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On a Construction of Amalgamation I

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This paper is concerned with a construction which generalizes some known constructions used in the theory of posets. We mention here e.g. the extensions in the sense of G. G. Boulaye [3] and especially the disjoint sums of M. F. Janowitz [7], the *pasting* of R. J. Greechie [4], [5], [6] who discovered the convenience of such constructions for the study of orthomodular posets and lattices. I hope that the present approach will be useful in other connections as well.

The purpose of this note is to investigate some of the basic questions about the amalgams. The results were partly reported in Mai 1969 [1] and in September 1969 [2].

I. Introduction

For two subposets $\mathcal{M}, \mathcal{N}$ of a poset $\mathcal{P}$ let $[\mathcal{M}, \mathcal{N}] = \{[x, y] \mid x \in M, y \in N$ or $x \in N, y \in M$ and (in both cases) $x \leqq y\}$.

Consider a system $\{\mathcal{S}_\lambda\}_{\lambda \in \Lambda}$ of posets $\mathcal{S}_\lambda = \langle S_\lambda, \leqq_\lambda \rangle$. (The subscripts distinguishing different partial ordering will be often omitted.) Suppose $\mathcal{S}_\lambda^\circ$ is a subposet of the poset $\mathcal{S}_\lambda$, $\lambda \in \Lambda$, and for each $\lambda_1, \lambda_2 \in \Lambda$ suppose $\mathcal{S}_{\lambda_1/\lambda_2}$ is an order isomorphism of $\mathcal{S}_{\lambda_1}^\circ$ onto $\mathcal{S}_{\lambda_2}^\circ$. A poset $\mathcal{S}$ is called an *amalgam* of the $\mathcal{S}_\lambda$'s relative to the isomorphisms $\varphi_{\lambda_1/\lambda_2}$ iff there exist order isomorphisms $\varphi_\lambda : \mathcal{S}_\lambda \to \mathcal{S}$ such that

(a) the union $\bigcup\{\varphi_\lambda(S_\lambda);\ \lambda \in \Lambda\}$ of the sets $\varphi_\lambda(S_\lambda) = \{\varphi_\lambda(x) \mid x \in S_\lambda\}$ equals to S;

(aa) for any two distinct $\varkappa, \lambda \in \Lambda$ and for each nonempty interval $\mathcal{J} = [\varphi_\varkappa(S_\varkappa), \varphi_\lambda(S_\lambda)]$ the intersection $\mathcal{J} \cap \varphi_\varkappa(S_\varkappa^\circ)$ is nonempty;

(aaa) for each $\alpha, \beta \in \Lambda$ the diagram

$$\begin{array}{ccccc} S_\alpha^\circ & & \xrightarrow{\varphi_{\alpha/\beta}} & & S_\beta^\circ \\ \downarrow \text{id} & & & & \downarrow \text{id} \\ S_\alpha & \xrightarrow{\varphi_\alpha} & S & \xleftarrow{\varphi_\beta} & S_\beta \end{array}$$

is commutative and $\varphi_\alpha(S_\alpha^\circ) \neq \emptyset$.

We will now confine our attention to some immediate consequences of the preceding definition.

Lemma 1,1. (**i**) *If* $\varphi_{\lambda_1}(s_1) = \varphi_{\lambda_2}(s_2)$ *and* $\lambda_1 \neq \lambda_2$, *then* $s_1 \in S^{\circ}_{\lambda_1}$, $s_2 \in S^{\circ}_{\lambda_2}$ *and* $s_2 = \varphi_{\lambda_1/\lambda_2}(s_1)$. (**ii**) *If card* $\Lambda \geqq 2$ *and* $\lambda \neq \mu$, $\lambda, \mu \in \Lambda$, *then*

$$\varphi_\lambda(S_\lambda) \cap \varphi_\mu(S_\mu) = \varphi_\lambda(S^{\circ}_\lambda) = \bigcap \{\varphi_\lambda(S_\lambda);\ \lambda \in \Lambda\}\ .$$

Proof. *Ad* (**i**): By (**aa**) $[\varphi_{\lambda_1}(s_1), \varphi_{\lambda_2}(s_2)] \cap \varphi_{\lambda_1}(S^{\circ}_{\lambda_1}) \neq \emptyset$ and so $\varphi_{\lambda_1}(s_1) = \varphi_{\lambda_1}(s^{\circ}_1)$ for an element $s^{\circ}_1 \in S^{\circ}_{\lambda_1}$. Hence $s_1 \in S^{\circ}_{\lambda_1}$ and similarly $s_2 \in S^{\circ}_{\lambda_2}$. In view of (**aaa**) this yields $\varphi_{\lambda_2}(s_2) = \varphi_{\lambda_1}(s_1) = \varphi_{\lambda_2}(\varphi_{\lambda_1/\lambda_2}(s_1))$, so that $s_2 = \varphi_{\lambda_1/\lambda_2}(s_1)$.
Ad (**ii**): If $s \in \varphi_\lambda(S_\lambda) \cap \varphi_\mu(S_\mu)$, then $s = \varphi_\lambda(s_1) = \varphi_\mu(s_2)$ and, by (**i**), $s = \varphi_\mu(\varphi_{\lambda/\mu}(s_1)) \in \varphi_\lambda(S^{\circ}_\lambda)$. Since $\varphi_\lambda(S^{\circ}_\lambda) \subset \varphi_\mu(S_\mu)$ this completes the proof.

