Acta Universitatis Carolinae. Mathematica et Physica

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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 13 (1972), No. 2, 43--52

Persistent URL: http://dml.cz/dmlcz/142279

Algebraicity of Endomorphisms of Some Relational Structures

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Received 23 November 1972

1. Introduction and preliminaries

By a transformation monoid on a set A we mean any set of transformations of A which is closed under composition and contains the identical transformation. A transformation monoid H on A is called algebraic if H is just the set of endomorphisms of some universal algebra (with arbitrary, not necessarily finitary operations); it is called finitary algebraic if H is the set of endomorphisms of some finitary algebra.

The problem of characterizing algebraic and finitary algebraic transformation monoids, stated in [5], is solved only in some special cases: see e.g. [1] for the case of the semigroup H being cyclic; [9] for H containing all permutations; [4], [6] and [5] for H consisting of permutations only; [3] for H consisting of injective and constant transformations only.

On the other hand, it follows from [7] that every transformation monoid H on A is the set of endomorphisms of a relational structure (with infinitary relations). H is the set of endomorphisms of a finitary relational structure iff H contains any transformation whose every finite restriction can be extended to a transformation belonging to H; this follows from [2], but an easy direct proof could be given.

This shows that the original problem of characterizing algebraic and finitary algebraic transformation monoids is closely related to the following problem: given a relational structure, decide whether the transformation monoid of its endomorphisms is algebraic (finitary algebraic, resp.). The two main results of the present paper, Theorems 2 and 6, are concerned with two special cases of this problem. In Theorem 6 we describe all unary relational structures whose set of endomorphisms is algebraic. Theorem 2 says that if H is an algebraic transformation monoid and any endomorphism of a given quasiordered set (which is not linearly ordered) belongs to H, then H contains all transformations. This was announced by M. Sekanina in [8] in the special case of H being finitary algebraic (and the quasi-ordered set being ordered). Sekanina's proof uses some deep results of [10] and, as it seems, can not be generalized to the infinitary case. The present paper was inspired by [8]. I am indebted to A. Pultr for valuable advice.

We identify an ordinal number j with the set of all ordinal numbers smaller than j. Let a set A and an ordinal number j be given. A^j is the set of all mappings of j into A. (We identify A^1 with A and A^2 with $A \times A$.) By a j-ary operation (or operation of arity j) on A we mean any mapping of A^j into A. By a j-ary relation on A we mean any subset of A^j.

By an algebra we mean an ordered pair $\langle A, F \rangle$ where A is a set and F is a set of operations on A. If every $f \in F$ is finitary (i.e. of finite arity), then $\langle A, F \rangle$ is called finitary. If every $f \in F$ is unary, then $\langle A, F \rangle$ is called unary. If f is an operation on A, then we write $\langle A, f \rangle$ instead of $\langle A, \{f\} \rangle$. By a relational structure we mean an ordered pair $\langle A, R \rangle$ where A is a set and R is a set of relations on A. We speak of finitary and unary relational structures and write $\langle A, r \rangle$ instead of $\langle A, \{r\} \rangle$.

Let an algebra $\langle A, F \rangle$ be given. A transformation h of A is called endomorphism of $\langle A, F \rangle$ if $h(f(x)) = f(h \circ x)$ for all $f \in F$ and $x \in A^j$ (j being the arity of f). The set of endomorphisms of $\langle A, F \rangle$ is denoted by $End(A, F)$. Let $\langle A, R \rangle$ be a relational structure. A transformation h of A is called endomorphism of $\langle A, R \rangle$ if $h \circ x \in r$ for all $r \in R$ and $x \in r$. The set of endomorphisms of $\langle A, R \rangle$ is denoted by $End(A, R)$.

Let $\langle A, F \rangle$ be an algebra. A set $X \subseteq A$ is called closed in $\langle A, F \rangle$ if $f(x) \in X$ for all $f \in F$ and $x \in X^j$ (j being the arity of f). An algebra $\langle A, F \rangle$ is called quasitrivial if every subset of A is closed in $\langle A, F \rangle$. An algebra $\langle A, F \rangle$ is called trivial if every $f \in F$ is trivial; a j-ary operation f on A is called trivial if there exists an $i \in j$ such that $f(x) = x(i)$ for all $x \in A^j$. If $\langle A, F \rangle$ is trivial, then $End(A, F) = A^A$, i.e. every transformation of A is an endomorphism of $\langle A, F \rangle$.

Theorem 1. Let an algebra $\langle A, F \rangle$ and two different elements a, b of A be given. If every mapping of A into $\{a, b\}$ is an endomorphism of $\langle A, F \rangle$, then $End(A, F) = A^A$.

Proof. Suppose that some transformation h of A is not an endomorphism. There exists an operation $f \in F$ of some arity j and an $x \in A^j$ such that $h(f(x)) \neq f(h \circ x)$. There exists a mapping k of A onto $\{a, b\}$ such that $k(h(f(x))) \neq k(f(h \circ x))$. As both $k \circ h$ and k are endomorphisms, we have $k(h(f(x))) = f(k \circ h \circ x) = k(f(h \circ x))$, a contradiction.

