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# On Certain Groups of Holomorphic Maps

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**0.** Consider the space $\mathbf{C}^2$ with the complex coordinates $(x, y)$. By $\Gamma_s$ denote the pseudogroup of local holomorphic diffeomorphisms of $\mathbf{C}^2$ $\tilde{x} = \tilde{x}(x, y)$, $\tilde{y} = \tilde{y}(x, y)$ satisfying $\partial(\tilde{x}, \tilde{y}) / \partial(x, y) = 1$. We are going to prove the following

**Theorem.** *Let $G \subset \Gamma_s$ be a Lie group such that $\dim G = 3$ and the orbits of $G$ are real hypersurfaces $M^3 \subset \mathbf{R}^4 \equiv \mathbf{C}^2$ with non-trivial Levi form. Then $G$ is locally $\Gamma_s$-equivalent to one of the following groups:*

$$\text{(I)} \quad \tilde{x} = x - \frac{a}{\alpha} y - \frac{1}{2} iBa^2 + c, \quad \tilde{y} = y + \alpha b + i\alpha Ba; \quad a, b, c \in \mathbf{R};$$

$$\text{(II)} \quad \tilde{x} = ax - \frac{1}{\alpha} by + c, \quad \tilde{y} = -\alpha Bbx + ay + \alpha d; \quad a, b, c, d \in \mathbf{R}; \quad a^2 - Bb^2 = 1;$$

$$\text{(III)} \quad \tilde{x} = \frac{(ax + b)(1 - bc - acx)\, y^2 - \alpha^2 a^2 c}{(1 - bc - acx)^2 y^2 + \alpha^2 a^2 c^2},$$

$$\tilde{y} = \frac{(1 - bc - acx)^2 y^2 + \alpha^2 a^2 c^2}{ay}; \quad a, b, c \in \mathbf{R};$$

(IV) *consider* (III) *with* $a \in i\mathbf{R}$, $b \in \mathbf{C}$, $c = \bar{b}$.

*Here,* $0 \neq \alpha \in \mathbf{C}$ *and* $0 \neq B \in \mathbf{R}$ *are parameters. The corresponding orbits* $M^3$ *are*

$$\text{(I')} \quad \left(\frac{y}{\alpha} - \frac{\bar{y}}{\bar{\alpha}}\right)^2 + 4iB(x - \bar{x}) = r,$$

$$\text{(II')} \quad \left(\frac{y}{\alpha} - \frac{\bar{y}}{\bar{\alpha}}\right)^2 - B(x - \bar{x})^2 = r,$$

$$\text{(III')} \quad (x - \bar{x})^2 y^2 \bar{y}^2 + (\alpha\bar{y} + \bar{\alpha}y)^2 + 4ry\bar{y} = 0$$

*with* $r \in \mathbf{R}$.

The groups (III) and (IV) will be studied elsewhere.

In the second part of this paper, I solve the equivalence problem for hypersurfaces of $\mathbf{C}^2$ with respect to the pseudogroup of all local biholomorphic mappings.

It is well known that two real hypersurfaces in $\mathbf{C}^2$ are not generally holomorphically equivalent. The problem of the construction of invariants of $M^3 \subset \mathbf{C}^2$ with respect to the pseudogroup of holomorphic mappings has been treated by E. Cartan (Annali di Mat., t. 11. 1932, 17–90); unfortunately, his treatment is very confused.

Let $V^3$ be a differentiable manifold together with a structure consisting of a choice of two tangent directions at each of its points. In what follows, I shall construct (in the general case) an $\{e\}$-structure on $V^3$ invariantly associated to the given structure; by means of this $\{e\}$-structure, the equivalence problem of the structures of the just described type will be solved. Further, I will show that the construction of an invariant $\{e\}$-structure on $M^3 \subset \mathbf{C}^2$ is equivalent to the preceding construction. The special cases will be treated in a forthcoming paper.

Parts of this paper have been written during my stays at the universities at Berlin (GDR) and Riga (USSR).

**1.** Consider the space $\mathbf{C}^2$, $\mathbf{C}$ being the complex numbers, with the complex coordinates $x = x^1 + iy^1$, $y = x^2 + iy^2$. Its real form is the space $\mathbf{R}^4$ ($\mathbf{R}$ being reals) with the coordinates $(x^1, y^1, x^2, y^2)$ together with the endomorphism $I$: $\mathbf{R}^4 \to \mathbf{R}^4$, $I^2 = -id$, defined by $(i = 1,2)$

$$I \frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i}, \quad I \frac{\partial}{\partial y^i} = -\frac{\partial}{\partial x^i}. \tag{1.1}$$

In general, on any complex vector space $V$, scalar multiplication by real numbers is, of course, defined. Relative to addition, and scalar multiplication by real numbers only, the elements of $V$ clearly form a real vector space, which will be denoted by $V_0$ and called the real vector space underlying the complex vector space $V$. If $V_0$ is the underlying real vector space of a complex space $V$, then there is an automorphism $I_0$ of $V_0$ satisfying $I_0^2 = -id$, induced by the automorphism $I$ of $V$ given by $IA = iA$, $A \in V$. Further, $dim_R V_0 = 2 dim_c V$. Let $V$ be a finite dimensional complex vector space and $A_1, \ldots, A_n$ its basis, then $A_1, I_0A_1, \ldots, A_n, I_0A_n$ give a basis for $V_0$. Let $W_0$ be a real vector space (of finite dimension). We say that a complex structure is given on $W_0$ if there is given an endomorphism $I_0$ of $W_0$ satisfying $I_0^2 = -id$; this endomorphism is an automorphism, since $I_0^{-1}$ exists and is given by $-I_0$. Let $W_0$ be a real vector space with a complex structure defined by $I_0$. Then: (i) There exists a basis for $W_0$ of the form $A_1, I_0A_1, \ldots, A_n, I_0A_n$; in particular, $dim_R W_0$ is even; (ii) there exists a complex space $W$ such that $W_0$ is the underlying real vector space of $W$ and $I_0$ is induced by the complex structure of $W$. Let us prove this last proposition. Since $dim_R W_0 > 0$, there exists a vector $A_1 \neq 0$ in $W_0$. Then $A_1$ and $I_0A_1$ are independent. In fact, if there exist real numbers $a$, $b$ such that $aA_1 + bI_0A_1 = 0$, then $aIA_1 - bA_1 = 0$ and $(a^2 + b^2) A_1 = 0$. This implies $a = b = 0$. We proceed by induction, and assume that an independent set $A_1$, $I_0A_1, \ldots, A_k, I_0A_k$ of vectors in $W_0$ has been found $(k \geq 1)$. If $dim_R W_0 = 2k$, there is nothing further to prove. If $dim_R W_0 > 2k$, then there is a non-zero vector $A_{k+1} \in W_0$ which is independent of the vectors $A_1, \ldots, I_0A_k$. The vectors $A_1$,

$I_0A_1, \ldots, A_{k+1}, I_0A_{k+1}$ form an independent set. In fact, if $a_1, \ldots, a_{k+1}, b_1, \ldots, b_{k+1}$ are real numbers such that

$$\sum_{j=1}^{k+1} a_j A_j + \sum_{j=1}^{k+1} b_j I_0 A_j = 0, \tag{1.2}$$

then

$$\sum_{j=1}^{k+1} a_j I_0 A_j - \sum_{j=1}^{k+1} b_j A_j = 0.$$

From these, we obtain

$$\sum_{j=1}^{k} (a_j a_{k+1} + b_j b_{k+1}) A_j + \sum_{j=1}^{k+1} (b_j a_{k+1} - a_j b_{k+1}) I_0 A_j + (a_{k+1}^2 + b_{k+1}^2) A_{k+1} = 0.$$

All coefficients being zero, we have $a_{k+1} = b_{k+1} = 0$, and (1.2) implies $a_1 = \ldots = a_k = b_1 = \ldots = b_k = 0$. The complex vector space $W$ is constructed from the elements of $W_0$ by defining the operation of scalar multiplication by a complex number $c = a + ib$ as $cA = aA + bI_0A$.