Proposition 1,2 *Let* $\{\mathscr{S}_\lambda\}_{\lambda\in\Lambda}$, $\mathscr{S}$, φ_λ *and* $\varphi_{\lambda_1/\lambda_2}$ *be defined as above and let* $\varphi^*_\lambda : \mathscr{S}_\lambda \to \mathscr{S}^* = \langle S^*, \leqq_* \rangle$ *be order isomorphisms satisfying the conditions* (**a**), (**aa**), (**aaa**). *Then the poset* $\mathscr{S}^*$ *is isomorphic to the poset* $\mathscr{S}$.

Proof. For every $s \in S$ there exists $s_\lambda \in S_\lambda$ such that $s = \varphi_\lambda(s_\lambda)$. Defining a mapping ψ of S into S^* by $\psi(s) = \varphi^*_\lambda(s_\lambda)$, we shall see that ψ is an isomorphism of $\mathscr{S}$ onto $\mathscr{S}^*$. Clearly, ψ is well-defined. Furthermore, if $s^* \in S^*$, then $s^* = \varphi^*_\lambda(s_2)$, $s_2 \in S_{\lambda_2}$ and for $s = \varphi_\lambda(s_\lambda)$ we have $s^* = \psi(s)$. Thus ψ maps S onto S^*. If $s = \varphi_\lambda(s_\lambda)$, $t = \varphi_\mu(t_\mu)$, then in the case $\lambda = \mu$ it is obvious that $s \leqq t$ is equivalent to $\psi(s) \leqq_* \psi(t)$; in the case $\lambda \neq \mu$ we can use the following argument: The assumption $\lambda \neq \mu$ implies that there exist $s^{\circ}_\lambda \in S^{\circ}_\lambda$, $s^{\circ}_\mu \in S^{\circ}_\mu$ such that $s = \varphi_\lambda(s_\lambda) \leqq \varphi_\lambda(s^{\circ}_\lambda) = \varphi_\mu(s^{\circ}_\mu) \leqq \varphi_\mu(t_\mu)$. Hence $s_\lambda \leqq s^{\circ}_\lambda$ and $s^{\circ}_\mu \leqq t_\mu$. Consequently,

$$\psi(s) = \varphi^*_\lambda(s_\lambda) \leqq_* \varphi^*_\lambda(s^{\circ}_\lambda) = \varphi^*_\mu(s^{\circ}_\mu) \leqq_* \varphi^*_\mu(t_\mu) = \psi(t)\ .$$

Replacing here the φ's by φ^*'s we see that also the implication $\psi(s) \leqq_* \psi(t) \Rightarrow s \leqq t$ is valid and this proves the proposition.

Let $\psi_{\lambda/\mu}$ be the mappings defined by $\psi_{\lambda/\mu} = \varphi_{\lambda/\mu}$ iff $\lambda \neq \mu$, $\lambda, \mu \in \Lambda$ and for $\lambda = \mu$ let $\psi_{\lambda/\mu}$ be the identity mapping on S_λ.

We now associate with each element $s \in S_\lambda$ a symbol $s^{(\lambda)}$ and we next define $A_\Lambda = A_\Lambda(S_\lambda) = \bigcup_{\lambda\in\Lambda} \{l^{(\lambda)} \mid l \in S_\lambda\}$. Let R be the relation on $A_\Lambda(S_\lambda)$ defined by

$$l_1^{(\lambda)}\, R\, l_2^{(\mu)} \Leftrightarrow \psi_{\lambda/\mu}(l_1) = l_2\ .$$

Lemma 1,3. (**i**) *If* $\psi_{\lambda/\mu}(l_1) = l_2$ *and* $\psi_{\mu/\varkappa}(l_2) = l_3$, *then* $\psi_{\lambda/\varkappa}(l_1) = l_3$.
(**ii**) R *is an equivalence relation on* A_Λ.

Proof. The second statement is a corollary of the first. It therefore remains to show that $\psi_{\lambda/\varkappa}(l_1) = l_3$ whenever $\lambda \neq \mu$ and $\mu \neq \varkappa$, since otherwise the assertion is trivial. But, $\varphi_\lambda(l_1) = \varphi_\mu(\varphi_{\lambda/\mu}(l_1)) = \varphi_\mu(l_2)$ and, similarly, $\varphi_\varkappa(l_3) = \varphi_\mu(l_2)$, completing the proof.

The quotient set A_Λ/R will be denoted by $\mathfrak{A}_\Lambda = \mathfrak{A}_\Lambda(S_\lambda)$ and for the equivalence class of s we use the notation $[s]$.

Lemma 1,4. *Let* Φ_λ *be the projection mapping*, $\Phi_\lambda : s_\lambda \mapsto [s_\lambda]$. *Then*

(i) $\bigcup \{\Phi_\lambda(S_\lambda);\ \lambda \in \Lambda\} = \mathfrak{A}_\Lambda$.
(ii) *For every* $\alpha, \beta \in \Lambda$ *and every* $s_0 \in S_\alpha^\circ$

$$\Phi_\beta(\varphi_{\alpha/\beta}(s_0)) = \Phi_\alpha(s_0)\ .$$

Proof of (**ii**): If $[s] = \Phi_\alpha(s_0)$, then $[s_0] = [s_0^{(\alpha)}]$. Now let $s_2 = \varphi_{\alpha/\beta}(s_0)$. Since $s_2 \in S_\beta^\circ$, we have

$$\Phi_\alpha(s_0) = [s_0^{(\alpha)}] = [(\varphi_{\beta/\alpha}(s_2))^{(\alpha)}] = [s_2^{(\beta)}] = \Phi_\beta(s_2) = \Phi_\beta(\varphi_{\alpha/\beta}(s_0))\ .$$

We are now able to show that $\mathfrak{A}_\Lambda$ can be made (in a natural way) into a poset. Actually, one can construct the relation $\leqq$ as follows: If $[m^{(\mu)}], [n^{(\nu)}] \in \mathfrak{A}_\Lambda$ we define $[m^{(\mu)}] \leqq [n^{(\nu)}]$ iff there exist $m_0 \in S_\mu, n_0 \in S_\nu$ such that $m \leqq_\mu m_0,\ n_0 \leqq_\nu n$ and $\psi_{\mu/\nu}(m_0) = n_0$.