2. Quasiordered sets

Let r be a binary relation on A. If r is reflexive and transitive, then the relational structure $\langle A, r \rangle$ is called quasiordered set. If r is reflexive, transitive and antisymmetrical, then $\langle A, r \rangle$ is called ordered set. An ordered set $\langle A, r \rangle$ is linearly ordered if for any two elements $a, b \in A$ either $\langle a, b \rangle \in r$ or $\langle b, a \rangle \in r$ holds. An ordered set $\langle A, r \rangle$ is called antichain if $\langle a, b \rangle \in r$ implies $a = b$.

Theorem 2. Let $\langle A, r\rangle$ be a quasiordered set which is not linearly ordered. Let $\langle A, F\rangle$ be an algebra such that $End(A, r) \subseteq End(A, F)$. Then $End(A, F) = A^A$.

Proof. If $\langle A, r\rangle$ is not ordered, then we may apply Theorem 1. If $\langle A, r\rangle$ is an antichain, then $End(A, r) = A^A$. Thus we shall suppose that $\langle A, r\rangle$ is an ordered set which is neither an antichain nor a linearly ordered set. We have $Card(A) \geq 3$. By an endomorphism we shall mean an endomorphism of $\langle A, r\rangle$. We shall write $a \leq b$ instead of $\langle a, b\rangle \in r$. Let $f \in F$ be j-ary. It is sufficient to prove $End(A, f) = A^A$.

Lemma 1. $\langle A, f\rangle$ is quasitrivial.

Proof. Notice first that whenever $b < c$, then $\{b, c\}$ is closed in $\langle A, f\rangle$; in fact, $\{b, c\}$ is the range of the endomorphism h defined by $h(y) = b$ if $y \leq b$ and $h(y) = c$ if $y \not\leq b$. Suppose that some $X \subseteq A$ is not closed in $\langle A, f\rangle$. There exists an $x \in X^j$ such that $f(x) \notin X$. Put $f(x) = a$. Suppose $b < a < c$ for some b and c. Define an endomorphism k by $k(y) = b$ if $y < a$; $k(y) = a$ if $y = a$; $k(y) = c$ if $y \not\leq a$. We have $a = k(a) = k(f(x)) = f(k \circ x)$ where $k \circ x \in \{b, c\}^j$, a contradiction, as $\{b, c\}$ is closed. Consequently, a is either maximal or minimal. Choose an element $d \neq a$ such that if a is not both maximal and minimal, then d is comparable with a. Define an endomorphism g by $g(a) = a$ and $g(y) = d$ if $y \neq a$. We have $a = g(a) = g(f(x)) = f(g \circ x)$ where $g \circ x \in \{d\}^j$, a contradiction, as any one-element subset of A is the range of a constant endomorphism and thus closed.

Lemma 2. There exist three pairwise different elements $a, b, c \in A$ such that a, b are comparable and one of the following two conditions is satisfied:

(1) Every mapping k of $\{a, b, c\}$ into $\{a, b, c\}$ such that $k(b) = b$ can be extended to an endomorphism.

(2) Every mapping k of $\{a, b, c\}$ into $\{a, b, c\}$ such that either $k(a) = k(b)$ or simultaneously $k(a) = a$ and $k(b) = b$ can be extended to an endomorphism.

Proof. We have four possibilities: (i) There exist four elements a, b, c, d such that $d < a < b$, $d < c < b$ and a, c are not comparable. Define a transformation h of A in this way: if $y \leq a$ and $y \leq c$, then $h(y) = d$; if $y \leq a$ and $y \not\leq c$, then $h(y) = a$; if $y \not\leq a$ and $y \leq c$, then $h(y) = c$; if $y \not\leq a$ and $y \not\leq c$, then $h(y) = b$. Evidently, h is an endomorphism. If k is as in (1), extend k to a mapping k' with domain $\{a, b, c, d\}$ by $k'(d) = d$; evidently, $k' \circ h$ is an extension of k to an endomorphism. (ii) There exist three elements a, b, c such that $a < b$, $c < b$ and a, c have no common lower bound. Define an endomorphism h: if $y \leq a$, then $h(y) = a$; if $y \leq c$, then $h(y) = c$; if $y \not\leq a$ and $y \not\leq c$, then $h(y) = b$. If k is as in (1), then $k \circ h$ is an extension of k to an endomorphism. (iii) There exist a, b, c such that $b < a$, $b < c$ and a, c have no common upper bound; this is dual to the previous possibility. (iv) None of the previous three possibilities takes place and there exist three elements a, b, c such that $a < b$ and c is comparable with neither a nor b. Define an endomorphism h: if y is comparable with c, then $h(y) = c$; if y is not comparable with c, then $h(y) = a$ if $y \leq a$ and $h(y) = b$ if $y \not\leq a$. If k is as in (2), then $k \circ h$ is an extension of k to an endomorphism.

