim}\, \mathscr{S}_B$ and $\mathscr{I} = \underrightarrow{\lim}\, \mathscr{S}$ are f.s. If moreover every $\tilde{F}_\alpha$ separates points from closed sets of $\mathscr{X}_\alpha$ then $\mathscr{T}$ is a compact hull of $\mathscr{S}_B$.

Proof. If B is cofinal in $\langle A \leqq \rangle$ then $\langle B \leqq \rangle$ is right directed (and $\underrightarrow{\lim}\, \mathscr{S}$ is isomorphic to $\underrightarrow{\lim}\, \mathscr{S}_B$). If $\langle A \leqq \rangle$ is ordered, $\langle A - B \leqq \rangle$ well ordered and $A - B \subset \mathscr{L}(\mathscr{S})$ then it is easy to see that B is cofinal in $\langle A[\delta] \leqq \rangle$ with δ being the smallest element of $\{\beta \in A \,|\{\gamma \in A \mid \gamma \geqq \beta\} \subset A - B\}$ (and $\underrightarrow{\lim}\, \mathscr{S}$ is again isomorphic to $\underrightarrow{\lim}\, \mathscr{S}_B$ because, as it can be easily seen from [2, Lemma 1.4.1(5)], $\underrightarrow{\lim}\, \mathscr{S}$ is isomorphic to $\underrightarrow{\lim}\, \mathscr{S}_{A[\delta]}$). As $\delta \in \mathscr{L}(\mathscr{S})$ and $\langle A[\delta] \leqq \rangle$ is by 7 right directed, $\langle B \leqq \rangle$ is such, too, being cofinal in $\langle A[\delta] \leqq \rangle$. In any case $\langle B \leqq \rangle$ is right directed (and $\underrightarrow{\lim}\, \mathscr{S}$ is isomorphic to $\underrightarrow{\lim}\, \mathscr{S}_B$), and from the condition (3) it follows that there is a cofinal subset C of $\langle B \leqq \rangle$ of the type ω_0. Thus $\underrightarrow{\lim}\, \mathscr{S}_B$ is isomorphic to $\underrightarrow{\lim}\, \mathscr{S}_C$ and as the condition (1) holds for all $\alpha, \beta \in C$, we may assume that $B = C$ i.e. that $\langle B \leqq \rangle$ itself is of the type ω_0.

$\mathscr{T} = \{(B_\alpha, w^*_\alpha) \,|\varrho^{**}_{\alpha\beta}|\, \langle B \leqq \rangle\}$ (see 6) is a weak compact hull (a compact hul! if moreover the $\tilde{F}_\alpha$'s separate points from closed sets of $\mathscr{X}_\alpha$) of $\mathscr{S}_B$, where $\varrho^{**}_{\alpha\beta}$ are homeomorphisms of (B_α, w^*_α) into (B_β, w^*_β), being continuous and 1-1 by Lemma 5, and the (B_α, w^*_α) being compact. Hence $\mathscr{T}$ fulfils the conditions of [2, Cor. 1.5.5 or Remark 1.5.6B]. Thus $\mathscr{K}$ is f.s. Further, as $\mathscr{T}$ is a weak compact hull of $\mathscr{S}_B$, there is a 1-1 continuous map $j : \mathscr{J} \to \mathscr{K}$, whence $\mathscr{J}$ is f.s. As $\mathscr{I}$ is isomorphic to $\mathscr{J}$, $\mathscr{I}$ is f.s., too.

9. *Theorem.* Let $\mathscr{S} = \{(X_U, t_U) |\varrho_{UV}| X\}$ be a sheaf from the category of topological spaces over a separated topological space X such that every $x \in X$ is of a countable local character. Suppose that for each open $U \subset X$ we have a set $F_U \subset \subset C^*((X_U, t_U) \to R)$ (see 1) which separates points and points from closed sets of (X_U, t_U), such that ϱ^*_{UV} maps F_V into F_U for any $V \subset U$, and that

(1) Every $x \in X$ has a filter base Ax of open neighborhoods such that if we consider the sup-norm in the sets F_U then $\varrho^*_{UV} F_V$ is norm dense in F_U for all $U, V \in \in Ax$, $V \subset U$.

(2) If $U \subset X$ is open and $\mathscr{V}$ is an open cover of U then $\text{sp} \cup \{\varrho^*_{UV} F_V \mid V \in \mathscr{V}\}$ is norm dense in sp F_U (sp M is the smallest module containing M).