Lemma 1,5. $\langle \mathfrak{A}_\Lambda, \leqq \rangle$ *is a poset.*

Proof. If $[m^{(\lambda)}] = [m_1^{(\lambda_1)}], [n^{(\mu)}] = [n_1^{(\mu_1)}]$ and if there are m_0, n_0 such that

$$m \leqq_\lambda m_0, \quad \psi_{\lambda/\mu}(m_0) = n_0 \leqq_\mu n,$$

then, by Lemma 3 (**i**),

$$\psi_{\lambda_1/\mu_1}(\psi_{\mu/\lambda_1}(n_0)) = \psi_{\lambda/\mu_1}(m_0)$$

and, by the definition of the ψ's,

$$\psi_{\lambda_1/\lambda}(m_1) = m\ , \quad \psi_{\mu_1/\mu}(n_1) = n\ .$$

Hence

$$\psi_{\lambda_1/\lambda}(m_1) \leqq \psi_{\mu/\lambda}(n_0)\ , \quad \psi_{\lambda/\mu}(m_0) \leqq \psi_{\mu_1/\mu}(n_1)$$

and

$$m_1 \leqq \psi_{\mu/\lambda_1}(n_0), \quad \psi_{\lambda_1/\mu_1}(\psi_{\mu/\lambda_1}(n_0)) = \psi_{\lambda/\mu_1}(m_0) \leqq n_1\ .$$

This proves that $\leqq$ is well-defined.

To prove transitivity suppose $[m^{(\lambda)}] \leqq [n^{(\mu)}]$ and $[n^{(\mu)}] \leqq [p^{(\tau)}]$. Then there exist m_0, n_0, p_0 such that

$$m \leqq_\lambda m_0, \quad n_0 \leqq_\mu n, \quad n \leqq_\mu n_1, \quad p_0 \leqq_\tau p,$$
$$\psi_{\lambda/\mu}(m_0) = n_0, \quad \psi_{\mu/\tau}(n_1) = p_0.$$

Since $n_0 \leqq n \leqq n_1$, we get $\psi_{\lambda/\tau}(m_0) = \psi_{\mu/\tau}(n_0) \leqq p$.
Thus $[m^{(\lambda)}] \leqq [p^{(\tau)}]$.

We next check the antisymmetry: If $[m^{(\lambda)}] \leqq [n^{(\mu)}] \leqq [m^{(\lambda)}]$, then there are m_0, n_0, n_1, m_1 such that

$$\psi_{\lambda/\mu}(m_0) = n_0, \quad \psi_{\mu/\lambda}(n_1) = m_1$$
$$m \leqq m_0, \quad n_0 \leqq n, \quad n \leqq n_1, \quad m_1 \leqq m.$$

Since $n_0 \leqq n$,

$$m \leqq m_0 = \psi_{\mu/\lambda}(n_0) \leqq \psi_{\mu/\lambda}(n) \leqq \psi_{\mu/\lambda}(n_1) = m_1 \leqq m\ ,$$

hence $[n^{(\mu)}] = [m^{(\lambda)}]$.

Lemma 1,6. *The mappings* Φ_λ *defined in Lemma 1,4 are order isomorphisms satisfying the conditions* (**a**), (**aa**), (**aaa**).

Proof. By Lemma 1,4 it only remains to prove (**aa**). Suppose that

$\Phi_\varkappa(s_1) \leqq \Phi_\lambda(s_2)$, $\varkappa \neq \lambda$, $s_1 \in S_\varkappa$, $s_2 \in S_\lambda$. Then $[s_1^{(\varkappa)}] \leqq [s_2^{(\lambda)}]$ and there exist s_{10}, s_{20} such that

$$s_1 \leqq s_{10}, \quad \varphi_{\varkappa/\lambda}(s_{10}) = s_{20} \leqq s_2.$$

Therefore $[s_1^{(\varkappa)}] \leqq [s_{10}^{(\varkappa)}] = [s_{20}^{(\lambda)}] \leqq [s_2^{(\lambda)}]$ and $\Phi_\varkappa(s_{10}) \in [[s_1^{(\varkappa)}], [s_2]] \cap \Phi_\varkappa(S_\varkappa^\circ)$.

We can summarize the results proved above as follows:

Theorem 1,7. *For any system $\{\mathscr{S}_\lambda\}_{\lambda \in \Lambda}$ of posets there exists an amalgam relative to the given system of the isomorphisms $\varphi_{\lambda_1/\lambda_2}$ and it is uniquelly determined (up to isomorphism).*

For the amalgam we shall use the notation $:\varphi_{\lambda/\mu} : \mathscr{S}_\lambda$, $\lambda \in \Lambda$. Since the amalgam of two posets $\mathscr{S}_1, \mathscr{S}_2$ is determined by a unique order isomorphism $\varphi : \mathscr{S}_1^\circ \to \mathscr{S}_2^\circ$, we write in this case simply $\mathscr{S}_1 : \varphi : \mathscr{S}_2$.