Let us fix three elements a, b, c for which the just proved Lemma 2 holds. In both cases there exist three endomorphisms k_1, k_2 and k_3 such that

$$\begin{aligned} &k_1(a) = a; \quad k_1(b) = b; \quad k_1(c) = b; \\ &k_2(a) = a; \quad k_2(b) = b; \quad k_2(c) = a; \\ &k_3(a) = b; \quad k_3(b) = b; \quad k_3(c) = a. \end{aligned}$$

For every $X \subseteq j$ define a mapping z_X of j into $\{a, b\}$ in this way: if $i \in X$, then $z_X(i) = a$; if $i \in j - X$, then $z_X(i) = b$. Denote by D the set of all $X \subseteq j$ such that $f(z_X) = a$.

Lemma 3. D is an ultrafilter on j.

Proof. As $\langle A, f \rangle$ is quasitrivial, the set j belongs and the empty set does not belong to D. Let $X \in D$ and $X \subseteq Y \subseteq j$. Define a mapping x of j into $\{a, b, c\}$ in this way: if $i \in X$, then $x(i) = a$; if $i \in Y - X$, then $x(i) = c$; if $i \in j - Y$, then $x(i) = b$. We have $k_1(f(x)) = f(k_1 \circ x) = f(z_X) = a$; this, together with $f(x) \in \{a, b, c\}$, gives $f(x) = a$. Consequently, $f(z_Y) = f(k_2 \circ x) = k_2(f(x)) = k_2(a) = a$, so that $Y \in D$. To prove that D is a filter, we must show that if $X \in D$ and $Y \in D$, then $X \cap Y \in D$, too. Define an $x \in \{a, c\}^j$ in this way: *if* $i \in Y - X$, then $x(i) = a$; if $i \notin Y - X$ (i.e. if $i \in X \cup (j - Y)$), then $x(i) = c$. Define $y \in \{a,b,c\}^j$ in this way: if $i \in X \cap Y$, then $y(i) = a$; if $i \in j - Y$, then $y(i) = b$; if $i \in Y - X$, then $y(i) = c$. If it were $f(x) = a$, we would have $f(z_{X \cup (j - Y)}) = f(k_3 \circ x) = k_3(f(x)) = k_3(a) = b$, which is impossible as $X \in D$ implies $X \cup (j - Y) \in D$. We get $f(x) = c$. If it were $f(y) = c$, we would have $a = k_3(c) = k_3(f(y)) = f(k_3 \circ y) = f(k_1 \circ x) = k_1(f(x)) = k_1(c) = b$, a contradiction; if it were $f(y) = b$, we would have $f(z_Y) = f(k_2 \circ y) = k_2(f(y)) = k_2(b) = b$, a contradiction again. We get $f(y) = a$ and consequently $f(z_{X \cap Y}) = f(k_1 \circ y) = k_1(f(y)) = k_1(a) = a$, so that $X \cap Y \in D$. We have proved that D is a filter. It remains to show that if $X \subseteq j$ and $X \notin D$, then $j - X \in D$. Define an $x \in \{a, c\}^j$ by $x(i) = a$ if $i \in X$ and $x(i) = c$ if $i \in j - K$. If it were $f(x) = a$, we would have $f(z_X) = f(k_1 \circ x) = k_1(f(x)) = k_1(a) = a$, a contradiction. Hence, $f(x) = c$, so that $f(z_{j - X}) = f(k_3 \circ x) = k_3(f(x)) = k_3(c) = a$ and $j - X \in D$.

Lemma 4. Let $x \in A^j$ and $d \in A$. Then $f(x) = d$ iff $\{i \in j;\ x(i) = d\} \in D$.

Proof. We shall suppose $a < b$, as a, b are comparable and in the case $b < a$ the proof would be analogous. Let $f(x) = d$. Define two transformations h_1 and h_2 of A in this way: if $y \leq d$, then $h_1(y) = a$; if $y \not\leq d$, then $h_1(y) = b$; if $y \geq d$, then $h_2(y) = b$; if $y \not\leq d$, then $h_2(y) = a$. Evidently, h_1 and h_2 are endomorphisms. Put $X = \{i \in j;\ x(i) \leq d\}$ and $Y = \{i \in j;\ x(i) \geq d\}$. We have $f(z_X) = f(h_1 \circ x) = h_1(f(x)) = h_1(d) = a$ and $f(z_{j - Y}) = f(h_2 \circ x) = h_2(f(x)) = h_2(d) = b$, so that $X \in D$ and $j - Y \notin D$, so that $X \cap Y \in D$, i.e. $\{i \in j;\ x(i) = d\} \in D$. The direct implication is thus proved. Let, conversely, $\{i \in j;\ x(i) = d\} \in D$. We have proved above $\{i \in j;\ x(i) = f(x)\} \in D$. As the intersection of the two sets belonging to D belongs to D and is thus non-empty, there exists an i such that $x(i) = d$ and $x(i) = f(x)$, i.e. $d = f(x)$.