Let $\Gamma$ be the pseudogroup of all local holomorphic diffeomorphisms of $\mathbf{C}^2$. Each $\gamma \in \Gamma$ induces a diffeomorphism of $\mathbf{R}^4$ denoted by $\gamma$, too. The local diffeomorphism $\gamma$ of $\mathbf{R}^4$ given by

$$\tilde{x}^i = f^i(x^j, y^j), \quad \tilde{y}^i = g^i(x^j, y^j); \quad i = 1,2; \tag{1.3}$$

is an element of $\Gamma$ if and only if the functions $f^i, g^i$ satisfy the Cauchy-Riemann equations

$$\frac{\partial f^i}{\partial x^j} = \frac{\partial g^i}{\partial y^j}, \quad \frac{\partial f^i}{\partial y^j} = -\frac{\partial g^i}{\partial x^j}; \quad i, j = 1,2. \tag{1.4}$$

Let $\Gamma_s \subset \Gamma$ be the pseudogroup of diffeomorphisms $\tilde{x} = \tilde{x}(x, y)$, $\tilde{y} = \tilde{y}(x, y)$ of the space $\mathbf{C}^2$ or $\mathbf{R}^4$ resp. satisfying

$$\frac{\partial(\tilde{x}, \tilde{y})}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial \tilde{x}}{\partial x} & \dfrac{\partial \tilde{x}}{\partial y} \\ \dfrac{\partial \tilde{y}}{\partial x} & \dfrac{\partial \tilde{y}}{\partial y} \end{vmatrix} = 1. \tag{1.5}$$

It is easy to see that $\gamma \in \Gamma$ is an element of $\Gamma_s$ if and only if $\gamma$ preserves the 2-form

$$\Phi = dx \wedge dy; \tag{1.6}$$

indeed,

$$\tilde{\Phi} = d\tilde{x} \wedge d\tilde{y} = \frac{\partial(\tilde{x}, \tilde{y})}{\partial(x, y)} \Phi.$$

Define

$$\varphi = dx^1 \wedge dx^2 - dy^1 \wedge dy^2, \quad \psi = dx^1 \wedge dy^2 + dy^1 \wedge dx^2, \tag{1.7}$$

obviously, $\Phi = \varphi + i\psi$. Of course, we may write

$$\varphi = \tfrac{1}{2}(dx \wedge dy + d\bar{x} \wedge d\bar{y}), \quad \psi = -\tfrac{1}{2}i(dx \wedge dy - d\bar{x} \wedge d\bar{y}). \tag{1.8}$$

We have

$$\varphi(v, w) = -\varphi(Iv, Iw), \quad \psi(v, w) = -\varphi(v, Iw) \text{ for } v, w \in \mathbf{R}^4. \tag{1.9}$$

Indeed, let

$$v = a^1 \frac{\partial}{\partial x^1} + b^1 \frac{\partial}{\partial y^1} + a^2 \frac{\partial}{\partial x^2} + b^2 \frac{\partial}{\partial y^2},$$

$$w = c^1 \frac{\partial}{\partial x^1} + d^1 \frac{\partial}{\partial y^1} + c^2 \frac{\partial}{\partial x^2} + d^2 \frac{\partial}{\partial y^2}. \tag{1.10}$$

Then

$$Iv = -b^1 \frac{\partial}{\partial x^1} + a^1 \frac{\partial}{\partial y^1} - b^2 \frac{\partial}{\partial x^2} + a^2 \frac{\partial}{\partial y^2},$$

$$Iw = -d^1 \frac{\partial}{\partial x^1} + c^1 \frac{\partial}{\partial y^1} - d^2 \frac{\partial}{\partial x^2} + c^2 \frac{\partial}{\partial y^2}$$

and

$$\begin{aligned} \varphi(v, w) &= a^1c^2 - a^2c^1 - b^1d^2 + b^2d^1 = -\varphi(Iv, Iw), \\ \psi(v, w) &= a^1d^2 - b^2c^1 + b^1c^2 - a^2d^1 = -\varphi(v, Iw). \end{aligned} \tag{1.11}$$

In $\mathbf{C}^2$, this may be rewritten as follows. Introduce the well known vector fields

$$\frac{\partial}{\partial x} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} - i\frac{\partial}{\partial y^1}\right), \quad \frac{\partial}{\partial \bar{x}} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial y^1}\right), \ldots.$$

Then

$$\frac{\partial}{\partial x^1} = \frac{\partial}{\partial x} + \frac{\partial}{\partial \bar{x}}, \quad \frac{\partial}{\partial y^1} = i\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \bar{x}}\right), \ldots,$$

and the vectors $v$, $w$ may be written as

$$v = A^1 \frac{\partial}{\partial x} + A^2 \frac{\partial}{\partial y} + \bar{A}^1 \frac{\partial}{\partial \bar{x}} + \bar{A}^2 \frac{\partial}{\partial \bar{y}},$$

$$w = C^1 \frac{\partial}{\partial x} + C^2 \frac{\partial}{\partial y} + \bar{C}^1 \frac{\partial}{\partial \bar{x}} + \bar{C}^2 \frac{\partial}{\partial \bar{y}}$$

with

$$A^i = a^i + ib^i, \quad C^i = c^i + id^i; \quad i = 1,2.$$

It is easy to check that

$$Iv = iA^1 \frac{\partial}{\partial x} + iA^2 \frac{\partial}{\partial y} - i\bar{A}^1 \frac{\partial}{\partial \bar{x}} - i\bar{A}^2 \frac{\partial}{\partial \bar{y}},$$

$$Iw = iC^1 \frac{\partial}{\partial x} + iC^2 \frac{\partial}{\partial y} - i\bar{C}^1 \frac{\partial}{\partial \bar{x}} - i\bar{C}^2 \frac{\partial}{\partial \bar{y}}$$

and

$$\varphi(v, w) = \tfrac{1}{2}(A^1C^2 - A^2C^1 + \bar{A}^1\bar{C}^2 - \bar{A}^2\bar{C}^1) = -\varphi(Iv, Iw),$$

$$\psi(v, w) = -\tfrac{1}{2}i(A^1C^2 - A^2C^1 - \bar{A}^1\bar{C}^2 + \bar{A}^2\bar{C}^1) = -\varphi(v, Iw).$$

Let $X = X(x, y)$, $Y = Y(x, y)$ be a local holomorphic diffeomorphism of $\mathbf{C}^2$. Then

$$dX \wedge dY + d\bar{X} \wedge d\bar{Y} = \frac{\partial(X, Y)}{\partial(x, y)} dx \wedge dy + \overline{\frac{\partial(X, Y)}{\partial(x, y)}} d\bar{x} \wedge d\bar{y}.$$

Thus: *Let $\gamma$ be a local diffeomorphism of $\mathbf{R}^4$ defined on $U \subset \mathbf{R}^4$. Then $\gamma \in \Gamma_s$ if and only if*

$$(d\gamma_a \,.\, I)(v_a) = (I \,.\, d\gamma_a)(v_a), \tag{1.12}$$

$$\varphi(v_a, w_a) = \varphi(d\gamma_a(v_a), d\gamma_a(w_a))$$

*for each* $\quad a \in U; \quad v_a, w_a \in T_a(\mathbf{R}^4) \equiv \mathbf{R}^4.$

From now on, consider the following situation: In $\mathbf{R}^4$ with the coordinates $(x^1, y^1, x^2, y^2)$ be given a complex structure $I$ (1.1) and the form (1.7$_1$); let $\Gamma_s$ be the pseudogroup of local diffeomorphisms of $\mathbf{R}^4$ satisfying (1.12).