2. Amalgams of lattices

To avoid repeating that a poset is a lattice, we make the assumption that *all symbols, in the remainder of the paper, denoted by $\mathscr{L}$ having possibly subscripts or superscripts will denote lattices.*

Simple examples show that the amalgam $:\varphi_{\lambda/\mu} : \mathscr{L}_\lambda$, $\lambda \in \Lambda$ need not be a lattice even if we suppose $\mathscr{S}_\lambda^\circ = \mathscr{L}_\lambda^\circ$ are sublattices of the lattices $\mathscr{L}_\lambda$. We shall now treat the question under what conditions is such an amalgam a lattice.

We start off with a relatively simple but, at the same time, highly effective and useful result about the basic relations in an amalgam.

Lemma 2,1. **(i)** *If $\mathscr{S}_\lambda^\circ = \mathscr{P}_\lambda^\circ$ are (meet) subsemilattices of the lattices $\mathscr{L}_\lambda$, then $[a^{(\alpha)}] \cap [b^{(\alpha)}] = [(a \cap b)^{(\alpha)}]$ for any $[a^{(\alpha)}], [b^{(\alpha)}] \in \mathfrak{A}_\Lambda(L_\Lambda)$.*

(ii) *If $\mathscr{S}_\lambda^\circ = \mathscr{L}_\lambda^\circ$ are sublattices of the lattices $\mathscr{L}_\lambda$ and if $\mathfrak{A}_\Lambda(\mathscr{L}_\lambda)$ is a lattice, then the lattices $\langle \varphi_\lambda(L_\lambda), \leqq \rangle$ are sublattices of the amalgam.*

(iii) *Let $\mathscr{S}_\lambda^\circ = \mathscr{L}_\lambda^\circ$ and suppose that $[a^{(\alpha)}] \cap [b^{(\beta)}]$ exists. Then there is an element d belonging to L_α or to L_β such that $[d] = [a^{(\alpha)}] \cap [b^{(\beta)}]$.*

Proof. *Ad* **(i)**: Suppose that $[a^{(\alpha)}] \geqq [c^{(\gamma)}]$ and $[b^{(\alpha)}] \geqq [c^{(\gamma)}]$. Without loss of generality we may assume that $\gamma \neq \alpha$. Hence there exist a_0, c_0, b_0, c_1 such that

$$a \geqq a_0, \quad c_0 \geqq c, \quad b \geqq b_0, \quad c_1 \geqq c$$
$$\psi_{\alpha/\gamma}(a_0) = c_0, \quad \psi_{\alpha/\gamma}(b_0) = c_1 .$$

Since $a_0 \cap b_0 \in P_\alpha^\circ$, $c_0 \cap c_1 \in P_\gamma^\circ$, we have

$$\psi_{\alpha/\gamma}(a_0 \cap b_0) = \varphi_{\alpha/\gamma}(a_0 \cap b_0) = \varphi_{\alpha/\gamma}(a_0) \cap \varphi_{\alpha/\gamma}(b_0) = c_0 \cap c_1$$

and $a \cap b \geqq a_0 \cap b_0$, $c_0 \cap c_1 \geqq c$. It follows that $[(a \cap b)^{(\alpha)}] \geqq [c^{(\gamma)}]$.

Ad **(ii)**: This is clear from **(i)**.

Ad **(iii)**: Set $[d^{(\varkappa)}] = [a^{(\alpha)}] \cap [b^{(\beta)}]$. We can suppose that $\alpha \neq \beta$, $\varkappa \neq \alpha$ and $\varkappa \neq \beta$. Since $[a^{(\alpha)}] \geqq [d^{(\varkappa)}]$ and $[b^{(\beta)}] \geqq [d^{(\varkappa)}]$, there are b_0, c_0, a_0, c_1 such that

$$b \geqq b_0, \ \varphi_{\beta/\varkappa}(b_0) = c_0 \geqq d, \ a \geqq a_0 \ \varphi_{\alpha/\varkappa}(a_0) = c_1 \geqq d.$$

A straightforward computation yields $[d^{(\varkappa)}] = [(\varphi_{\varkappa/\alpha}(c_0 \cap c_1))^{(\alpha)}] = [a^{(\alpha)}] \cap [b^{(\beta)}]$.

Throughout the rest of this paper, unless otherwise specified, by an amalgam we shall mean an amalgam where $\mathcal{S}_\lambda^\circ = \mathcal{L}_\lambda^\circ$ are sublattices of the lattices $\mathcal{L}_\lambda$.

We write $(c] = \{z \mid z \leqq c\}$, $[c) = \{v \mid v \geqq c\}$ and similarly $\varphi(c] = \{y \mid \exists x \; x \leqq c, \; \varphi(x) = y\}$ and we use this notation below.