Let h be an arbitrary transformation of A. Let $x \in A^j$ and put $d = f(x)$. By Lemma 4 we have $\{i \in j;\ x(i) = d\} \in D$. As evidently $\{i \in j;\ x(i) = d\} \subseteq \{i \in j;\ h(x(i)) = h(d)\}$, the latter set belongs to D, too, so that $f(h \circ x) = h(d)$ by Lemma 4. We get $h \in End(A, f)$.

This completes the proof of Theorem 2.

Corollary. Let $\langle A, T\rangle$ be a topological space such that the union of any system of closed sets of $\langle A, T\rangle$ is closed. (Any finite topological space satisfies this condition.) Let $\langle A, F\rangle$ be an algebra such that any continuous mapping of $\langle A, T\rangle$ into itself is an endomorphism of $\langle A, F\rangle$. If the open sets of $\langle A, T\rangle$ do not constitute a chain, then $End(A, F) = A^A$.

Proof. Define a binary relation r on A: $\langle a, b\rangle \in r$ iff a belongs to the closure of $\{b\}$ in $\langle A, T\rangle$. Evidently, $\langle A, r\rangle$ is a quasiordered set which is not linearly ordered and the endomorphisms of $\langle A, r\rangle$ are just the continuous mappings of $\langle A, T\rangle$ into itself.

Theorem 3. Let r be a transitive relation on A such that whenever $\langle a, b\rangle \in r$, then $\langle a, a\rangle \in r$ and $\langle b, b\rangle \in r$. The transformation monoid $End(A, r)$ is algebraic iff one of the following five cases takes place:
(1) $\langle A, r\rangle$ is a linearly ordered set;
(2) $\langle A, r\rangle$ is an antichain;
(3) $r = A \times A$;
(4) r is empty;
(5) $r = \{\langle a, a\rangle\}$ for some $a \in A$.

Proof. In cases (2), (3) and (4) $\langle A, r\rangle$ has the same endomorphisms as the algebra with no operations; in the case (5) as the algebra with one constant a; in the case (1) as the algebra with one binary operation f defined in this way: $f(a, b)$ is the maximum of a and b. Let conversely $End(A, r) = End(A, F)$ for some algebra $\langle A, F\rangle$. Denote by B the set of all $a \in A$ such that $\langle a, a\rangle \in r$. If $B = A$, then it follows from Theorem 2 that either (1) or (2) or (3) takes place. Let $B \neq A$; it is sufficient to suppose that B is non-empty and to prove $Card(B) = 1$. Fix an element $c \in A - B$. As the constant transformation with value c does not belong to $End(A, r) = End(A, F)$, there exists an operation $f \in F$ of some arity j such that $f(x) \neq c$ where x is the only element of $\{c\}^j$. Let a be an arbitrary element of B. Define a transformation h of A by $h(c) = c$ and $h(y) = a$ if $y \neq c$. We have $h \in End(A, r) = End(A, F)$, so that $f(x) = f(h \circ x) = h(f(x)) = a$. As $a \in B$ was arbitrary, $Card(B) = 1$.

3. The condition $End(A, F) = A^A$

Evidently, if $\langle A, F\rangle$ is an algebra, then $End(A, F) = A^A$ iff $End(A, f) = A^A$ for any $f \in F$.

Theorem 4. Let f be a j-ary operation on a set A of cardinality ≥ 3. Then $End(A, f) = A^A$ iff there exists an ultrafilter D on j such that the following holds for all $x \in A^j$ and $a \in A$:

$$f(x) = a \quad \text{iff} \quad \{i \in j;\ x(i) = a\} \in D.$$

Proof follows from the proof of Theorem 2: the converse implication could be verified analogously as in the end of that proof, and the direct implication follows from Lemmas 3 and 4.

Corollary 1. Let $\langle A, F\rangle$ be a finitary algebra, $Card(A) \geq 3$. If $End(A, F) = A^A$, then $\langle A, F\rangle$ is trivial.

Proof. It is well-known that every ultrafilter on a finite set is trivial, i.e. it corresponds to an element of the set.

Corollary 2. Let f be a j-ary operation on an infinite set A. If $End(A,f) = A^A$ and f is not trivial, then $Card(j) > Card(A)$ and $Card(j)$ is a measurable cardinal.