Now, let $M^3 \subset \mathbf{R}^4$ be a hypersurface. At each point $m \in M^3$, consider the space

$$\tau_m = T_m(M^3) \cap IT_m(M^3). \tag{1.13}$$

Obviously, $\dim \tau_m = 2$ and $I(\tau_m) = \tau_m$. The pseudogroup $\Gamma_s$ induces on $M^3$ the following structure: at each point $m \in M^3$, we have a tangent plane $\tau_m$ and its endomorphism $I_m : \tau_m \to \tau_m$ satisfying $I_m^2 = - id$; further, there is given a 2-form $\varphi^\star$ (the restriction of $\varphi$) on $M^3$ such that

$$\varphi^\star(v_m, w_m) = - \varphi^\star(I_m v_m, I_m w_m) \qquad \text{for } v_m, w_m \in \tau_m.$$

Of course, $\varphi^\star \not\equiv 0$.

**2.** Let us suppose that the field of planes $\tau_m$ is non-integrable. Let us investigate this supposition more carefully. Define a partial complex structure on a manifold $X$, $\dim X = p$, as an assignment of a tangent space $\tau_x \subset T_x(X)$ and an endomorphism $I_x : \tau_x \to \tau_x$, $I_x^2 = - id$, to each point $x \in X$; let $\dim \tau_x = 2q$. Consider a fixed point $x_0 \in X$ and its neighbourhood $U$ such that there are tangent vector fields $v_1, \ldots, v_q, w_1, \ldots, w_q, u_1, \ldots, u_{p-2q}$ in $U$ satisfying $v_i(x), w_i(x) \in \tau_x$ and $I_x v_i(x) = w_i(x)$ in $U$; write $i, j, \ldots = 1, \ldots, q$; $\alpha, \beta, \ldots = 1, \ldots, p - 2q$. Then

$$\begin{aligned} [v_i, v_j] &= a_{ij}^k v_k + b_{ij}^k w_k + c_{ij}^a u_a, \\ [v_i, w_j] &= d_{ij}^k v_k + e_{ij}^k w_k + f_{ij}^a u_a, \\ [w_i, w_j] &= g_{ij}^k v_k + h_{ij}^k w_k + k_{ij}^a u_a. \end{aligned} \tag{2.1}$$

Let $V_0 \in \tau_{x_0}$ be a fixed vector. On $U$, consider an arbitrary vector field $V$ such that $V(x_0) = V_0$ and $V(x) \in \tau_x$ for each $x \in U$. Then there are functions $p^i$, $q^i$ (on $U$) such that

$$V = p^i v_i - q^i w_i. \tag{2.2}$$

At each point $x \in U$, consider the vector $IV$; of course,

$$IV = q^i v_i + p^i w_i. \tag{2.3}$$

We have

$$
\begin{aligned}
[V, IV] &= [p^i v_i - q^i w_i, q^j v_j + p^j w_j] = \qquad (2.4)\\
&= (p^i \,.\, v_i q^k - q^i \,.\, w_i q^k - q^i \,.\, v_i p^k - p^i \,.\, w_i p^k + a^k_{ij} p^i q^j + \\
&+ d^k_{ij} p^i p^j + d^k_{ij} q^i q^j - g^k_{ij} p^j q^i)\, v_k + (p^i \,.\, v_i p^k - q^i \,.\, w_i p^k + \\
&+ q^i \,.\, v_i q^k + p^i \,.\, w_i q^k + b^k_{ij} p^i q^j + e^k_{ij} p^i p^j + e^k_{ij} q^i q^j - \\
&- h^k_{ij} q^i p^j)\, w_k + (c^a_{ij} p^i q^j + f^a_{ij} p^i p^j + f^a_{ij} q^i q^j - k^a_{ij} p^j q^i)\, u_a.
\end{aligned}
$$

Let $\pi_x : T_x(X) \to T_x(X)/\tau_x$ be the natural projection. We see from (2.4) that

$$L_{x_0}(V_0) = \pi_{x_0}([V, IV](x_0)) \in T_{x_0}(X)/\tau_{x_0} \qquad (2.5)$$

does not depend on the choice of the field $V$ extending the vector $V_0$. Thus we get a well defined map

$$L_x : \tau_x \to T_x(X)/\tau_x \qquad (2.6)$$

which is called the *Levi map* of the given partial complex structure (at the point $x \in X$). If $v_i, w_i, u_a \in T_x(X)$ as above and $\tilde{u}_a = \pi(u_a) \in T_x(X)/\tau_x$, then

$$
\begin{aligned}
L_x(V) &\equiv L_x(p^i v_i - q^i w_i) = \qquad (2.7)\\
&= (c^a_{ij} p^i q^j + f^a_{ij} p^i p^j + f^a_{ij} q^i q^j - k^a_{ij} p^j q^i)\, \tilde{u}_a.
\end{aligned}
$$

From this and (2.1), we see that the field $\{\tau_x\}$ is integrable if and only if $L_x(V) = 0$ for each $x \in X$ and each $V \in \tau_x$.

To compare our notion of the Levi map with the well established notion of the Levi map used in the literature, let us calculate the Levi map of a real hypersurface $X^{2n-1} \subset \mathbf{C}^n$. Suppose that $X^{2n-1}$ is given by the equation

$$F(z^1, \ldots, z^n, \bar{z}^1, \ldots, \bar{z}^n) = 0 \qquad (2.8)$$

in the neighbourhood of the point $z^1 = 0, \ldots, z^n = 0$. Of course,

$$F(z^1, \ldots, z^n, \bar{z}^1, \ldots, \bar{z}^n) = \overline{F(z^1, \ldots, z^n, \bar{z}^1, \ldots, \bar{z}^n)}, \qquad (2.9)$$

$F(z^i, \bar{z}^i)$ being a real function. In a suitable small neighbourhood of the origin of $\mathbf{C}^n$, consider the one-paramateric set of hypersurfaces

$$F(z^1, \ldots, z^n, \bar{z}^1, \ldots, \bar{z}^n) = \alpha, \quad \alpha \in (-\varepsilon, \varepsilon). \qquad (2.10)$$

Let $v$ be a real vector field around the origin of $\mathbf{C}^n$. Then

$$v = A^i \frac{\partial}{\partial z^i} + \bar{A}^i \frac{\partial}{\partial \bar{z}^i}\,, \qquad (2.11)$$

and the vector field $Iv$ is given by

$$Iv = iA^i \frac{\partial}{\partial z^i} - i\bar{A}^i \frac{\partial}{\partial \bar{z}^i}\,. \qquad (2.12)$$

Indeed, write $z^i = x^i + iy^i$, and (as usual)

$$\frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - i\frac{\partial}{\partial y^i}\right), \quad \frac{\partial}{\partial \bar{z}^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} + i\frac{\partial}{\partial y^i}\right).$$

Then

$$\frac{\partial}{\partial x^i} = \frac{\partial}{\partial z^i} + \frac{\partial}{\partial \bar{z}^i}, \quad \frac{\partial}{\partial y^i} = i\left(\frac{\partial}{\partial z^i} - \frac{\partial}{\partial \bar{z}^i}\right)$$

and

$$I\frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i}, \quad I\frac{\partial}{\partial y^i} = -\frac{\partial}{\partial x^i}.$$

Then

$$v = a^i\frac{\partial}{\partial x^i} + b^i\frac{\partial}{\partial y^i} = (a^i + ib^i)\frac{\partial}{\partial z^i} + (a^i - ib^i)\frac{\partial}{\partial \bar{z}^i},$$

$$Iv = -b^i\frac{\partial}{\partial x^i} + a^i\frac{\partial}{\partial y^i} = (-b^i + ia^i)\frac{\partial}{\partial z^i} + (-b^i - ia^i)\frac{\partial}{\partial \bar{z}^i} =$$

$$= i(a^i + ib^i)\frac{\partial}{\partial z^i} - i(a^i - ib^i)\frac{\partial}{\partial \bar{z}^i}.$$