Lemma 2,2. *If $\mathfrak{A}_\Lambda(\mathcal{L}_\lambda)$ is a meet-semilattice, then*

$$\text{(A)} \qquad \begin{cases} \forall \lambda \neq \mu \; \forall c_1 \in L_\lambda \; \forall c_2 \in L_\mu \\ \varphi_{\lambda/\lambda}(c_1] \neq \emptyset \quad \text{or} \quad \varphi_{\mu/\mu}(c_2] \neq \emptyset . \end{cases}$$

Proof. Let $[d^{(\varkappa)}] = [c_1^{(\lambda)}] \cap [c_2^{(\mu)}]$. Then $[c_1^{(\lambda)}] \geqq [d^{(\varkappa)}]$, $[c_2^{(\mu)}] \geqq [d^{(\varkappa)}]$ and there exist $c_{10}, d_{10}, c_{20}, d_{20}$ such that

$$c_1 \geqq c_{10}, \; \psi_{\lambda/\varkappa}(c_{10}) = d_{10} \geqq d, \; c_2 \geqq c_{20}, \; \psi_{\mu/\varkappa}(c_{20}) = d_{20} \geqq d.$$

By hypothesis, we have $\psi_{\lambda/\mu} = \varphi_{\lambda/\mu}$ or $\psi_{\mu/\varkappa} = \varphi_{\mu/\varkappa}$. Say $\psi_{\lambda/\mu} = \varphi_{\lambda/\mu}$. Then $\varphi_{\lambda/\varkappa}(c_{10}) = d_{10}$ implies $\varphi_{\lambda/\lambda}(c_{10}) = \varphi_{\varkappa/\lambda}(d_{10})$ and we therefore conclude that $\varphi_{\lambda/\lambda}(c_1] \neq \emptyset$.

Lemma 2,3. *If $\mathfrak{A}_\Lambda(\mathcal{L}_\lambda)$ is a meet-semilattice, then*

$$\text{(B)} \quad \begin{cases} \forall \lambda \neq \mu \; (\varphi_{\lambda/\lambda}(c_1] \neq \emptyset \text{ and } \varphi_{\mu/\mu}(c_2] \neq \emptyset) \Rightarrow \\ \Rightarrow [(\forall x \in \varphi_{\mu/\lambda}(c_2] \; c_1 \cap x \in \varphi_{\mu/\lambda}(c_2]) \text{ or } (\forall y \in \varphi_{\lambda/\mu}(c_1] \; c_2 \cap y \in \varphi_{\lambda/\mu}(c_1])] . \end{cases}$$

Proof. Suppose c_1 and c_2 are any two elements such that

$$\begin{gathered} \varphi_{\lambda/\lambda}(c_1] \neq \emptyset, \quad \varphi_{\mu/\mu}(c_2] \neq \emptyset \quad \exists x \in \varphi_{\mu/\lambda}(c_2] \\ c_1 \cap x \notin \varphi_{\mu/\lambda}(c_2] \quad \exists y \in \varphi_{\lambda/\mu}(c_1] \quad c_2 \cap y \notin \varphi_{\lambda/\mu}(c_1] . \end{gathered}$$

Since $[c_1^{(\lambda)}] \cap [c_2^{(\mu)}] = [d^{(\varkappa)}]$ exists, there are $c_{10}, d_{10}, c_{20}, d_{20}$ for which

$$\begin{gathered} c_1 \geqq c_{10} \quad \psi_{\lambda/\varkappa}(c_{10}) = d_{10} \geqq d \\ c_2 \geqq c_{20} \quad \psi_{\mu/\varkappa}(c_{20}) = d_{20} \geqq d. \end{gathered}$$

We shall show that we can consider only the case where $\varkappa = \lambda$ or $\varkappa = \mu$. For suppose $\varkappa \neq \lambda$ and $\varkappa \neq \mu$. Then $[(d_{10} \cap d_{20})^{(\varkappa)}] = [d^{(\varkappa)}]$ and therefore $d_{10} \cap d_{20} = d$. Now the assumption $\lambda \neq \varkappa \neq \mu$ implies that $d_{10} \in L_\varkappa^\circ$, $d_{20} \in L_\varkappa^\circ$ and so $d \in L_\varkappa^\circ$. But this implies $[d_1^{(\lambda)}] = [c_1^{(\lambda)}] \cap [c_2^{(\mu)}]$ where $d_1 = \varphi_{\varkappa/\lambda}(d)$.

In the case $\varkappa = \lambda \neq \mu$ we get easily

$$[(c_1 \cap x)^{(\lambda)}] \leqq [c_1^{(\lambda)}] , \quad [(c_1 \cap x)^{(\lambda)}] \leqq [x^{(\lambda)}] \leqq [c_2^{(\mu)}] ;$$

hence

$$[(c_1 \cap x)^{(\lambda)}] \leqq [c_1^{(\lambda)}] \cap [c_2^{(\mu)}] = [d^{(\lambda)}] .$$

On the other hand, since $[d^{(\lambda)}] \leqq [c_2^{(\mu)}]$, there exist d_0, c_{30} with

$$d \leqq d_0 , \quad \varphi_{\lambda/\mu}(d_0) = c_{30} \leqq c_2 .$$

From the fact that $\mathcal{L}_\lambda^\circ$ is a sublattice of $\mathcal{L}_\lambda$, we conclude $d_3 = d_0 \cup x \in L_\lambda^\circ$. Moreover, $[(c_2 \cap y)^{(\mu)}] \leqq [y^{(\mu)}] \leqq [c_1^{(\lambda)}]$ and therefore

$$[(c_2 \cap y)^{(\mu)}] \leqq [c_1^{(\lambda)}] \cap [c_2^{(\mu)}] = [d^{(\lambda)}] .$$

Thus there are p_0, q_0 such that

$$c_2 \cap y \leqq p_0\,, \quad \psi_{\mu/\lambda}(p_0) = q_0 \leqq d\,.$$

Let $p_1 = p_0 \cap y$, $q_1 = \varphi_{\mu/\lambda}(y) \cap q_0$. Because of

$$[c_2^{(\mu)}] \geqq [c_3^{(\mu)}] = [d_3^{(\lambda)}] \geqq [d^{(\lambda)}] \geqq [q_1^{(\lambda)}] = [p_1^{(\mu)}]\,,$$

we have $p_1 = y \cap c_2$ and finally

$$y \cap c_2 = \varphi_{\lambda/\mu}(q_1) \in \varphi_{\lambda/\mu}(c_1]\,.$$

This contradiction completes the proof.