Then there is a separated closure $\hat{t}$ in the covering space P of $\mathscr{S}$ such that for the set $\Gamma(U, \hat{t})$ of all continuous sections over U, for the natural map p_U sending $a \in X_U$ onto the section $\hat{a}$ over U, where $\hat{a}(x)$ is the germ of a in the stalk over x, and for the topology $b_U(\hat{t})$ projectively defined in $A_U = p_U(X_U)$ by the maps $\{r_{U,x} : : A_U \to (I_x, \hat{t}/I_x) \mid x \in U\}$, where $r_{U,x}(\hat{a}) = \hat{a}(x)$ for $x \in U$, we have for any open $U \subset X$
(*a*) $p_U : (X_U, t_U) \to (A_U, b_U(\hat{t}))$ is a homeomorphism.
(b) $\Gamma(U, \hat{t}) = A_U$.

Proof. Let $\mathscr{T} = \{(B_U, w^*_U) |\varrho^{**}_{UV}| X\}$ be the $\mathscr{E}$-dual hull of $\mathscr{S}$ by $\mathscr{E} = \{\text{sp } F_U \mid \emptyset \neq \neq U \subset X \text{ open}\}$. By Th. 8, $\mathscr{T}_x = \mathscr{T}_{Ax}$ is a compact hull of $\mathscr{S}_{Ax}$, and it is easy to see that Th. 8 can be applied to show that $(H_x, h_x) = \varinjlim \mathscr{T}_x$ is f.s.

For each open $U \subset X$ let A'_U be the set of the sections in the covering space H of $\mathscr{T}$ which canonically correspond to B_U, and for $x \in U$ let ξ'_{Ux} or η'_{Ux} be the canonical maps of B_U or A'_U into (H_x, h_x), respectively. (Thus for $f \in B_U$, $\xi'_{Ux}(f) = \eta'_{Ux}(\hat{f}) =$ the germ of f at x ($\hat{f}$ is the section from A'_U corresponding to f, i.e. $\hat{f}(x) =$ germ of f at x for each $x \in U$).) We denote by h the topology in H defined by "$O \subset H$ is h-open iff $O \cap H_x$ is h_x-open for all $x \in X$"; clearly $h/H_x = h_x$. For open $U \subset X$ let $b'_U(h)$ be the topology in A'_U projectively defined by the maps $\{\eta'_{Ux} : A'_U \to (H_x, h_x) \mid x \in U\}$. As the (H_x, h_x)'s are f.s., $(A'_U, b_U(h))$ is Hausdorff for any open $U \subset X$.

For open $U \subset X$ let $p'_U : B_U \to A'_U$ be the map sending $f \in B_U$ onto the section $\hat{f} \in A'_U$, where $\hat{f}(x) =$ germ of f at x, for $x \in U$. Then we easily show that $p'_U : (B_U, w^*_U) \to (A'_U, b_U(h))$ is continuous for any open $U \subset X$; for this end the continuity of the maps $\eta'_{Ux} p'_U : (B_U, w^*_U) \to (H_x, h_x)$ for any fixed open $U \subset X$ and each $x \in U$ is enough owing to the projective definition of $b_U(h)$ by the maps $\{\eta'_{Ux} : A'_U \to (H_x, h_x) \mid x \in \in U\}$. But plainly $\eta'_{Ux} p'_U = \xi_{Ux}$ and $\xi_{Ux} : (B_U, w^*_U) \to (H_x, h_x)$ is continuous for each $x \in U$.

Further, each p'_U is 1-1. To show it, take $f, g \in B_U$ so that $p'_U(f) = p'_U(g)$. It means that there is an open cover $\mathscr{V}$ of U such that $\varrho^{**}_{UV}(f) = \varrho^{**}_{UV}(g)$, i.e. $f \circ \varrho^*_{UV} = = g \circ \varrho^*_{UV}$, for each $V \in \mathscr{V}$. The latter equality means that, given any $V \in \mathscr{V}$, for each $\varphi \in F_V$ we have $f(\varrho^*_{UV}(\varphi)) = g(\varrho^*_{UV}(\varphi))$ i.e. $f = g$ on $\varrho^*_{UV} F_V$. Thus $f = g$ on each $\varrho^*_{UV} F_V$ for $V \in \mathscr{V}$, hence $f = g$ on $\cup \{\varrho^*_{UV} F_V \mid V \in \mathscr{V}\}$ and the condition (2) yields that $f = g$ on F_U.