We are looking now for the vector fields $v$ (2.11) which are tangent to the hypersurfaces (2.10), the vector fields $Iv$ (2.12) having the same property. This yields

$$A^i\frac{\partial F}{\partial z^i} + \bar{A}^i\frac{\partial F}{\partial \bar{z}^i} = 0, \quad iA^i\frac{\partial F}{\partial z^i} - i\bar{A}^i\frac{\partial F}{\partial \bar{z}^i} = 0,$$

i.e.,

$$A^i\frac{\partial F}{\partial z^i} = 0, \quad \bar{A}^i\frac{\partial F}{\partial \bar{z}^i} = 0. \tag{2.13}$$

Because of $F = \bar{F}$, we have

$$\frac{\partial F}{\partial \bar{z}^i} = \overline{\frac{\partial F}{\partial z^i}};$$

indeed, write $F(z^i, \bar{z}^i) = f(x^1, \ldots, x^n, y^1, \ldots, y^n)$, then

$$\frac{\partial F}{\partial z^i} = \frac{1}{2}\left(\frac{\partial f}{\partial x^i} - i\frac{\partial f}{\partial y^i}\right), \quad \frac{\partial F}{\partial \bar{z}^i} = \frac{1}{2}\left(\frac{\partial f}{\partial x^i} + i\frac{\partial f}{\partial y^i}\right);$$

Thus the system (2.13) is equivalent to

$$A^i\frac{\partial F}{\partial z^i} = 0. \tag{2.14}$$

It is easy to see that the coordinates $z^i$ in $\mathbf{C}^n$ may be chosen in such a way (by a linear change) that

$$\begin{gathered} F(z^1, \ldots, z^n, \bar{z}^1, \ldots, \bar{z}^n) = z^n + \bar{z}^n + G(z^1, \ldots, z^{n-1}, \bar{z}^1, \ldots, \bar{z}^{n-1}, \bar{z}^n - z^n); \\ G(0, \ldots, 0) = 0; \\ \frac{\partial G(0, \ldots, 0)}{\partial z^\alpha} = 0, \quad \frac{\partial G(0, \ldots, 0)}{\partial \bar{z}^\alpha} = 0, \quad \frac{\partial G(0, \ldots, 0)}{\partial(\bar{z}^n - z^n)} = 0 \end{gathered} \tag{2.15}$$

for $\alpha = 1, \ldots, n-1$.

The geometrical meaning is very simple: The tangent hyperplane $T_0(X^{2n-1})$ at the origin is given by $z^n + \bar{z}^n = 0$, i.e., $x^n = 0$.

Of course, $\partial F(z^i, \bar{z}^i)/\partial z^n \neq 0$ in a neighbourhood of the origin, and (2.14) may be written as

$$A^\alpha \frac{\partial F}{\partial z^\alpha} + A^n \frac{\partial F}{\partial z^n} = 0 \qquad (\alpha, \beta, \ldots = 1, \ldots, n-1). \tag{2.16}$$

Its general solution is given by

$$A^\alpha = B^\alpha \frac{\partial F}{\partial z^n}, \quad A^n = - B^\alpha \frac{\partial F}{\partial z^\alpha},$$

$B^1, \ldots, B^{n-1}$ being arbitrary complex-valued functions, and we get

$$v = B^\alpha \frac{\partial F}{\partial z^n}\frac{\partial}{\partial z^\alpha} - B^\alpha \frac{\partial F}{\partial z^\alpha}\frac{\partial}{\partial z^n} + \overline{B^\alpha}\frac{\partial F}{\partial \bar{z}^n}\frac{\partial}{\partial \bar{z}^\alpha} - \overline{B^\alpha}\frac{\partial F}{\partial \bar{z}^\alpha}\frac{\partial}{\partial \bar{z}^n}, \tag{2.17}$$

$$Iv = iB^\alpha \frac{\partial F}{\partial z^n}\frac{\partial}{\partial z^\beta} - iB^\beta \frac{\partial F}{\partial z^\beta}\frac{\partial}{\partial z^n} - i\overline{B^\beta}\frac{\partial F}{\partial \bar{z}^n}\frac{\partial}{\partial \bar{z}^\beta} + i\overline{B^\beta}\frac{\partial F}{\partial \bar{z}^\beta}\frac{\partial}{\partial \bar{z}^n}.$$

At the origin of $\mathbf{C}^n$, we have

$$\frac{\partial F(0, \ldots, 0)}{\partial z^n} = 1, \quad \frac{\partial F(0, \ldots, 0)}{\partial \bar{z}^n} = 1, \quad \frac{\partial F(0, \ldots, 0)}{\partial z^\alpha} = 0, \quad \frac{\partial F(0, \ldots, 0)}{\partial \bar{z}^\alpha} = 0,$$

and the vectors $(2.17_1)$ are given by

$$v = B^\alpha \frac{\partial}{\partial z^\alpha} + \overline{B^\alpha}\frac{\partial}{\partial \bar{z}^\alpha}. \tag{2.18}$$

Thus the space $\tau_0 \subset T_0(X^{2n-1})$ is spanned by the vectors

$$\frac{\partial}{\partial z^\alpha} + \frac{\partial}{\partial \bar{z}^\alpha}, \quad i\frac{\partial}{\partial z^\alpha} - i\frac{\partial}{\partial \bar{z}^\alpha}; \quad \alpha = 1, \ldots, n-1.$$

From (2.17), we get

$$[v, Iv]_0 = B^\alpha . iB^\beta \frac{\partial^2 F(0)}{\partial z^\alpha \partial z^n}\frac{\partial}{\partial z^\beta} - B^\alpha . iB^\beta \frac{\partial^2 F(0)}{\partial z^\alpha \partial z^\beta}\frac{\partial}{\partial z^n} -$$

$$- B^\alpha . i\overline{B^\beta}\frac{\partial^2 F(0)}{\partial z^\alpha \partial \bar{z}^n}\frac{\partial}{\partial \bar{z}^\beta} + B^\alpha . i\overline{B^\beta}\frac{\partial^2 F(0)}{\partial z^\alpha \partial \bar{z}^\beta}\frac{\partial}{\partial \bar{z}^n} + \overline{B^\alpha} . iB^\beta \frac{\partial^2 F(0)}{\partial \bar{z}^\alpha \partial z^n}\frac{\partial}{\partial z^\beta} -$$

$$- \overline{B^\alpha} . iB^\beta \frac{\partial^2 F(0)}{\partial \bar{z}^\alpha \partial z^\beta}\frac{\partial}{\partial z^n} - \overline{B^\alpha} . i\overline{B^\beta}\frac{\partial^2 F(0)}{\partial \bar{z}^\alpha \partial \bar{z}^n}\frac{\partial}{\partial \bar{z}^\beta} + \overline{B^\alpha} . i\overline{B^\beta}\frac{\partial^2 F(0)}{\partial \bar{z}^\alpha \partial \bar{z}^\beta}\frac{\partial}{\partial \bar{z}^n} -$$

$$- iB^\beta B^\alpha \frac{\partial^2 F(0)}{\partial z^\beta \partial z^n}\frac{\partial}{\partial z^\alpha} + iB^\beta B^\alpha \frac{\partial^2 F(0)}{\partial z^\alpha \partial z^\beta}\frac{\partial}{\partial z^n} - iB^\beta \overline{B^\alpha}\frac{\partial^2 F(0)}{\partial z^\beta \partial \bar{z}^n}\frac{\partial}{\partial \bar{z}^\alpha} +$$

$$+ iB^\beta \overline{B^\alpha}\frac{\partial^2 F(0)}{\partial z^\beta \partial \bar{z}^\alpha}\frac{\partial}{\partial \bar{z}^n} + i\overline{B^\beta} B^\alpha \frac{\partial^2 F(0)}{\partial \bar{z}^\beta \partial z^n}\frac{\partial}{\partial z^\alpha} - i\overline{B^\beta} B^\alpha \frac{\partial^2 F(0)}{\partial z^\alpha \partial \bar{z}^\beta}\frac{\partial}{\partial z^n} +$$