Proof. Let D be as in Theorem 4. As f is not trivial, D is not trivial, too, i.e. no finite set belongs to D. Suppose that there exists an injective mapping x of j into A. Put $a = f(x)$. The set $\{i \in j;\ x(i) = a\}$ has at most one element and belongs to D, a contradiction. We get $Card(j) > Card(A)$. In order to prove that $Card(j)$ is measurable it is sufficient to prove that whenever j is the union of some countable sequence $X_1, X_2, X_3, \ldots$ of pairwise disjoint sets, then one of its members belongs to D. Choose a sequence $a_1, a_2, a_3, \ldots$ of pairwise different elements of A and define a mapping y of j into A by $y(i) = a_n$ whenever $i \in X_n$. There exists an m such that $f(y) = a_m$; we get $X_m \in D$.

If $\langle A, F\rangle$ is an algebra, denote by F' the set of all finitary algebraic operations (polynomials) of $\langle A, F\rangle$. We have evidently $End(A, F) \subseteq End(A, F')$. On the other hand, it can happen that $\langle A, F'\rangle$ is trivial but $End(A, F) \neq A^A$:

Example. Let D be any non-trivial ultrafilter on $\omega = \{0, 1, 2, \ldots\}$. Let A be an infinite set. Define an ω-ary operation f on A in this way: if $x \in A$ and there exists an $a \in A$ such that $\{i \in \omega;\ x(i) = a\} \in D$, then $f(x) = a$; if such an a does not exist, then $f(x) = x(0)$. It is easy to show that $\langle A, \{f\}'\rangle$ is trivial but $End(A,f) \neq A^A$.

4. Inversion of almost identical endomorphisms of algebras

If a and b are two elements of A, then we define a transformation w_{ab} of A in this way: $w_{ab}(a) = b$; $w_{ab}(x) = x$ for all $x \in A - \{a\}$.

Theorem 5. Let an algebra $\langle A, F\rangle$ and two different elements $a, b \in A$ be given; let $A - \{a\}$ be closed in $\langle A, F\rangle$. If w_{ba} is an endomorphism of $\langle A, F\rangle$, then w_{ab} is an endomorphism, too.

Proof. Notice first that $w_{ba} = w_{ba} \circ w_{ab}$. Suppose that w_{ba} is an endomorphism but w_{ab} is not. There exists an operation $f \in F$ of some arity j and an $x \in A^j$ such that $w_{ab}(f(x)) \neq f(w_{ab} \circ x)$.

Suppose, $(x) = b$. We have $b \neq f(w_{ab} \circ x)$; this and the endomorphism property of w_{ba} gives $f(w_{ba} \circ w_{ab} \circ x) = f(w_{ab} \circ x)$, so that $f(w_{ab} \circ x) = f(w_{ba} \circ w_{ab} \circ x) =$

$= f(w_{ba} \circ x) = w_{ba}(f(x)) = w_{ba}(b) = a$ which is a contradiction because $w_{ab} \circ x \in \in (A - \{a\})^j$ and $A - \{a\}$ is closed.

We get $f(x) \neq b$. Consequently $w_{ba}(f(x)) = f(x)$, so that $f(w_{ba} \circ x) = f(x)$.

Suppose $f(w_{ba} \circ x) = f(x) \neq a$. As $f(x) = w_{ab}(f(x)) \neq f(w_{ab} \circ x)$, we have $f(w_{ba} \circ x) \neq f(w_{ab} \circ x)$, so that $f(w_{ab} \circ x) \neq f(w_{ba} \circ x) = f(w_{ba} \circ w_{ab} \circ x) = w_{ba}(f(w_{ab} \circ x))$, which gives $f(w_{ab} \circ x) = b$; we get $f(w_{ba} \circ x) = f(w_{ba} \circ w_{ab} \circ x) = w_{ba}(f(w_{ab} \circ x)) = w_{ba}(b) = a$, a contradiction with the assumption.

The case $f(w_{ba} \circ x) = f(x) = a$ remains. We have $b \neq f(w_{ab} \circ x)$, so that $w_{ba}(f(w_{ab} \circ x)) = f(w_{ab} \circ x)$. Hence, $f(w_{ab} \circ x) = w_{ba}(f(w_{ab} \circ x)) = f(w_{ba} \circ w_{ab} \circ x) = f(w_{ba} \circ x) = a$ where $w_{ab} \circ x \in (A - \{a\})^j$, a contradiction again.

5. Unary relational structures

Let $\langle A, R \rangle$ be a unary relational structure, so that R is an arbitrary set of subsets of A. Endomorphisms of $\langle A, R \rangle$ are just the transformations h of A such that whenever $a \in r \in R$, then $h(a) \in r$.

For every $a \in A$ denote by $\bar{a}$ the set of all $b \in A$ such that $a \in r \in R$ implies $b \in r$. In other words, $\bar{a}$ is the intersection of all $r \in R$ such that $a \in r$; if $a \in r$ for no $r \in R$, then $\bar{a} = A$. We have $a \in \bar{a}$.

Denote by B the set of all $a \in A$ such that $\bar{a} = \{a\}$.