From that what we have known we gather that each $p'_U : (B_U, w_U^*) \to (A'_U, b_U(h))$ is a homeomorphism as it is 1-1, continuous, the former space is compact and the latter Hausdorff.

Now, for each open $U \subset X$ we have by Lemma 5 the natural evaluation $j_U : (X_U, t_U) \to (B_U, w_U^*)$ which is by 5 (b), (c) continuous and 1-1. There thus is a continuous 1-1 map $j_x : (I_x, t_x^*) = \varinjlim \mathscr{S}_{Ax} \to (H_x, h_x)$ for each $x \in X$. Let t_x be the topology in I_x projectively defined by $j_x : I_x \to (H_x, h_x)$, and t the topology in the covering space P of $\mathscr{S}$ defined by "$O \subset P$ is t-open iff $O \cap I_x$ is t_x-open for each $x \in X$"; thus $t/I_x = t_x$. Let $b_U(t)$ be the topology projectively defined in A_U by the maps $\{\eta_{Ux} : A_U \to (I_x, t_x) \mid x \in U\}$. Now we have the following commutative diagram:

$$\begin{array}{ccccccc}
(A_U, b_U(t)) & \xleftarrow{p_U} & (X_U, t_U) & \xrightarrow{j_U} & (B_U, w_U^*) & \xrightarrow{p'_U} & (A'_U, b'_U(h)) \\
\Big| \eta_{Ux} & & \Big\downarrow \xi_{Ux} & & \Big\downarrow \xi'_{Ux} & & \eta'_{Ux} \Big| \\
\longrightarrow & & (I_x, t_x) & \xrightarrow{j_x} & (H_x, h_x) & & \longleftarrow
\end{array}$$

Firstly we shall show that p_U is continuous. To this end it is enough that $\eta_{Ux} p_U$ be continuous for any $x \in U$ owing to the definition of $b_U(t)$. But the latter maps are continuous iff ξ_{Ux} is for each $x \in U$. Owing to the way of definition of t_x this happens iff $j_x \xi_{Ux}$ is continuous for each $x \in U$. But $j_x \xi_{Ux} = \xi'_{Ux} j_U$ which is continuous because such is j_U and ξ'_{Ux} for each $x \in U$.

Now we shall show the continuity of p_U^{-1}. Since p'_U is a homeomorphism and j_U as well (onto $j_U(X_U)$), it is enough to show the continuity of $q_U = p'_U j_U p_U^{-1}$. Owing to the definition of $b_U(h)$ it will be assured by the continuity of $\eta'_{Ux} q_U$ for each $x \in U$. But it is $j_x \eta_{Ux}$ which is continuous for all $x \in U$ iff such is η_{Ux} for all $x \in U$ (mind the definition of t_x). But it is just the case owing to the definition of $b_U(t)$.

If $U \subset X$ is open, $x \in U$, $a \in X_U$, $\alpha = \xi_{Ux}(a)$, we set graph $(a; U) = \{\xi_{Uy}(a) \mid y \in U\}$. For $\alpha \in P$, $\alpha \in I_x$ we set $H(\alpha) = \{N \cup \text{graph}\,(a; U) \mid N$ is a t_x-neighborhood of α in I_x, $U \subset X$ open with $x \in U$, $a \in X_U$ with $\xi_{Ux}(a) = \alpha\}$. Then $H(\alpha)$ is a base of a filter round α. These bases define a closure $\hat{t}$ in P. Plainly $\hat{t}/I_x = t_x$ for all $x \in X$ whence $b_U(t) = b_U(\hat{t})$ for all open $U \subset X$, which finishes the proof of (a).