$$+ i\overline{B^\beta}\overline{B^\alpha}\frac{\partial^2 F(0)}{\partial \bar{z}^\beta \partial \bar{z}^n}\frac{\partial}{\partial \bar{z}^\alpha} - i\overline{B^\beta}\overline{B^\alpha}\frac{\partial^2 F(0)}{\partial \bar{z}^\alpha \partial \bar{z}^\beta}\frac{\partial}{\partial \bar{z}^n} =$$

$$= 2iB^\alpha \overline{B}^\beta \frac{\partial^2 F(0)}{\partial \bar{z}^\beta \partial z^n} \frac{\partial}{\partial z^\alpha} - 2iB^\beta \overline{B}^\alpha \frac{\partial^2 F(0)}{\partial z^\beta \partial \bar{z}^n} \frac{\partial}{\partial \bar{z}^\alpha} -$$

$$- 2i\overline{B}^\alpha B^\beta \frac{\partial^2 F(0)}{\partial \bar{z}^\alpha \partial z^\beta} \frac{\partial}{\partial z^n} + 2iB^\alpha \overline{B}^\beta \frac{\partial^2 F(0)}{\partial z^\alpha \partial \bar{z}^\beta} \frac{\partial}{\partial \bar{z}^n} =$$

$$= i \left( B^\alpha \bar{B}^\beta \frac{\partial^2 F(0)}{\partial \bar{z}^\beta \partial z^n} - B^\beta \overline{B}^\alpha \frac{\partial^2 F(0)}{\partial z^\beta \partial \bar{z}^n} \right) \left( \frac{\partial}{\partial z^\alpha} + \frac{\partial}{\partial \bar{z}^\alpha} \right) +$$

$$+ \left( B^\alpha \overline{B}^\beta \frac{\partial^2 F(0)}{\partial \bar{z}^\beta \partial z^n} + B^\beta \overline{B}^\alpha \frac{\partial^2 F(0)}{\partial z^\beta \partial \bar{z}^n} \right) . i \left( \frac{\partial}{\partial z^\alpha} - \frac{\partial}{\partial \bar{z}^\alpha} \right) -$$

$$- 2i B^\alpha \overline{B}^\beta \frac{\partial^2 F(0)}{\partial z^\alpha \partial \bar{z}^\beta} \left( \frac{\partial}{\partial z^n} - \frac{\partial}{\partial \bar{z}^n} \right) .$$

Of course,

$$\frac{\partial}{\partial y^n} = i \left( \frac{\partial}{\partial z^n} - \frac{\partial}{\partial \bar{z}^n} \right) .$$

Consider once again the natural projection $\pi_0 : \tau_0 \to T_0(X^{2n-1})/\tau_0$, and write

$$\pi_0 \left( \frac{\partial}{\partial y^n} \right) = u;$$

then

$$L_0 \left( B^\alpha \frac{\partial}{\partial z^\alpha} + \overline{B}^\alpha \frac{\partial}{\partial \bar{z}^\alpha} \right) = - 2 B^\alpha \overline{B}^\beta \frac{\partial^2 F(0)}{\partial z^\alpha \partial \bar{z}^\beta} u.$$

This is the classical formula for the Levi map. It is easy to prove that $L_x = 0$ at each point $x \in X^{2n-1}$ is equivalent to the condition that $X^{2n-1}$ is locally holomorphically equivalent to a hyperplane of $\mathbf{C}^n$.

**3.** Let us consider a manifold $M^3$ with the structure described at the end of No 1. At each point $m \in M^3$, let us choose a frame $(v_1, v_2, v_3)$, $v_i \in T_m(M^3)$, such that $\tau_m$ is spanned by $v_1, v_2$ and $I_m v_1 = v_2$. Each other frame of the same type is given by

$$w_1 = \alpha v_1 - \beta v_2, \quad w_2 = \beta v_1 + \alpha v_2, \qquad (3.1)$$
$$w_3 = \gamma v_1 + \delta v_2 + \varphi v_3; \quad (\alpha^2 + \beta^2)\varphi \neq 0.$$

Let $v, v' \in T_m(M^3)$,

$$v = a v_1 + b v_2 + c v_3, \quad v' = a' v_1 + b' v_2 + c' v_3. \qquad (3.2)$$

Then

$$\varphi^\star(v, v') = A(ab' - a'b) + B(ac' - a'c) + C(bc' - b'c), \qquad (3.3)$$

where $A, B, C$ are reals. For $v, v' \in \tau_m$, we have $c = c' = 0$ and

$$\varphi^\star(v, v') = A(ab' - a'b), \quad \varphi^\star(Iv, Iv') = A(ab' - a'b).$$

From the condition $\varphi^\star(v_m, w_m) = - \varphi^\star(I_m v_m, I_m w_m)$, we get $A = 0$. Let

$$v = \tilde{a} w_1 + \tilde{b} w_2 + \tilde{c} w_2, \quad v' = \tilde{a}' w_1 + \tilde{b}' w_2 + \tilde{c}' w_3,$$

$w_1, w_2, w_3$ being given by (3.1). Write

$$\varphi^\star(v, v') = \tilde{B}(\tilde{a}\tilde{c}' - \tilde{a}'\tilde{c}) + \tilde{C}(\tilde{b}\tilde{c}' - \tilde{b}'\tilde{c}).$$

Then

$$a = \alpha\tilde{a} + \beta\tilde{b}, \quad b = -\beta\tilde{a} + \alpha\tilde{b}, \quad c = \varphi\tilde{c}$$

and

$$\tilde{B} = \varphi(\alpha B - \beta C), \quad \tilde{C} = \varphi(\beta B + \alpha C). \tag{3.4}$$

The case $B = C = 0$ being excluded (otherwise $\varphi^\star \equiv 0$), there exist frames $(w_1, w_2, w_3)$ with $\tilde{B} = 1$, $\tilde{C} = 0$, and we have the following result: On $M^3$, the considered structure induces a $G$-structure $B_G(M^3)$ such that $(v_1, v_2, v_3)_m \in B_G(M^3)$ if and only if $v_1, v_2 \in \tau_m$, $I_m v_1 = v_2$ and $\varphi^\star(v, v') = ac' - a'c$, $v$ and $v'$ being given by (3.2); if $\{w_1, w_2, w_3\}_m \in B_G(M^3)$, then

$$w_1 = \alpha v_1, \quad w_2 = \alpha v_2, \quad w_3 = \gamma v_1 + \delta v_2 + \alpha^{-1} v_3; \quad \alpha \neq 0. \tag{3.5}$$

The last assertion follows easily from (3.4); indeed, we should have $1 = \alpha\varphi$, $0 = \beta\varphi_x$.

Consider a $G$-structure $B_G(M^3)$ of this type, i.e., $G$ is the group of the matrices

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ \gamma & \delta & \alpha^{-1} \end{pmatrix}, \quad \alpha \neq 0. \tag{3.6}$$

In a domain $V \subset M^3$, choose a section $(v_1, v_2, v_3)$ of $B_G(M^3)$; then

$$\begin{aligned} [v_1, v_2] &= a_1 v_1 + a_2 v_2 + a_3 v_3, \\ [v_1, v_3] &= b_1 v_1 + b_2 v_2 + b_3 v_3, \\ [v_2, v_3] &= c_1 v_1 + c_2 v_2 + c_3 v_3, \end{aligned} \tag{3.7}$$

$a_1, \ldots, c_3$ being functions on $V$. In what follows, let us restrict ourselves to manifolds with non-integrable field of planes $\tau_m$; thus $a_3 \neq 0$ on $V$. From the Jacobi identity

$$[v_1,[v_2, v_3]] + [v_2,[v_3, v_1]] + [v_3,[v_1, v_2]] = 0,$$

we get

$$\begin{aligned} v_1 c_1 - v_2 b_1 + v_3 a_1 + a_1 c_2 + b_1 c_3 - b_3 c_1 - a_2 c_1 &= 0, \\ v_1 c_2 - v_2 b_2 + v_3 a_2 + b_2 c_3 + a_2 b_1 - b_3 c_2 - a_1 b_2 &= 0, \\ v_1 c_3 - v_2 b_3 + v_3 a_3 + a_3 c_2 + a_3 b_1 - a_1 b_3 - a_2 c_3 &= 0. \end{aligned} \tag{3.8}$$