The conditions (**A**) and (**B**) are not sufficient that $\mathfrak{A}_\Lambda(\mathscr{L}_\lambda)$ be a meet-semilattice. For, let $[0,1] : \varphi : [0, 1]$ be the amalgam of two copies $\mathscr{L}_1$, $\mathscr{L}_2$ of $[0, 1] \subset \mathbf{R}$ where $L_1^\circ = L_2^\circ = \{x \mid 0 \leqq x < 1, x \in \mathbf{Q}\}$ and φ is the identity mapping of L_1°. Then the conditions (**A**) and (**B**) hold, and yet $[1^{(1)}] \cap [1^{(2)}]$ does not exist.

Since (**A**) and (**B**) represent no guarantee for the existence of meets, we shall still consider an additional condition. (To denote the fact that $\sup_P M \in M$, we write $\sup_P M = \max M$, and in this case we say that *the maximum of the set M exists.*)

Lemma 2,4. *If* $\mathfrak{A}_\Lambda(\mathscr{L}_\lambda)$ *is a meet-semilattice, then*

$$(\mathbf{C}) \quad \begin{cases} \forall\, \lambda \neq \mu \quad \forall c_1 \in L_\lambda \quad \forall c_2 \in L_\mu \\ [(y \in \varphi_{\lambda/\mu}(c_1] \Rightarrow c_2 \cap y \in \varphi_{\lambda/\mu}(c_1]) \Rightarrow \\ \Rightarrow (\textit{the maximum of the set } \{c_1 \cap x \mid x \in \varphi_{\mu/\lambda}(c_2]\} \\ \textit{exists and } \varphi_{\mu/\lambda}(c_2] \neq \emptyset)]\,. \end{cases}$$

Proof. Let $[d^{(\varkappa)}] = [c_1^{(\lambda)}] \cap [c_2^{(\mu)}]$. By Lemma 2,1 (**iii**), we may assume that $\varkappa = \lambda$ or $\varkappa = \lambda$.

Suppose first that $\varphi_{\lambda/\mu}(c_1] = \emptyset$. Then $[d^{(\varkappa)}] = [d_2^{(\mu)}]$ implies the existence of c_{10}, d_{20} such that

$$c_1 \geqq c_{10}\,, \quad \varphi_{\lambda/\mu}(c_{10}) = d_{20} \geqq d_2$$

and $\varphi_{\lambda/\mu}(c_1] \neq \emptyset$, a contradiction. Hence $[d^{(\varkappa)}] = [d_1^{(\lambda)}]$ and since $\varphi_{\lambda/\mu}(c_1] = \emptyset$, also $\varphi_{\lambda/\lambda}(c_1] = \emptyset$. By Lemma 2,2, $\varphi_{\mu/\mu}(c_2] \neq \emptyset$ and therefore $\varphi_{\mu/\lambda}(c_2] \neq \emptyset$. Note that the considerations we are going to use in the following depend only on the assumption $\varphi_{\mu/\lambda}(c_2] \neq \emptyset$. We shall refer to this fact in the end of the demonstration.

If $x \in \varphi_{\mu/\lambda}(c_2]$, then $[(c_1 \cap x)^{(\lambda)}] \leqq [d_1^{(\lambda)}]$. Consequently $[d_1^{(\lambda)}] \leqq [c_2^{(\mu)}]$. Since $[d_1^{(\lambda)}] \leqq [c_2^{(\mu)}]$, there are e, f such that $[d_1^{(\lambda)}] \leqq [e^{(\lambda)}] = [f^{(\mu)}] \leqq [c_2^{(\mu)}]$, and it is easy to see that $[d_1^{(\lambda)}] = [(c_1 \cap e)^{(\lambda)}]$. This shows, however, that $d_1 = c_1 \cap e$, and the condition (**C**) is in this case valid.

Next assume that $\varphi_{\lambda/\mu}(c_1] \neq \emptyset$ and that the implication $y \in \varphi_{\lambda/\mu}(c_1] \Rightarrow$ $\Rightarrow c_2 \cap y \in \varphi_{\lambda/\mu}(c_1]$ is true. Since $c_2 \cap y \in \varphi_{\lambda/\mu}(c_1]$, there is an element $z \leqq c_1$ such that $\varphi_{\lambda/\mu}(z) = c_2 \cap y$. It follows that $\varphi_{\mu/\lambda}(c_2] \neq \emptyset$.

It remains only to show that in the case $\varphi_{\lambda/\mu}(c_1] \neq \emptyset$ there exists the maximum of the set $\{c_1 \cap x \mid x \in \varphi_{\mu/\lambda}(c_2]\}$. As above, we may suppose that $\varkappa = \lambda$ or $\varkappa = \mu$.