To prove (b) we first notice that p_U is 1-1 for any open $U \subset X$. Namely if for $a, b \in X_U$, $a \neq b$ we have $p_U(a) = p_U(b)$ then (look at the diagram) $j_x \eta_{Ux}\, p_U(a) = j_x \eta_{Ux}\, p_U(b)$ for any $x \in U$. But it means that $\eta'_{Ux} p'_U\, j_U(a) = \eta'_{Ux} p'_U\, j_U(b)$ for all $x \in U$. But p'_U, j_U and η'_{Ux} are 1-1 hence it is impossible (the latter map is 1-1 because, given $p, q \in A'_U$, $p \neq q$, we can find for any $y \in U$ a $V_y \in Ay$, $V_y \subset U$, and then at least for one V_y we must have $u = \varrho_{UV_y}^{**}(p) \neq \varrho_{UV_y}^{**}(q) = v$; therefore $\eta'_{Uy}(p) =$ (germ of u at y) $\neq$ (germ of v at y) $= \eta'_{U_y}(q)$ as ϱ_{VW}^{**} are 1-1 for $V, W \in Ay$).

It is clear that for any open $U \subset X$ and any $a \in X_U$ the section $p_U(a) = \hat{a} : U \to (P, \hat{t})$ is continuous, hence $A_U \subset \Gamma(U, \hat{t})$. And if $r \in \Gamma(U, \hat{t})$ then by the definition of $\hat{t}$, for each $x \in U$ there is an open $V_x \subset U$ and an $a_{V_x} \in X_{V_x}$ such that $r(y) = \hat{a}_{V_x}(y)$

for all $y \in V_x$. If $z \in V_x \cap V_y$ then $\hat{a}_{V_x}(z) = r(z) = \hat{a}_{V_y}(z)$, hence $\hat{a}_{V_x} = \hat{a}_{V_y}$ on $W = V_x \cap V_y$. It means that $p_W(a_W) = p_W(b_W)$ for $a_W = \varrho_{V_xW}(a_{V_x})$, $b_W = \varrho_{V_yW}(a_{V_y})$. As p_W is 1-1, we have $a_W = b_W$, i.e. the family $\{a_{V_x} \mid x \in U\}$ is smooth (see 3). There thus is $a \in X_U$ such that $\varrho_{UV_x}(a) = a_{V_x}$ for all $x \in U$. Thus $\hat{a}(y) = \hat{a}_{V_y}(y) = r(y)$ for all $y \in U$ hence $p_U(a) = \hat{a} = r$ and $\Gamma(U, \hat{t}) \subset A_U$ which proves (b).

10. Remark. It might be worth to notice that in the proof of the assertion (a) of Th. 9 $\mathcal{S}$ need not be a sheaf; it is enough that it be only a presheaf, i.e. it need not fulfil the conditions a, b of Def. 3. In fact, if a presheaf $\mathcal{S}$ fulfils all the conditions of Th. 9 then it fulfils the condition (a) of Def. 3, because we have shown in the proof that the maps are 1-1 which is equivalent to the condition (a) of Def. 3. But it need not fulfil the cond. (b). The latter we needed to prove the inclusion $\Gamma(U, \hat{t}) \subset A_U$ in the assertion (b) of Th. 9. In fact, we have proven the following: "Let a presheaf $\mathcal{S}$ fulfil the conditions of Th. 9. Then there is a topology $\bar{t}$ in the covering space P of $\mathcal{S}$ such that $p_U : (X_U, t_U) \to (A_U, b_U(\bar{t}))$ is a homeomorphism and $A_U \subset \Gamma(U, \bar{t})$ for any open $U \subset X$. Moreover, the maps $p_U : (X_U, t_U) \to (A_U, b_U(\bar{t}))$ are jointly continuous (it is to say that $\tilde{p}_U : (X_U, t_U) \times U \to (A_U, b_U(\bar{t}))$ sending $(a, x) \in X_U \times U$ onto $\hat{a}(x)$ are continuous)". Namely, setting for $\alpha \in P$, $\alpha \in I_x : K(\alpha) = \{\cup\} I_y \mid y \in U$, $y \neq x\} \cup N \mid x \in U \subset X$ open, N is a t_x-neighborhood of $\alpha\}$ then it is easy to see that $K(\alpha)$ is a base of a filter round α in P and that these bases yield a Hausdorff topology $\bar{t}$ in P such that $\bar{t}/I_x = t_x$ for all $x \in X$. Bearing in mind all what we have already proven it is easy to see that $\bar{t}$ has all the properties we have claimed. If $(I_x, t_x^*) = \underrightarrow{\lim}\, \mathcal{S}_{Ax}$ then $\bar{t}/I_x$ is coarser than t_x^* for any $x \in X$. If $\mathcal{S}$ is moreover a sheaf then the full assertion of Th. 9 holds. Notice that $\hat{t}$ is only a closure, not a topology.

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