Let $(w_1, w_2, w_3)$ be another section of $B_G(M^3)$, let us have (3.5) with $\alpha, \gamma, \delta$ real-valued functions on $V$. Then

$$\begin{aligned} [w_1, w_2] &= A_1 w_1 + A_2 w_2 + A_3 w_3, \\ [w_1, w_3] &= B_1 w_1 + B_2 w_2 + B_3 w_3, \\ [w_2, w_3] &= C_1 w_1 + C_2 w_2 + C_3 w_3. \end{aligned} \tag{3.9}$$

We have

$$\begin{aligned} [w_1, w_2] = [\alpha v_1, \alpha v_2] &= \alpha \,.\, v_1\alpha \,.\, v_2 - \alpha \,.\, v_2\alpha \,.\, v_1 + \alpha^2(a_1 v_1 + a_2 v_2 + a_3 v_3) = \\ &= A_1 \alpha v_1 + A_2 \alpha v_2 + A_3(\gamma v_1 + \delta v_2 + \alpha^{-1} v_3), \end{aligned}$$

i.e.

$$-\alpha \,.\, v_2\alpha + \alpha^2 a_1 = \alpha A_1 + \gamma A_3, \ \alpha \,.\, v_1\alpha + \alpha^2 a_2 = \alpha A_2 + \delta A_3, \ \alpha^2 a_3 = \alpha^{-1} A_3. \tag{3.10}$$

Thus there exists a section $(w_1, w_2, w_3)$ satisfying $A_3 = 1$, $A_1 = A_2 = 0$, and we have the following result: There exists (locally) exactly one section $(v_1, v_2, v_3)$ of $B_G(M^3)$ satisfying

$$\begin{aligned} [v_1, v_2] &= v_3, \\ [v_1, v_3] &= b_1 v_1 + b_2 v_2 + b_3 v_3, \\ [v_2, v_3] &= c_1 v_1 + c_2 v_2 + c_3 v_3. \end{aligned} \tag{3.11}$$

The integrability conditions (3.8) reduce to

$$\begin{aligned} v_1 c_1 - v_2 b_1 + b_1 c_3 - b_3 c_1 &= 0, \\ v_1 c_2 - v_2 b_2 + b_2 c_3 - b_3 c_2 &= 0, \\ v_1 c_3 - v_2 b_3 + c_2 + b_1 &= 0. \end{aligned} \tag{3.12}$$

Now, let $B_G(M^3)$ be transitive. Then $b_1, \ldots, c_3$ are constants, and the equations (3.12) reduce to

$$b_1 c_3 - b_3 c_1 = 0, \quad b_2 c_3 - b_3 c_2 = 0, \quad c_2 + b_1 = 0. \tag{3.13}$$

Let $b_3 c_3 \neq 0$. Then there are real numbers $A, B, C$ such that

$$\begin{aligned} [v_1, v_2] &= v_3, \\ [v_1, v_3] &= ABCv_1 - AB^2 v_2 + Bv_3, \\ [v_2, v_3] &= AC^2 v_1 - ABCv_2 + Cv_3; \qquad BC \neq 0. \end{aligned} \tag{3.14}$$

Let $b_3 \neq 0$, $c_3 = 0$. Then $c_1 = c_2 = b_1 = 0$ and (3.11) are of the form

$$\begin{aligned} [v_1, v_2] &= v_3, \\ [v_1, v_3] &= Av_2 + Bv_3, \\ [v_2, v_3] &= 0; \qquad B \neq 0; \end{aligned} \tag{3.15}$$

the case $b_3 = 0$, $c_3 \neq 0$ is symmetric. For $b_3 = c_3 = 0$, we get

$$\begin{aligned} [v_1, v_2] &= v_3, \\ [v_1, v_3] &= Av_1 + Bv_2, \\ [v_2, v_3] &= Cv_1 - Av_2. \end{aligned} \tag{3.16}$$

The following result follows: *The Lie algebra of G* (see the Theorem) *is of the type* (3.14) *or* (3.15) *or* (3.16) resp.

Finally, let us prove the existence of the transitive $G$-structures of the types (3.14)–(3.16). A simple check shows that the vector fields

$$\begin{aligned} u_1 &= \frac{1}{2}(1 + 2y - 3x^2)\frac{\partial}{\partial x} + \frac{1}{2}(2x + z - 3xy)\frac{\partial}{\partial y} + \frac{3}{2}(y - xz)\frac{\partial}{\partial z}, \\ u_2 &= \frac{1}{2}(1 - 2y + 3x^2)\frac{\partial}{\partial x} + \frac{1}{2}(2x - z + 3xy)\frac{\partial}{\partial y} + \frac{3}{2}(y + xz)\frac{\partial}{\partial z}, \\ u_3 &= x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 3z\frac{\partial}{\partial z} \end{aligned} \tag{3.17}$$

on $\mathbf{R}^3$ satisfy

$$[u_1, u_2] = u_3, \quad [u_1, u_3] = u_2, \quad [u_2, u_3] = u_1. \tag{3.18}$$

In a suitable neighbourhood of the point $(\frac{1}{4}\pi, 0, 0) \in \mathbf{R}^3$, consider the vector fields

$$w_1 = \sin(y+z)\frac{\partial}{\partial x} + \frac{\cos x}{\sin x}\cos(y+z)\frac{\partial}{\partial y} - \frac{\sin x}{\cos x}\cos(y+z)\frac{\partial}{\partial z},$$

$$w_2 = \cos(y+z)\frac{\partial}{\partial x} - \frac{\cos x}{\sin x}\sin(y+z)\frac{\partial}{\partial y} + \frac{\sin x}{\cos x}\sin(y+z)\frac{\partial}{\partial z},$$

$$w_3 = \frac{\partial}{\partial y} + \frac{\partial}{\partial z}; \tag{3.19}$$

the direct check proves

$$[w_1, w_2] = 2w_3, \quad [w_1, w_3] = -2w_2, \quad [w_2, w_3] = 2w_1. \tag{3.20}$$

Now, consider the $G$-structure (3.14). Obviously, $[Cv_1 - Bv_2, v_3] = 0$. On a neighbourhood of a point $m_0 \in M^3$, consider local coordinates $(x, y, z)$ such that

$$Cv_1 - Bv_2 = \frac{\partial}{\partial y}, \quad v_3 = \frac{\partial}{\partial x},$$

this being always possible. Let

$$v_2 = \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y} + \gamma\frac{\partial}{\partial z}, \quad \text{i.e.,} \quad Cv_1 = B\alpha\frac{\partial}{\partial x} + (B\beta + 1)\frac{\partial}{\partial y} + B\gamma\frac{\partial}{\partial z}.$$

From $(3.14_{1,2})$, we get

$$\frac{\partial\alpha}{\partial y} = C; \quad \frac{\partial\beta}{\partial y} = 0, \quad \frac{\partial\gamma}{\partial y} = 0, \quad \frac{\partial\alpha}{\partial x} = -C, \quad \frac{\partial\beta}{\partial x} = -AC, \quad \frac{\partial\gamma}{\partial x} = 0.$$

Consider the particular solution $\alpha = C(y-x)$, $\beta = -ACx$, $\gamma = 1$. Then

$$v_1 = B(y-x)\frac{\partial}{\partial x} + (C^{-1} - ABx)\frac{\partial}{\partial y} + BC^{-1}\frac{\partial}{\partial z}, \tag{3.21}$$

$$v_2 = C(y-x)\frac{\partial}{\partial x} - ACx\frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

$$v_3 = \frac{\partial}{\partial x};$$

this vectors being linearly independent and satisfying (3.14), they generate a $G$-structure of the type (3.14) on $\mathbf{R}^3$. Similarly, the vector fields

$$v_1 = -(Bx+y)\frac{\partial}{\partial x} - Ax\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \quad v_2 = \frac{\partial}{\partial y}, \quad v_3 = \frac{\partial}{\partial x} \tag{3.22}$$

generate a $G$-structure of the type (3.15) on $\mathbf{R}^3$. The type (3.16) is a little more complicated. First of all, suppose $A = B = 0$; the $G$-structure of this type on $\mathbf{R}^3$ is generated by the vector fields

$$v_1 = \frac{\partial}{\partial x}, \quad v_2 = -Cy\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \quad v_3 = \frac{\partial}{\partial y}. \tag{3.23}$$

