Acta Universitatis Carolinae. Mathematica et Physica

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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 14 (1973), No. 2, 17--21

Persistent URL: http://dml.cz/dmlcz/142309

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1973 ACTA UNIVERSITATIS CAROLINAE - MATHEMATICA ET PHYSICA VOL. 14, NO. 2 Pag. 17-21

Remark About a Construction of Some Tournaments With Points of Certain Projective Planes

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Received 25 January 1973

This paper present a study of the collineations on finite cyclic projective planes on $N = n^2 + n + 1$ points, where n is power of prime, these collineations being called collineations of period N [1]. It is shown, that for any finite projective plane there exist collineations, which have analogical properties as collineations of period N, so that we can say that these are collineations of a period smaller than N. All of these collineations of a given finite cyclic projective plane form a group, which is transitive on points and on lines of this plane or on subsets with points or lines of this plane; these collineations induce cycles of points of this plane. It is possible to construct a tournament with points of finite cyclic projective plane or tournament on vertices of regular N-polygon. Hence we can say when regular N-polygon breaks up or not to decomposition, i.e. when we can the circumference of N-polygon draft by one closed way.

Let φ be collineation of finite projective plane π over the Galois field $GF(n)$, where n is power of prime, that

$$\varphi(P_0) = P_1, \varphi^2(P_0) = \varphi(\varphi(P_0)) = \varphi(P_1) = P_2, \dots$$
$$\varphi^{N-1}(P_0) = P_{N-1}, \varphi^N(P_0) = P_0 ,$$

where

$$P_i \neq P_j \text{ for } i \neq j (i, j = 0, 1, \dots, N-1)$$
$$N = n^2 + n + 1 , \quad P_i \in \pi .$$

Then we say φ is a collineation of plane π of period N [1]. The collineation φ induces a cycle of the points of the plane π

$$P_0, P_1, \dots, P_{N-1}$$

and φ is the generator of finite cyclic group

$$\Phi = \{\varphi, \varphi^2, \dots, \varphi^{N-1}, \varphi^N = \varphi^0\} ,$$

where

$$\varphi^j(P_i) = P_t(t \equiv i + j \ (mod\ N)) .$$

The group Φ is transitive on the points and on the lines of the plane π and we say that plane π is cyclic with respect to the collineation φ [2].

In the following text we designate all of the points of the plane π by its subscripts. It is

$$\varphi^j(i) = t(t \equiv i + j \ (mod\ N)) \,. \tag{1}$$

Theorem 1: The collineation φ^d of the plane π, where $d \not\equiv 0\,(mod\ N)$, is of period N if, and only if, d is relatively prime to N.

Proof. It follows by the theorem: Let φ be a generator of finite cyclic group of order N. His generators are powers of φ if, and only if, its exponents are relatively prime to N [3]. Hereby it follows from (1)

$$\varphi^d(0) = d,\ \varphi^{2d}(0) = 2d,\ \dots,\ \varphi^{(N-1)d}(0) = (N-1)\,d,$$
$$\varphi^{Nd}(0) = 0,$$

for all id $(mod\ N)$ $(i = 1, 2, \dots, N)$.

Corollary 1: The collineation φ^d of the theorem 1 induces a cycle on the points of the plane

$$0,\ d,\ 2d,\ \dots,\ (N-1)\,d \ \ (mod\ N) \tag{2}$$

and is

$$id \not\equiv jd \ \text{for}\ i \not\equiv j \ \ (i, j = 0, 1, \dots, N-1)\,.$$

Theorem 2: The collineation φ^d of the plane π, where $d \not\equiv 0{,}1\ (mod\ N)$, is generator of a subgroup $\bar{\Phi}$ of the group Φ if, and only if, d is not relatively prime to N.

Proof. It follows by the theorem: Order of every subgroup of finite group Φ is divisor of order of the group Φ [3]. It is

$$\bar{\Phi} = \{\varphi^d, \varphi^{2d}, \dots, \varphi^{(k-1)d}, \varphi^{kd} = \varphi^0\}\,,$$

where positive integer k is the smallest of all positive integers satisfying the conguruence

$$kd \equiv 0 \ \ (mod\ N)\,.$$

Otherwise holds the theorem: Every subgroup of the cyclic group is cyclic [3].

