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Boolean and Orthomodular Lattices — a Short Characterization via Commutativity

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A three-axiom description of Boolean algebras and orthomodular lattices is given.

Получено описание булевых алгебр и ортомодулярных структур системами состоящими из трех аксиом.

Je podán tříaxiomatický popis Booleových algeber a ortomodulárních svazů.

1. Introduction

This note owes its inspiration to the remarkable paper of Sobociński [3]. It turns out that the concepts and methods developed there play an important role in the theory of ortholattices (cf. [1], [4] and [5]); our discussion here will yield two new consequences of such an approach.

First of all, we give a brief outline of two results from the theory of orthomodular lattices. For further details the reader is referred to Birkhoff's book [2].

We shall use the following theorem, the proof of which may be found in [2, Theorem 21, p. 53]. Recall that two elements a, b of an ortholattice *commute*, written aCb if $a = (a \cap b) \cup (a \cap b^{\perp})$.

Thoerem 1. An ortholattice $\mathfrak{A} = (A, \cup, \cap, {}^{\perp})$ is a Boolean algebra if and only if

$$[ab] : a, b \in A \,.\supset .\, aCb \,.$$

We now turn our attention to a similar statement about orthomodular lattices.

Theorem 2. An ortholattice $\mathfrak{A} = (A, \cup, \cap, {}^{\perp})$ is an orthomodular lattice if and only if it satisfies the following postulate

$$M\ [ab] : a, b, c \in A \,.\supset .\, a \cup b = ((a \cup b) \cap a) \cup ((a \cup b) \cap a^{\perp}) \,.$$

Proof. 1) Since in any lattice $a \leqq a \cup b$, by [2, Lemma 1 and Theorem 21, pp. 52, 53] we have $a \cup bCa$.

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2) Assume $\mathfrak{A}$ satisfies M. If $a \leqq b$, one then obtains $b = (b \cap a) \cup (b \cap a^{\perp}) = a \cup (b \cap a^{\perp})$.

2. Main Theorems

Theorem 3. Any algebraic system $\mathfrak{A} = (A, \cup, \cap, {}^{\perp})$ where $\cup$ and $\cap$ are two binary operations and ${}^{\perp}$ is a unary operation is a Boolean algebra if it satisfies the axioms

BA 1 $[ab] : a, b \in A \,.\supset .\, a = a \cup (b \cap b^{\perp})$;
BA 2 $[abc] : a, b, c \in A \,.\supset .\, (a \cup b) \cup c = (c^{\perp} \cap b^{\perp})^{\perp} \cup a$;
BA 3 $[abc] : a, b, c \in A \,.\supset .\, a = (a \cap (b \cup c)) \cup (a \cap b^{\perp})$.

Remark. Using Theorem 23 of [2, p. 53] we find that the axioms BA 1, BA 2 and BA 3 hold in any Boolean algebra.

Proof of Theorem 3. Put $c = b \cap b^{\perp} = 0$, use Theorem 1 and [1].

Thoerem 4. Any algebraic system $\mathfrak{A} = (A, \cup, \cap, {}^{\perp})$ where $\cup$ and $\cap$ are two binary operations and ${}^{\perp}$ is a unary operation is an orthomodular lattice if it satisfies the postulates

OM 1 $[ab] : a, b \in A \,.\supset .\, a = a \cup (b \cap b^{\perp})$;
OM 2 $[abc] : a, b, c \in A \,.\supset .\, (a \cup b) \cup c = (c^{\perp} \cap b^{\perp})^{\perp} \cup a$;
OM 3 $[abc] : a, b, c \in A \,.\supset .\, a \cup b = ((a \cup b) \cap (a \cup c)) \cup ((a \cup b) \cap a^{\perp})$.

Remark. Since in any orthomodular lattice $a^{\perp} C a \cup b$ and $a^{\perp} C a \cup c$, we obtain, using [2, Theorem 23, p. 53], that

$$((a \cup b) \cap (a \cup c)) \cup ((a \cup b) \cap a^{\perp}) = (a \cup b) \cap (a \cup c \cup a^{\perp}) = a \cup b\,.$$

From this fact we conclude that the postulates OM 1, OM 2 and OM 3 hold in any orthomodular lattice.

Proof of Theorem 4. Put $b = 0$ in OM 3. Then

$$\begin{aligned} a &= a \cup 0 && \text{OM 1} \\ &= ((a \cup 0) \cap (a \cup c)) \cup ((a \cup 0) \cap a^{\perp}) && \text{OM 3} \\ &= (a \cap (a \cup c)) \cup (a \cap a^{\perp}) && \text{OM 1} \\ &= a \cap (a \cup c)\,. && \text{OM 1} \end{aligned}$$

By [1], $\mathfrak{A}$ is an ortholattice. The postulate OM 3 for $c = 0$ gives

$$\begin{aligned} a \cup b &= ((a \cup b) \cap (a \cup 0)) \cup ((a \cup b) \cap a^{\perp}) \\ &= ((a \cup b) \cap a) \cup ((a \cup b) \cap a^{\perp})\,. && \text{OM 1} \end{aligned}$$

Theorem 2 shows that $\mathfrak{A}$ is orthomodular.

References

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