Similarly, the $G$-structure of the type (3.16) with $A = C = 0$ is generated by the vector fields

$$v_1 = -By\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \quad v_2 = \frac{\partial}{\partial x}, \quad v_3 = \frac{\partial}{\partial y}. \tag{3.24}$$

Now, consider the case $A^2 + BC = 0$, $AB \neq 0$, i.e.,

$$[v_1, v_2] = v_3, \quad [v_1, v_3] = Av_1 + Bv_2, \quad [v_2, v_3] = -\frac{A^2}{B}v_1 - Av_2.$$

We see that $[Av_1 + Bv_2, v_3] = 0$, and the vector fields

$$v_1 = -By\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \tag{3.25}$$

$$v_2 = ABy\frac{\partial}{\partial x} + (1 + Ax)\frac{\partial}{\partial y} - A\frac{\partial}{\partial z},$$

$$v_3 = \frac{\partial}{\partial x}$$

generate the $G$-structure of this type on $\mathbf{R}^3$. If $A^2 + BC \neq 0$ then the Lie algebra (3.16) $L$ satisfies $[L, L] = L$ and it contains a basis $(u_1, u_2, u_3)$ satisfying (3.18) or a basis $(w_1, w_2, w_3)$ satisfying (3.20).

**4.** Consider the space $\mathbf{C}^2$ and the pseudogroup $\Gamma$. The relation between the 1-parametric local subgroups of $\Gamma$ and the holomorphic vector fields on $\mathbf{C}^2$ is well known. Let

$$v = a(x, y)\frac{\partial}{\partial x} + b(x, y)\frac{\partial}{\partial y} \tag{4.1}$$

be a (locally defined) holomorphic vector field; the corresponding local group $G_v$ consists of the maps

$$\varphi_t\colon \quad \tilde{x} = f(x, y, t), \quad \tilde{y} = g(x, y, t), \quad t \in (-\varepsilon, \varepsilon) \tag{4.2}$$

given by

$$\frac{\partial f(x, y, t)}{\partial x} = a(f(x, y, t), g(x, y, t)), \frac{\partial g(x, y, t)}{\partial t} = b(f(x, y, t), g(x, y, t)),$$

$$f(x, y, 0) = x, \quad g(x, y, 0) = y. \tag{4.3}$$

*We have* $G_v \subset \Gamma_s$ *if and only if*

$$\frac{\partial a(x, y)}{\partial x} + \frac{\partial b(x, y)}{\partial y} = 0. \tag{4.4}$$

Indeed, let us write

$$D(x, y, t) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.$$

We have $D(x, y, 0) = 1$. From (4.3), we get

$$\frac{\partial D}{\partial t} = \left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y}\right) D,$$

and the result follows easily. Denote by $L_s$ the Lie algebra of holomorphic vector fields (4.1) on $\mathbf{C}^2$ satisfying (4.4).

*Let* $w_1, w_2 \in L_s$, $w_1 \neq 0 \neq w_2$, $[w_1, w_2] = 0$; *then there are (locally)* $\Gamma_s$-*coordinates* $(u, v)$ *such that*

$$w_1 = \frac{\partial}{\partial u}, w_2 = \alpha \frac{\partial}{\partial v} (0 \neq \alpha \in \mathbf{C}) \text{ or } w_2 = a(v) \frac{\partial}{\partial u} \text{ resp.} \tag{4.5}$$

Here, the $\Gamma_s$-coordinates $(u, v)$ are defined (locally) as holomorphic coordinates $u = u(x, y)$, $v = v(x, y)$ satisfying $\partial(u, v)/\partial(x, y) = 1$. Indeed, we may choose (at least locally) $\Gamma_s$-coordinates $r = r(x, y)$, $s = s(x, y)$ such that $w_1 = \partial/\partial r$. Let

$$w_2 = b(r, s) \frac{\partial}{\partial r} + c(r, s) \frac{\partial}{\partial s}, \quad \frac{\partial b}{\partial r} + \frac{\partial c}{\partial s} = 0.$$

From $[w_1, w_2] = 0$, we get

$$\frac{\partial b}{\partial r} = 0, \quad \frac{\partial c}{\partial r} = 0.$$

Thus $b = b(s), c = \alpha \in \mathbf{C}$. Now, consider the $\Gamma_s$-coordinates $u = u(r, s)$, $v = v(r, s)$. Then

$$w_1 = \frac{\partial u}{\partial r} \frac{\partial}{\partial u} + \frac{\partial v}{\partial r} \frac{\partial}{\partial v},$$

$$w_2 = b(s) \left( \frac{\partial u}{\partial r} \frac{\partial}{\partial u} + \frac{\partial v}{\partial r} \frac{\partial}{\partial v} \right) + \alpha \left( \frac{\partial u}{\partial s} \frac{\partial}{\partial u} + \frac{\partial v}{\partial s} \frac{\partial}{\partial v} \right).$$

We have

$$\frac{\partial u}{\partial r} = 1, \quad \frac{\partial v}{\partial r} = 0 \quad \text{and} \quad \frac{\partial v}{\partial s} = 1,$$

i.e., $u = r + g(s)$, $v = s + \varrho$, $\varrho \in \mathbf{C}$, and

$$w_2 = \left( b + \alpha \frac{dg}{ds} \right) \frac{\partial}{\partial u} + \alpha \frac{\partial}{\partial v}.$$

If $\alpha \neq 0$, let us choose $g(s)$ such that

$$\frac{dg(s)}{ds} = - \frac{b(s)}{\alpha}.$$

**5.** Let $L$ be a Lie algebra of the type (3.14), suppose $L \subset L_s$. Then

$$\left[ v_2 - \frac{C}{B} v_1, v_3 \right] = 0,$$

and we may choose (locally) $\Gamma_s$-coordinates $(u, v)$ such that

$$v_3 = \frac{\partial}{\partial u}, \quad v_2 - \frac{C}{B} v_1 = \alpha \frac{\partial}{\partial v}; \quad 0 \neq \alpha \in \mathbf{C}; \tag{5.1}$$

or

$$v_3 = \frac{\partial}{\partial u}, \quad v_2 - \frac{C}{B} v_1 = a(v) \frac{\partial}{\partial u} \tag{5.2}$$

resp. Suppose (5.1) and

$$v_1 = b(u, v)\frac{\partial}{\partial u} + c(u, v)\frac{\partial}{\partial v}, \quad \frac{\partial b}{\partial u} + \frac{\partial c}{\partial v} = 0.$$

From $(3.14_2)$, we get

$$-\frac{\partial b}{\partial u} = B, \quad \frac{\partial c}{\partial u} = AB^2\alpha,$$

i.e.,

$$b = -Bu + \beta(v), \quad c = AB^2\alpha u + Bv + \gamma_0; \quad \gamma_0 \in \mathbf{C}.$$

From $(3.14_1)$,

$$\left[v_1, \frac{C}{B}v_1 + \alpha\frac{\partial}{\partial v}\right] = -\alpha\frac{\partial b}{\partial v}\frac{\partial}{\partial u} - \alpha\frac{\partial c}{\partial v}\frac{\partial}{\partial v} = \frac{\partial}{\partial u},$$

i.e., $\partial c/\partial v = B = 0$. Thus we should have (5.2) because of $B \neq 0$. Let further $L$ be of the type (3.15) and $L \subset L_s$. Then there are (locally) $\Gamma_s$-coordinates $(u, v)$ such that

$$v_3 = \frac{\partial}{\partial u}, \quad v_2 = \alpha\frac{\partial}{\partial v}; \quad 0 \neq \alpha \in \mathbf{C}; \tag{5.3}$$

or

$$v_3 = \frac{\partial}{\partial u}, \quad v_2 = a(v)\frac{\partial}{\partial u} \tag{5.4}$$

resp. Suppose (5.3) and let us write

$$v_1 = b(u, v)\frac{\partial}{\partial u} + c(u, v)\frac{\partial}{\partial v}, \quad \frac{\partial b}{\partial u} + \frac{\partial c}{\partial v} = 0.$$

From $(3.15_{1,2})$,

$$\alpha\frac{\partial b}{\partial v} = -1, \quad \frac{\partial c}{\partial v} = 0, \quad \frac{\partial b}{\partial u} = -B, \quad \frac{\partial c}{\partial u} = -A\alpha.$$

Because of $B \neq 0$, we have (5.4).