Then necessarily $\varkappa = \lambda$. For if $\varkappa = \mu$, then there exist e, f such that $c_1 \geqq e$, $\varphi_{\lambda/\mu}(e) = f \geqq d$ which implies $d = f \cap c_2$. Since $f \in \varphi_{\lambda/\mu}(c_1]$, we have $d = f \cap c_2 \in$ $\in \varphi_{\lambda/\mu}(c_1]$ and $[d^{(\mu)}] = [(\varphi_{\mu/\lambda}(f \cap c_2))^{(\lambda)}]$.

Now, if $\varkappa = \lambda$ and $\varphi_{\lambda/\mu}(c_1] \neq \emptyset$, then a repetition of the argument used above clearly leads to the desired result. Q.E.D.

Corollary. *If the maximum mentioned in the lemma* 2,4 *exists, then*

$$[(\max \{c_1 \cap x \mid x \in \varphi_{\mu/\lambda}(c_2]\})^{(\lambda)}] = [c_1^{(\lambda)}] \cap [c_2^{(\mu)}]\,.$$

Let $\Gamma_1{\downarrow}$ denote the set $\varphi_{\lambda/\mu}(c_1]$ and let, similarly, $\Gamma_2{\downarrow}$ be the set $\varphi_{\mu/\lambda}(c_2]$. The set $\{c_i \cap y \mid y \in \Gamma_j{\downarrow}\}$ will be denoted by $c_i \wedge \Gamma_j{\downarrow}$.

Theorem 2,5. *An amalgam* $\mathfrak{A}_\Lambda(\mathscr{L}_\lambda)$ *is a lattice iff it satisfies the following condition* (**D**) *and its dual*:

$$(\mathbf{D}) \quad \begin{cases} \forall\, \lambda \neq \mu \quad \forall c_1 \in L_\lambda \quad \forall\, c_2 \in L_\mu \\ [(\Gamma_2{\downarrow} \neq \emptyset\,,\ \textit{the maximum of the set } c_1 \wedge \Gamma_2{\downarrow} \\ \textit{exists and } c_2 \wedge \Gamma_1{\downarrow} \subset \mathring{L}_\mu) \textit{ or} \\ (\Gamma_1{\downarrow} \neq \emptyset\,,\ \textit{the maximum of the set } c_2 \wedge \Gamma_1{\downarrow} \\ \textit{exists and } c_1 \wedge \Gamma_2{\downarrow} \subset \mathring{L}_\lambda)]. \end{cases}$$

Proof. 1. Suppose first that $\varphi_{\lambda/\mu}(c_1] \neq \emptyset$ and $\varphi_{\mu/\mu}(c_2] \neq \emptyset$. By (**B**), either $\{c_2 \cap y \mid y \in \varphi_{\lambda/\mu}(c_1]\} \subset \varphi_{\lambda/\mu}(c_1]$ or $\{c_1 \cap x \mid x \in \varphi_{\mu/\lambda}(c_2]\} \subset \varphi_{\mu/\lambda}(c_2]$. In accordance with the notation defined above, this means that either $c_2 \wedge \Gamma_1{\downarrow} \subset \mathring{L}_\mu$ or $c_1 \wedge \Gamma_2{\downarrow} \subset \mathring{L}_\lambda$. In the first case $[c_1^{(\lambda)}] \cap [c_2^{(\mu)}] = [(\max c_1 \wedge \Gamma_2{\downarrow})^{(\lambda)}]$, by (**C**) and the same argument applies to the second case.

Suppose further that $\varphi_{\lambda/\mu}(c_1] = \emptyset$. By (**C**), it is clear that $\Gamma_2{\downarrow} \neq \emptyset$ and that the corresponding maximum exists. Since $\Gamma_1{\downarrow} = \emptyset$, it follows trivially that $c_2 \wedge \Gamma_1{\downarrow} \subset \mathring{L}_\mu$.

Finally, if $\varphi_{\mu/\lambda}(c_2] = \emptyset$, we may repeat the same argument by replacing λ by μ. Therefore, by (**A**), the necessity of (**D**) is proved.

2. For the converse, suppose that $\varphi_{\mu/\lambda}(c_2] \neq \emptyset$, $c_2 \wedge \Gamma_1{\downarrow} \subset \mathring{L}_\mu$ and that the maximum m of the set $\{c_1 \cap x \mid x \in \varphi_{\mu/\lambda}(c_2]\}$ exists. We shall prove that then $[m^{(\lambda)}]$ has the properties of the greatest lower bound of $\{[c_1^{(\lambda)}], [c_2^{(\mu)}]\}$.

Indeed, if $m = c_1 \cap c_{20}$ where $c_{20} \in \varphi_{\mu/\lambda}(c_2]$, then $[(c_1 \cap c_{20})^{(\lambda)}] \leqq$ $\leqq [c_1{}^{(\lambda)}], [c_2^{(\mu)}]$. Suppose we have $[d^{(\varkappa)}] \leqq [c_1^{(\lambda)}]$ and $[d^{(\varkappa)}] \leqq [c_2^{(\mu)}]$. If $\lambda \neq \varkappa$, $\mu \neq \varkappa$, then there exist d_0, c_{10}, d_1, c_{22} such that

$$[(d_0 \cap d_1)^{(\varkappa)}] \leqq [d_0^{(\varkappa)}] = [c_{10}^{(\lambda)}] \leqq [c_1^{(\lambda)}]$$
$$[(d_0 \cap d_1)^{(\varkappa)}] \leqq [d_1^{(\varkappa)}] = [c_{22}^{(\mu)}] \leqq [c_2^{(\mu)}]\,.$$

On the other hand, $d_0 \cap d_1 \in \mathring{L}_\varkappa$, and so $[(\varphi_{\varkappa/\lambda}(d_0 \cap d_1))^{(\lambda)}] = [(d_0 \cap d_1)^{(\varkappa)}]$.