Corollary 2: The collineation φ^d of the theorem 2 induces disjoint cycles on points of the plane

$$a,\ a+d,\ \dots,\ a+(k-1)\,d \ \ (mod\ N) \tag{3}$$

where for appropriate integer a is

$$0 \leqq a < d\,.$$

Remark 1: The number of disjoint cycles (3) is N/k, where N/k is index of the subgroup $\bar{\Phi}$ of the group Φ.

Remark 2: Subgroup $\bar{\Phi}$ of the group Φ is transitive on every set $\{a, a+d, \dots, a+(k-1)\,d\}$ $(mod\ N)$.

Remark 3: As $n^2 + n + 1 = n(n+1) + 1$ is always odd, $d = 2^a$, a being positive integer, leads always to collineations of the plane π of period N. $d = 2\varkappa$,

$\varkappa = 3, 5, 7, \ldots$ does not lead to collineations of the plane π of period N in every case, when $\varkappa$ is divisor of N.

Remark 4: As

$$- d \equiv N - d \ (mod\ N) \ (d = 1, \ldots, N)$$

is $\varphi^{-d} = \varphi^{N-d}$ and the pair of φ^d, φ^{N-d} is a pair of mutually inverse collineations of the plane π. The collineation φ^0 of the plane π is by itself inverse and induce no of cycles (2), (3).

From foregoing considerations it follows:
On points of the plane π we can construct a tournament [4] so, that some of our cycles [4] are given by cycles (2), (3). Hereby, when we take a cycle which induces the collineation φ^d of the plane π we cannot take a cycle which is induced by the collineation φ^{N-d} of the plane π. As $|\Phi| = |\pi|$ and from (1) it follows that for every two points of the plane π exist all lines [4] and the tournament is constructed.

When the collineation φ^d of the plane π induces a cycle (2) we have two possibilities for orientation of cycle [4] of the tournament (in other words we take either collineation φ^d or collineation φ^{N-d}). When the collineation φ^d of the plane π induces cycles (3) we can change every cycle of length k of the tournament with cycle of converse orientation. By this way we can construct with points of the plane π exactly

$$2^{\binom{n+1}{2} + \frac{N}{k_1} + \frac{N}{k_2} + \ldots + \frac{N}{k_r} - r}, \quad r = N - e(N)$$

tournaments, where $e(N)$ is function of Euler. With regard to remark 3 every of this tournaments has a complete cycle [4], namely, every cycle of tournament given by cycle (2).

Theorem 3: The tournament constructed by described way with points of the plane π is strong.

Proof. [4]: The directed graph is called strong, if every pair of points are mutually reachable. The theorem follows from: A tournament is strong if, and only if, it has a complete cycle [4]. Let us note that if a tournament is strong, then it contains a cycle of each length $l = 3, 4, \ldots, N$ [4].

Remark 5: The tournament of theorem 3 contains cycles of each length $l = 3, 4, \ldots, N$.

From foregoing consideration it follows:
We can constructed a tournament on the vertices of the regular and convex N-polygon. Its vertices we denote by numbers

$$0, 1, \ldots, N - 1$$

so, that any of the vertex of the regular convex N-polygon we denote by 0 and other of vertices we denote from one to another successively by trace of the circumference of this N-polygon. It is indifferent where we start and in which direction we go on. Some of the cycles of the tournament are given by cycles (2), (3). Hereby, when we take a cycle induced by the collineation φ^d of the plane π, we cannot take a cycle which is induced by the collineation φ^{N-d} of the plane π. The tournament on the vertices of the regular and convex N-polygon is again constructed.

Theorem 4: The tournament constructed by this way with the vertices of the regular and convex N-polygon can be drafted by one stroke.