Theorem 6. Let $\langle A, R \rangle$ be a unary relational structure. *End*(A, R) is algebraic iff the following four conditions are satisfied:

(1) If B is empty, then every $r \in R$ is either empty or equal to A;

(2) If *Card*$(B) = 1$, then $B \subseteq r$ for any non-empty $r \in R$;

(3) Whenever $a, b \in A - B$ and $b \in \bar{a}$, then $\bar{a} = \bar{b}$;

(4) Whenever $a_1, \ldots, a_n$ (where $n \geq 2$) are pairwise different elements of $A - B$ and $u_1, \ldots, u_n$ pairwise different elements of B such that $u_1 \in \bar{a}_1 \cap \bar{a}_2$, $u_2 \in \bar{a}_2 \cap \bar{a}_3, \ldots, u_n \in \bar{a}_n \cap \bar{a}_1$, then $\bar{a}_1 = \ldots = \bar{a}_n$.

If *End*(A, R) is algebraic, then there exists a unary algebra $\langle A, F \rangle$ such that *End*$(A, R) =$ *End*(A, F).

Proof. Put $H =$ *End*(A, R). We shall first prove the direct implication. Let $\langle A, F \rangle$ be an algebra such that *End*$(A, F) = H$.

Lemma A. If $a \in A - B$, then $A - \{a\}$ is closed in $\langle A, F \rangle$.

Proof. There exists a $b \in \bar{a}$ such that $b \neq a$. The mapping w_{ab} (defined in Section 4) evidently belongs to H and consequently to *End*(A, F), so that its range, the set $A - \{a\}$, is closed in $\langle A, F \rangle$.

Lemma B. If $B \subseteq X \subseteq A$, then X is closed in $\langle A, F \rangle$.

Proof follows immediately from Lemma A.

Let us prove (1). Suppose that B is empty but some non-empty $r \in R$ is different from A. Choose an $a \in A - r$. By Lemma B the set $\{a\}$ is closed, so that the constant transformation with value a is an endomorphism of $\langle A, F \rangle$ and belongs thus to H, which is evidently impossible.

Let us prove (2). Denote by b the only element of B. By Lemma B the set $\{b\}$ is closed in $\langle A, F\rangle$, so that the constant transformation with value b is an endomorphism; this implies $b \in r$ for any non-empty $r \in R$.

Let us prove (3). Evidently $\bar{b} \subseteq \bar{a}$. As $b \in \bar{a}$, we have $w_{ab} \in H$; by Theorem 5 and Lemma A this implies $w_{ba} \in H$. Consequently $a \in \bar{b}$, i.e. $\bar{a} \subseteq \bar{b}$.

Lemma C. Let a and b be two different elements of $A - B$; let $\bar{a} \cap \bar{b} \cap B$ contain two different elements u, v. Then $\bar{a} = \bar{b}$.

Proof. It is sufficient to prove $b \in \bar{a}$. Suppose $b \notin \bar{a}$. For every pair p, q of elements of A define a transformation k_{pq} of A in this way: $k_{pq}(a) = p$; $k_{pq}(b) = q$; if $y \in A - \{a, b\}$, then $k_{pq}(y) = y$. We have $k_{bb} \notin H$, so that there exists an operation $f \in F$ of some arity j and an $x \in A_j$ such that $k_{bb}(f(x)) \neq f(k_{bb} \circ x)$.

Suppose $f(x) \in \{a, b\}$, so that $b \neq f(k_{bb} \circ x)$. We have $k_{au}(f(k_{bb} \circ x)) = f(k_{au} \circ k_{bb} \circ x) = f(k_{uu} \circ x) = k_{uu}(f(x)) = u$ and consequently $f(k_{bb} \circ x) = u$. Quite similarly $f(k_{bb} \circ x) = v$. $u \neq v$ gives a contradiction.

We get $f(x) \notin \{a, b\}$, so that $f(x) \neq f(k_{bb} \circ x)$. We have $k_{au}(f(k_{bb} \circ x)) = f(k_{au} \circ k_{bb} \circ x) = f(k_{uu} \circ x) = k_{uu}(f(x)) = f(x)$; this is possible only if $f(k_{bb} \circ x) = b$. Thus $f(x) = k_{au}(b) = u$; quite similarly $f(x) = v$ and $u \neq v$ gives a contradiction. This completes the proof of Lemma C.

Let us prove (4). We may suppose $n \geqq 3$, as the case $n = 2$ is covered by Lemma C. For any integer i denote by $c(i)$ the only element of $\{1, \ldots, n\}$ which is congruent with i mod n. Define three transformations h, k, g of A in this way: if $i \in \{1, \ldots, n\}$, then $h(a_i) = u_{c(i+1)}$, $k(a_i) = a_{c(i+1)}$ and $g(a_i) = u_i$; if $y \notin \{a_1, \ldots, a_n\}$, then $h(y) = k(y) = g(y) = y$. Suppose $h \notin H$, so that there exists an operation $f \in F$ of some arity j and an $x \in A^j$ such that $h(f(x)) \neq f(h \circ x)$.