We now aim to prove that $[d^{(\varkappa)}] \leqq [(c_1 \cap c_{20})^{(\lambda)}]$. By the result just proved, we may assume, without loss of generality, that either $\varkappa = \lambda$ or $\varkappa = \mu$.

Case $\varkappa = \mu$. Since $[d^{(\mu)}] \leqq [c_1^{(\lambda)}]$, there exist c_{11}, c_{21} such that

$$c_1 \geqq c_{11},\ \varphi_{\lambda/\mu}(c_{11}) = c_{21} \geqq d\,.$$

It is clear that $[(c_{21} \cap c_2)^{(\mu)}] \gneqq [d^{(\mu)}]$ and so $d_1 = c_2 \cap c_{21} \geqq d$. By assumption $d_1 \in L_\mu^\bullet$. Let $e_1 = \varphi_{\mu/\lambda}(d_1)$. Then

$$e_1 = \varphi_{\mu/\lambda}(d_1) \leqq \varphi_{\mu/\lambda}(c_{21}) = c_{11} \leqq c_1 .$$

By definition of $c_1 \cap c_{20}$, we get $c_1 \cap c_{20} \geqq e_1$ and therefore

$$[(c_1 \cap c_{20})^{(\lambda)}] \geqq [e_1^{(\lambda)}] = [d_1^{(\mu)}] \geqq [d^{(\mu)}] .$$

Case $\varkappa = \lambda$. Since $[c_2^{(\mu)}] \geqq [d_1^{(\lambda)}]$, there exist d_{11}, c_{22} such that

$$c_2 \geqq c_{22} , \quad \varphi_{\mu/\lambda}(c_{22}) = d_{11} \geqq d .$$

Thus, we see that $d \leqq c_1 \cap d_{11} \leqq c_1 \cap c_{20}$. Hence $[d^{(\lambda)}] \leqq [(c_1 \cap d_{11})^{(\lambda)}] \leqq$ $\leqq [(c_1 \cap c_{20})^{(\lambda)}]$. This completes the proof of the theorem.

Corollary. *If the first possibility formulated in the condition* (**D**) *occurs, then*

$$[c_1^{(\lambda)}] \cap [c_2^{(\mu)}] = [(\max c_1 \wedge \Gamma_2{\downarrow})^{(\lambda)}] ;$$

in the case the second possibility occurs,

$$[c_1^{(\lambda)}] \cap [c_2^{(\mu)}] = [(\max c_2 \wedge \Gamma_1{\downarrow})^{(\mu)}] .$$

In what follows we shall deal with the cofinality and with the dual notion: A subset M of a poset $\mathscr{P}$ is said to be *dually cofinal* in $\mathscr{P}$ if for every $p \in P$ there exists an $m \in M$ such that $m \leqq p$.

Lemma 2,6. *The condition* (**A**) *for the amalgam* $\mathscr{L}_1 : \varphi : \mathscr{L}_2$ *is equivalent to the condition*

$$(\mathbf{A}^\star) \quad \begin{cases} (L_1^\circ \text{ is dually cofinal in } \mathscr{L}_1) & \text{or} \\ (L_2^\circ \text{ is dually cofinal in } \mathscr{L}_2) . \end{cases}$$

Proof. We observe first that if $c_1 \in L_1$, then $\varphi_{1/1}(c_1] = (c_1] \cap L_1^\circ$; similarly, $c_2 \in L_2$ implies that $\varphi_{2/2}(c_2] = (c_2] \cap L_2^\circ$. Let us now suppose that (**A**) is valid and that

$$\exists\, l_1 \in L_1 \quad \forall\, l_1^\circ \in L_1^\circ \quad l_1^\circ \text{ non } \leqq l_1$$
$$\exists\, l_2 \in L_2 \quad \forall\, l_2^\circ \in L_2^\circ \quad l_2^\circ \text{ non } \leqq l_2 .$$

Then either $(l_1] \cap L_1^\circ \neq \emptyset$ or $(l_2] \cap L_2^\circ \neq \emptyset$. If $(l_i] \cap L_i^\circ \neq \emptyset$, then for any $x \in (l_i] \cap L_i^\circ$ we have $x \in L_i^\circ$ and $x \leqq l_i$, a contradiction.

Next, assume (**A***) is true. If L_1° is dually cofinal in $\mathscr{L}_1$, then for any $c_1 \in L_1$, $c_2 \in L_2$ there exists an $l_1^\circ \in L_1^\circ$ with $l_1^\circ \leqq c$. Consequently, $l_1^\circ \in (c_1] \cap L_1^\circ$ and we conclude that (**A**) is valid.

Corollary. *If the amalgam* $\mathfrak{A}_\Lambda(\mathscr{L}_\lambda)$ *is a join-semilattice, then for all* $\lambda \in \Lambda$ (*possibly except one*) *the* L_λ° *is cofinal in* $\mathscr{L}_\lambda$.

Proof of Corollary follows from Lemma 2,6 and from the obvious fact that for every $\lambda \neq \mu \in \Lambda$ the amalgam $\mathfrak{A}_\Lambda(\mathscr{L}_\lambda)$ induces the amalgam $\mathscr{L}_\lambda : \varphi_{\lambda/\mu} : \mathscr{L}_\mu$ which is also a join-semilattice.

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