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The Category of Totally Symmetric Quasigroups is Binding

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It is proved that the category of totally symmetric quasigroups is binding.

В статье доказывается, что всякая алгебраическая категория изоморфно вкладывается в категорию вполне симметрических квазигрупп.

V článku se dokazuje, že každou kategorii algeber lze vnořit do kategorie totálně symetrických kvazigrup.

1. We use the terminology of [1]. If G is a halfgroupoid then G' denotes the set of all triples $\langle a_1, a_2, a_3 \rangle$ of elements of G such that $a_{p(1)}a_{p(2)} = a_{p(3)}$ for some permutation p of $\{1, 2, 3\}$. A halfgroupoid G is called a TS-halfgroupoid if the following holds for all $a, b, c, d \in G$:
If $\langle a, b, c \rangle \in G'$ and $\langle a, b, d \rangle \in G'$ then $c = d$.

2. Groupoids satisfying the identities $xy = yx$ and $x \,.\, xy = y$ are quasigroups. Such quasigroups are called totally symmetric. Hence TS-groupoids are nothing else than totally symmetric quasigroups. The category of totally symmetric quasigroups will be denoted by T.

3. Let L, G, A, B be sets, F be a system of mappings from L into G and H be a system of mappings from A into B. We shall say that F is an extension of H if $A \subseteq L$, $B \subseteq G$, $f \mid A \in H$ for every $f \in F$, and for every $h \in H$ there is exactly one $f \in F$ with $f \mid A = h$.

4. Let G be a TS-halfgroupoid. Denote by L the set of all non-ordered pairs $\{a, b\}$ of (not necessarily distinct) elements of G such that there is no $c \in G$ with $\langle a, b, c \rangle \in G'$. Further, denote by $S(G)$ the disjoint union $G \cup L$. Let $a, b \in G$. If there exists $c \in G$ with $\langle a, b, c \rangle \in G'$ then we define $ab = ba = c$, $ac = ca = b$, $bc = cb = a$. If such c does not exist, we put $ab = \{a, b\}$. Obviously, $S(G)$ is a TS-halfroupoid.

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5. Let G be a TS-halfgroupoid. Put $G_1 = G$, $G_{n+1} = S(G_n)$ and $F(G) = \bigcup_{n=1}^{\infty} G_n$, with the operation defined in an obvious manner.

6. Lemma. $F(G)$ is a totally symmetric quasigroup and G is its generating subhalfgroupoid. If H is an arbitrary totally symmetric quasigroup then every homomorphism of G into H can be extended in exactly one way to a homorphism of $F(G)$ into H.

Proof. is straightforward.

7. Lemma. Let G be a TS-halfgroupoid. Then

(i) if $a \in F(G)$ and there are $m > n \geq 0$ with $a^{2^m} = a^{2^n}$ then $a \in G$,

(ii) if $a, b, c \in G$ and $ab = c$ in $F(G)$ then $\langle a, b, c \rangle \in G'$.

Proof. is obvious.

8. We denote by R the category of symmetric graphs without loops and with at least one edge. Let $A = \langle A, r \rangle \in R$ and L be the set of all non-ordered pairs of (not necessarily different) elements of A. For every $x \in L$, we fix twelve elements $x_1, \ldots, x_{12}$ not belonging to A and define $L_x = \{x_i \mid i = 1, \ldots, n_x\}$ where $n_x = 4$ for $x = \{a, a\}$, $n_x = 3$ for $x = \{a, b\}$ with $a\, r\, b$ and $n_x = 12$ otherwise. Further, denote by $M(A)$ the disjoint union $A \cup \bigcup_{x \in L} L_x$ and define a partial binary operation on $M(A)$ as follows:

(i) for every $x \in L$ we put $x_{n_x} . x_{n_x} = x_1$ and $x_i x_i = x_{i+1}$, $i = 1, \ldots, n_x - 1$,

(ii) if $a, b \in A$ then we define $ab = \{a, b\}_1$.

9. Lemma. $M(A)$ is a TS-halfgroupoid.

Proof. is obvious.

10. If $A \in R$ then put $N(A) = F(M(A))$. We get a mapping N of R into T.

11. Theorem. If $A, B \in R$ then $\mathrm{Hom}_T(N(A), N(B))$ is an extension of $\mathrm{Hom}_R(A, B)$.

Proof. Let $A, B \in R$ and $f \in \mathrm{Hom}_T(N(A), N(B))$. We claim that $f(A) \subseteq B$. For consider the following two situations:

(i) Let $a, b \in A$ with $f(a) \in B, f(b) \notin B$. By 7., $\langle f(a), f(b), f(ab) \rangle$ is contained in $M(B)'$, and so $f(ab) \in B$, which is impossible, since $(ab)^{2^{12}} = ab$.

(iii) Let $a, b \in A$ be such that $f(a), f(b) \notin B$ and $a\, r\, b$. Then, using 7 again, we get $f(a)^8 = f(a)$, while $f(a^2) = f(a^{32})$, hence $f(a)^2 = f(a)^4$, a contradiction.

Thus we have proved our claim and the rest easily follows.

12. Using some results from [2], we see that T is binding.

References

[1] R. H. Bruck: A survey of binary systems, Springer Verlag, Berlin-Göttingen-Heidelberg, 1958.

[2] Z. Hedrlín, A. Pultr: Symmetric relations (undirected graphs) with given semigroups, Monatshefte f. Math. 69 (1965), 318–322.