Suppose $f(x) = a_i$ for some $i \in \{1, \ldots, n\}$, so that $u_{c(i+1)} \neq f(h \circ x)$. As $g \circ k^{n-1} \in H$, we have $f(g \circ k^{n-1} \circ x) = g(k^{n-1}(f(x))) = g(k^{n-1}(a_i)) = u_{c(i-1)}$. As $g \in H$, we have $g(f(k^{n-1} \circ x)) = u_{c(i-1)}$, so that $f(k^{n-1} \circ x)$ is equal either to $u_{c(i-1)}$ or to $a_{c(i-1)}$ and consequently $f(g \circ k^{n-2} \circ x) = f(g \circ k^{n-1} \circ k^{n-1} \circ x) = g(k^{n-1}(f(k^{n-1} \circ x)))$ is equal either to $u_{c(i-1)}$ or to $u_{c(i-2)}$. We get similarly $f(g \circ k^{n-3} \circ x) \in \{u_{c(i-1)}, u_{c(i-2)}, u_{c(i-3)}\}$, etc.; finally, $f(g \circ k \circ x) \in \{u_{c(i-1)}, u_{c(i-2)}, \ldots, u_{c(i-n+1)}\}$, i.e. $f(h \circ x) \neq u_i$. We have $g(k^{n-1}(f(k \circ x))) = f(g \circ k^{n-1} \circ k \circ x) = f(g \circ x) = g(f(x)) = u_i$, so that either $f(k \circ x) = u_i$ or $f(k \circ x) = a_{c(i+1)}$; we get in the first case $f(h \circ x) = f(g \circ k \circ x) = g(f(k \circ x)) = g(u_i) = u_i$ and in the second case $f(h \circ x) = u_{c(i+1)}$; however, both these cases are already excluded.

Suppose $f(x) = u_i$ for some i, so that $u_i \neq f(h \circ x)$. Similarly as in the previous case $f(g \circ k^{n-1} \circ x) = u_i$, so that $f(k^{n-1} \circ x) \in \{u_i, a_i\}$, from which we get $f(g \circ k^{n-2} \circ x) \in \{u_i, u_{c(i-1)}\}$; etc.; finally $f(g \circ k \circ x) \in \{u_i, u_{c(i-1)}, \ldots, u_{c(i-n+2)}\}$, i.e. $f(h \circ x) \neq u_{c(i+1)}$. We have $g(k^{n-1}(f(k \circ x))) = u_i$, so that either $f(k \circ x) = u_i$ or $f(k \circ x) = a_{c(i+1)}$; a contradiction in both cases.

Suppose $f(x) \notin \{a_1, \ldots, a_n, u_1, \ldots, u_n\}$. Similarly as in the previous cases $f(g \circ k^{n-1} \circ x) = f(x)$, so that $f(k^{n-1} \circ x) = f(x)$, so that $f(g \circ k^{n-2} \circ x) = f(x)$, etc.; finally $f(g \circ k \circ x) = f(x)$, i.e. $f(h \circ x) = f(x)$, a contradiction.

We get $h \in H$, so that $u_2 \in a_1, \ldots, u_n \in \bar{a}_{n-1}, u_1 \in a_n$. This gives $\bar{a}_1 = \ldots = a_n$ by Lemma C.

The direct implication is thus proved. Let us prove the converse.

If B is empty, then (1) gives $H = A^A$ and everything is evident.

Let $Card(B) = 1$; put $B = \{b\}$. For every $a \in A$ define a unary operation u_a on A in this way: if $y \in \bar{a}$, then $u_a(y) = y$; if $y \in A - \bar{a}$, then $u_a(y) = b$. It is easy to prove $H = End(A, \{u_a;\ a \in A\})$.

Let $Card(B) \geq 2$; choose two different elements $p, q \in B$. For every $a \in A - B$ such that $\bar{a} \cap B$ is empty define two unary operations f_a and g_a on A in this way: if $y \in \bar{a}$, then $f_a(y) = g_a(y) = y$; if $y \in A - \bar{a}$, then $f_a(y) = p$ and $g_a(y) = q$. For every $a \in A - B$ such that $a \cap B$ is non-empty choose an element $z_a \in \bar{a} \cap B$ and define a unary operation t_a on A in this way: if $y \in \bar{a}$, then $t_a(y) = y$; if $y \in A$ and there exists a finite sequence $a_1, \ldots, a_n$ (where $n \geq 2$) of elements of $A - B$ and a finite sequence $u_1, \ldots, u_{n-1}$ of pairwise different elements of B such that $\bar{a}_1, \ldots, \bar{a}_n$ are pairwise different, $u_1 \in \bar{a}_1 \cap \bar{a}_2, \ldots, u_{n-1} \in \bar{a}_{n-1} \cap \bar{a}_n$, $a_1 = a$ and $y \in \bar{a}_n$, then $t_a(y) = u_1$; in all other cases $t_a(y) = z_a$.