Proof: Now we start from vertex 0 by following the trace of complete cycle corresponding either with $d = 1$ or $d = N - 1$, i.e. the circumference of N-polygon. When we come to any vertex which is passed by any cycle of length smaller than N, and that we have not followed yet, we must follow its trace and than continue on the first cycle. After a finite number of steps, we reach the vertex 0. So we go on than on the other traces of complete cycles (of length N).

Theorem 5: The tournament from the theorem 4 is continuous isograph ("zusammenhängender gleichwertig gerichteten Graph" [5]).

Proof: Its follows from the theorem: Necessary and sufficient condition for the construction of the oriented graph by one closed way is that the graph is continuous isograph [5].

Remark 6: In the tournament from the theorem 4 for every point outdegree is like indegree [4] and it exists a walk [4] for every of two points of the tournament.

Remark 7: The tournament from the theorem 3 is also continuous isograph.

In constructing the tournament with vertices of the regular and convex N-polygon we can observe that the cycles of this tournament, which correspond to cycles of points of the plane π (these cycles being induced by collineations φ^d of the plane π) follow the trace of circumference of regular N-polygons, which are broken up exactly when d is not relatively prime to N and in this case into N/k regular k-polygons. Then is:

Theorem 6: When in tracing the circumference of the regular convex N-polygon we denote the vertices in succession from one to another by numbers

$$0, 1, \ldots, N-1,$$

then the regular N-polygon with vertices

$$0, d, 2d, \ldots$$

do not break up exactly when either $d = 1$ or $d \neq 1$ is relatively prime to N. In another case we get always N/k of regular k-polygons.

The sets

$$m_r = \{a_0 + r, a_1 + r, \ldots, a_n + r\} \quad (mod\ N),$$

where

$$\{a_0, a_1, \ldots, a_n\}$$

is the perfect difference set modulo N, are, with [2], the lines of the plane π and

$$\varphi^j(m_i) = m_t \ (t \equiv i + j\ (mod\ N)).$$

Each line m_r of the plane π in above mentioned tournament on vertices of regular convex N-polygon is represented by the vertices of $(n + 1)$-polygon M_r, whose all vertices are the vertices of named N-polygon, but any of its two legs is not of the same length. To the collineation φ^r of the plane π correspond in this tournament the rotations of the regular convex N-polygon about its center. When we know one of this $(n + 1)$-polygon M_r, we know all $(n + 1)$-polygons of a number $n^2 + n + 1$, which represent all lines of the plane π and we obtain them by all rotations of the N-polygon about its center, these rotations being in correspondence to all collineations φ^r of the plane π. These sets m_r are all perfect difference sets modulo N, and they only by themselves form a class of equivalent perfect difference sets in sense of Singer [1]. The set

$$m'_r = \{-(a_0 + r), -(a_1 + r), \ldots, -(a_n + r)\} \ (mod\ N)$$

is in the tournament with vertices of regular convex N-polygon represented by vertices of $(n + 1)$-polygon M'_r, which is symmetrical with respect to the axis given by point 0 and center of regular convex N-polygon to $(n + 1)$-polygon M_r. As we do not get M'_r from M_r by any rotations of the regular convex N-polygon about its center and

$$\{-a_0, -a_1, \ldots, -a_n\} \ (mod\ N)$$

is also a perfect difference set modulo N, we obtain from here another class of perfect difference sets equivalent among themselves. Hence, it exists an even number of classes of difference sets equivalent among themselves [1].

With [1] every perfect difference set modulo N contains exactly one pair of successing integers modulo N. Hence we can enlarge remark 2:

Remark 8: Subgroup $\bar{\Phi}$ of the group Φ is transitiv on every set of lines of the plane π

$$\begin{array}{ll} \{a, a + 1, \ldots\}, & \\ \{a + d, a + d + 1, \ldots\}, & \\ \ldots\ldots\ldots\ldots\ldots\ldots & (mod\ N) \\ \{a + (k - 1)\, d,\ a + (k - 1)\, d + 1, \ldots\}. & \end{array}$$

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