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Homogeneous Groupoids

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The structure of all finite, connected homogeneous tolerance groupoids with some one-sided identity element is completely described.

В стате получается полное описание всех конечных связных однородных толеранционных группоидов с некоторой односторонной единицой.

V článku jsou úplně popsány všechny konečné, souvislé homogenní grupoidy mající jednostranný jednotkový prvek.

There is a theorem on topological semigroups saying that every compact connected homogeneous semigroup with identity element and of a finite cohomological dimension is always a topological group ([3] p. 169). This note brings some similar results on tolerance groupoids.

For basic terminology on tolerance spaces and on tolerance algebras see [1]. See also [2] for tolerance semigroups. In what follows G will mean a tolerance groupoid with a tolerance t. Thus, if $a\ t\ b$ and $c\ t\ d$ for some a, b, c, d in G then $(ac)\ t\ (bd)$.

Proposition 1. If G is finite, connected and homogeneous and if G has an identity element then G is a simplex.

Remarks on terminology. G is said to be *connected* iff the transitive closure $\bar{t}$ of t is the universal relation. G is said to be *homogeneous* iff the automorphism group of the space G is transitive. G is said to be a *simplex* iff t is the universal relation.

From proposition 1 we easily obtain

Proposition 2. If G is finite and homogeneous and if G has an identity element then the tolerance t of G is a uniform congruence relation on G.

Remarks on terminology. An equivalence relation will be said to be *uniform* iff all its equivalence classes have the same cardinality.

Before proving proposition 1 we first state.

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Proposition 3. If G is finite, connected and homogeneous and if G has a right identity element then there is a connected and homogeneous left zero tolerance groupoid X and a surjective uniform homomorphism $f : G \to X$ such that for all a, b in G, $a\ t\ b$ iff $f(a)\ t\ f(b)$ in X.

Remarks on terminology. By a *uniform mapping* we mean any mapping with a uniform kernel. X is a *left zero* groupoid iff $yx = y$ for all x, y in X.

Proof of proposition 3. Let G have a right identity element u. A *t-neighbourhood* of some x in G is defined by $tx = \{y \in G;\ y\ t\ x\}$. Let $A = \{a \in G;\ tx = t(xa)$ for all $x \in G\}$. Clearly, $u \in A$. We shall prove that $A = G$. Suppose we have some $v \in G \setminus A$. Because of $u\ \bar{t}\ v$ there are some $a \in A$ and $b \in G \setminus A$ such that $a\ t\ b$. There is also some $x \in G$ such that $tx \neq t(xb)$. Take any $y \in tx$. Then $y\ t\ x$, $(ya)\ t\ (xb)$, $xb \in t(ya) = ty$ and $y \in t(xb)$. We conclude that $tx \subset t(xb)$. Because G is homogeneous every two t-neighbourhoods are isomorphic and we get $tx = t(xb)$, a contradiction. $A = G$ is proved. We have $tx = t(xa)$ for all $x, a \in G$.

Setting $x\ r\ y$ iff $tx = ty$ we see that r is a congruence relation on G such that G/r is a left zero groupoid. It is easy to see that the canonical map $f : G \to G/r = X$ has the desired properties of proposition 3. Also, X is connected with respect to the induced tolerance. It remains to prove that X is homogeneous.

Let us take any $x_1, x_2 \in X$. Then we have some $y_1, y_2 \in G$ with $f(y_1) = x_1$, $f(y_2) = x_2$. As G is homogeneous there is some automorphism α of the space G such that $y_1\alpha = y_2$. We shall construct an oriented multigraph on the set X as follows: For every $y \in G$ we draw an arc $\hat{y}$ from $f(y)$ to $f(y\alpha)$ and write simply $\hat{y} : (f(y), f(y\alpha))$ to describe this situation. It follows that $\hat{y}_1 : (x_1, x_2)$. These arcs $\hat{y}$ form a finite set S equivalent to G. We have two uniform equivalence relations (or partitions) on S, P_1 and P_2, the first identifying all arcs going out from the same source, the second identifying all arcs going to the same target. It is well known (see, e.g. [4] p. 11) that P_1 and P_2 have a common system R of representatives and this R is obviously an automorphism of X. More than that, we can suppose that $\hat{y}_1$ belongs to R. In this way we have constructed an automorphism which maps x_1 onto x_2. X is homogeneous.

Proof of proposition 1. We proceed as in the proof of proposition 3. We obtain a congruence relation r on G such that G/r is both left zero and right zero groupoid. It follows that r must be universal and G is a simplex.

Final remarks. The finiteness condition in proposition 1 can be replaced by the requirement that the t-neighbourhood of 1 is finite. The finiteness of G will then follow.

The last part of the proof of proposition 3 can be used for proving the following: Assume that X and G are finite tolerance spaces. If $X \times G$ is homogeneous and G a simplex then X is homogeneous. If $X \times G$ is homogeneous and the tolerance of G is an equivalence then X is homogeneous.

References

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