We must show that t_a was well defined. If $y \in \bar{a}$ (so that $t_a(y) = y$) and if at the same time $t_a(y)$ is defined by means of $a_1, \ldots, a_n$ and $u_1, \ldots, u_{n-1}$, then $y \in \bar{a}_n \cap \bar{a}_1$, so that $y \in B$; the condition (4), applied to $a_1, \ldots, a_n, u_1, \ldots, u_{n-1}, y$, gives $y = u_i$ for some i, so that $u_i \in \bar{a}_1$; the condition (4), applied to $a_1, \ldots, a_i, u_1, \ldots, u_i$, gives $i = 1$, so that $y = u_1$ and both definitions of $t_a(y)$ give the same result. Let $t_a(y)$ be defined by means of $a_1, \ldots, a_n, u_1, \ldots, u_{n-1}$ and at the same time by means of $b_1, \ldots, b_m, v_1, \ldots, v_{m-1}$. Suppose $u_1 \neq v_1$. As $y \in \bar{a}_n \cap \bar{b}_m$, there exists an $i \geq 2$ (take the smallest such i) such that there exists a $j \geq 2$ (again, take the smallest such j) such that $\bar{a}_i \cap \bar{b}_j$ is non-empty; choose some $w \in \bar{a}_i \cup \bar{b}_j$. If it were $\bar{a}_i = \bar{b}_j$, we would have evidently $i = j = 2$ and both u_1 and v_1 would belong to $\bar{a}_1 \cap \bar{a}_2$, so that $\bar{a}_1 = \bar{a}_2$, a contradiction. We get $\bar{a}_i \neq \bar{b}_j$. Evidently, the assumptions of (4), applied to $a_1, \ldots, a_i, b_j, \ldots, b_2, u_1, \ldots, u_{i-1}, w, v_{j-1}, \ldots, v_1$, are satisfied, but the conclusion is not. We get $u_1 = v_1$, so that both definitions of $t_a(y)$ give the same result.

Denote by F the set of all the operations f_a, g_a (where $\bar{a} \cap B$ is empty) and t_a (where $a \in A - B$ and $\bar{a} \cap B$ is non-empty), together with all constant unary operations with values belonging to B. We shall prove $H = End(A, F)$.

Let us prove $h(f_a(y)) = f_a(h(y))$ for all $h \in H, y \in A$ and $a \in A - B$ such that $\bar{a} \cap B$ is empty. If $y \in \bar{a}$, then $h(f_a(y)) = h(y) = f_a(h(y))$, because evidently $h(y) \in \bar{a}$, too. Let $y \notin \bar{a}$. If it were $h(y) \in \bar{a}$, then using (3) we would have $h(y) \in B$, a contradiction, as $\bar{a} \cap B$ is empty. We get $h(y) \notin \bar{a}$, too, so that $h(f_a(y)) = h(p) = p = f_a(p) = f_a(h(y))$.

Quite similarly $h(g_a(y)) = g_a(h(y))$.

Let us prove $h(t_a(y)) = t_a(h(y))$ for all $h \in H$, $y \in A$ and $a \in A - B$ such that $\bar{a} \cap B$ is non-empty. If $y \in \bar{a}$, then $h(t_a(y)) = h(y) = t_a(h(y))$, because evidently $h(y) \in \bar{a}$, too. If $t_a(y)$ is defined by means of $a_1, \ldots, a_n, u_1, \ldots, u_{n-1}$, so that $t_a(y) = u_1$ and $y \in \bar{a}_n$, then evidently $h(y) \in \bar{a}_n$ and $t_a(h(y))$ is defined by means of $a_1, \ldots, a_n, u_1, \ldots, u_{n-1}$, too, so that $h(t_a(y)) = h(u_1) = u_1 = t_a(h(y))$. If $y \notin \bar{a}$ and $t_a(y)$ is not defined by finite sequences, then using (3) it is easy to see that $h(y) \notin \bar{a}$ and $t_a(h(y))$ is not defined by finite sequences, too, so that $h(t_a(y)) = h(z_a) = z_a = t_a(h(y))$.

It remains to prove $End(A, F) \subseteq H$. Let $h \in End(A, F)$ and suppose $h \notin H$. There exists an $a \in A$ such that $h(a) \notin \bar{a}$. As any element of B is a fixed point of h, we have $a \in A - B$. Suppose that $\bar{a} \cap B$ is empty and $h(a) \neq p$. We have $h(a) = h(f_a(a)) = f_a(h(a)) = p$, a contradiction. We get a contradiction similarly in the case $h(a) \neq q$ (using only g_a instead of f_a). Hence, $\bar{a} \cap B$ is non-empty. We have $h(a) = h(t_a(a)) = t_a(h(a))$, a contradiction, as $h(a) \notin \bar{a}$, and the range of t_a is equal to $\bar{a}$.

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