Now, let $M^3 \subset \mathbf{C}^2 \equiv \mathbf{R}^4$ be the orbit of the group $G \subset \Gamma_s$ such that its Lie algebra $g$ is of the type (3.14) or (3.15) resp. Then we have shown that $g$ contains (in suitable $\Gamma_s$-coordinates) the vector fields $\partial/\partial x$, $a(y)\partial/\partial x$, and the vector fields

$$\frac{\partial}{\partial x^1}, \quad a_1(x^2, y^2)\frac{\partial}{\partial x^1} + a_2(x^2, y^2)\frac{\partial}{\partial y^1}; \quad a(y) = a_1(x^2, y^2) + ia_2(x^2, y^2);$$

are tangent to $M^3 \subset \mathbf{R}^4$. The plane $\tau_m$ is thus spanned by the vectors $\partial/\partial x^1$, $\partial/\partial y^1$, and the field $\tau_m$ is integrable. *The groups $G \subset \Gamma_s$ satisfying the suppositions of the* Theorem *and possessing the Lie algebra of the type* (3.14) *or* (3.15) *do not exist.*

**6.** Let us investigate the case $L \subset L_s$, $L$ being of the type (3.16). Suppose $dim[L, L] = 1$, i.e.,

$$[v_1, v_2] = v_3, \quad [v_1, v_3] = 0, \quad [v_2, v_3] = 0. \tag{6.1}$$

We may suppose the existence of $\Gamma_s$-coordinates $(u, v)$ such that

$$v_2 = \alpha \frac{\partial}{\partial v}, \quad v_3 = \frac{\partial}{\partial u}; \quad 0 \neq \alpha \in C.$$

Let

$$v_1 = b(u, v) \frac{\partial}{\partial u} + c(u, v) \frac{\partial}{\partial v}, \quad \frac{\partial b}{\partial u} + \frac{\partial c}{\partial v} = 0.$$

From $(6.1_{1,2})$, we get

$$\frac{\partial b}{\partial u} = 0, \quad \frac{\partial b}{\partial v} = \frac{1}{\alpha}, \quad \frac{\partial c}{\partial u} = \frac{\partial c}{\partial v} = 0,$$

i.e.,

$$b = -\frac{v}{\alpha} + \beta, \quad c = \gamma; \quad \beta, \gamma \in \mathbf{C};$$

we have $\gamma \neq 0$ because of the non-integrability of the field $\tau_m$. Consider the $\Gamma_s$-coordinates $x = u, y = v - \alpha\beta$. Then

$$v_2 = \alpha \frac{\partial}{\partial y}, \quad v_3 = \frac{\partial}{\partial x}, \quad v_1 = -\frac{v}{\alpha} \frac{\partial}{\partial x} + \gamma \frac{\partial}{\partial y},$$

and the general element of $L$ is

$$v = R\left(-\frac{v}{\alpha} \frac{\partial}{\partial x} + \gamma \frac{\partial}{\partial y}\right) + S\alpha \frac{\partial}{\partial y} + T \frac{\partial}{\partial x}; \quad R, S, T \in \mathbf{R}. \tag{6.2}$$

The associated local group $G_v$ is given by (4.3), i.e.,

$$\frac{\partial f}{\partial t} = -\frac{R}{\alpha} g + T, \quad \frac{\partial g}{\partial t} = R\gamma + S\alpha.$$

It is easy to see that its finite equations are

$$f = x - \frac{Rt}{\alpha} y - \frac{1}{2} RSt^2 - \frac{1}{2} \frac{\gamma}{\alpha} R^2 t^2 + Tt, \quad g = y + \gamma Rt + \alpha St.$$

Write $Rt = a, \quad St = b, \quad Tt = c;$ we get

$$f = x - \frac{a}{\alpha} y - \frac{1}{2} ab - \frac{1}{2} \frac{\gamma}{\alpha} a^2 + c, \quad g = y + \gamma a + \alpha b. \tag{6.3}$$

Thus

$$\bar{f} = \bar{x} - \frac{a}{\bar{\alpha}} \bar{y} - \frac{1}{2} ab - \frac{1}{2} \frac{\bar{\gamma}}{\bar{\alpha}} a^2 + c, \quad \bar{g} = \bar{y} + \bar{\gamma} a + \bar{\alpha} b,$$

i.e.,

$$f - \bar{f} = x - \bar{x} - a\left(\frac{y}{\alpha} - \frac{\bar{y}}{\bar{\alpha}}\right) - \frac{1}{2} a^2 \left(\frac{\gamma}{\alpha} - \frac{\bar{\gamma}}{\bar{\alpha}}\right),$$

$$\frac{g}{\alpha} - \frac{\bar{g}}{\bar{\alpha}} = \frac{y}{\alpha} - \frac{\bar{y}}{\bar{\alpha}} + a\left(\frac{\gamma}{\alpha} - \frac{\bar{\gamma}}{\bar{\alpha}}\right),$$

the elimination of $a$ yields

$$\left(\frac{y}{\alpha}-\frac{\bar{y}}{\bar{\alpha}}\right)^2+2\left(\frac{\gamma}{\alpha}-\frac{\bar{\gamma}}{\bar{\alpha}}\right)(x-\bar{x})=\left(\frac{g}{\alpha}-\frac{\bar{g}}{\bar{\alpha}}\right)^2+2\left(\frac{\gamma}{\alpha}-\frac{\bar{\gamma}}{\bar{\alpha}}\right)(f-\bar{f}),$$

and we get the type (I).

Let us investigate the case $L \subset L_s$, $L$ being of the type (3.16) with $dim[L, L] = 2$. Then $A^2 + BC = 0$. First of all, suppose $A = B = 0$, the case $A = C = 0$ being symmetric. The algebra $L$ is of the type

$$[v, v_2] = v_3, \quad [v_1, v_3] = Bv_2, \quad [v_2, v_3] = 0; \quad B \neq 0. \tag{6.4}$$

In $\mathbf{C}^2$, there are $\Gamma_s$-coordinates $(u, v)$ such that

$$v_2 = \alpha \frac{\partial}{\partial v}, \quad v_3 = \frac{\partial}{\partial u}; \quad \alpha \neq 0.$$

Let

$$v_1 = b(u, v)\frac{\partial}{\partial u} + c(u, v)\frac{\partial}{\partial v}, \quad \frac{\partial b}{\partial u} + \frac{\partial c}{\partial v} = 0.$$

From $(6.4_{1,2})$, we get

$$\frac{\partial b}{\partial v} = -\frac{1}{\alpha}, \quad \frac{\partial c}{\partial v} = 0, \quad \frac{\partial b}{\partial u} = 0, \quad \frac{\partial c}{\partial u} = -\alpha B,$$

i.e.,

$$v_1 = \left(-\frac{v}{\alpha} + b_0\right)\frac{\partial}{\partial u} + (-\alpha Bu + c_0)\frac{\partial}{\partial v}; \quad b_0, c_0 \in \mathbf{C}.$$

Consider the $\Gamma_s$-coordinates $x = u - c_0\alpha^{-1}B^{-1}$, $y = v - \alpha b_0$. Then

$$v_1 = -\frac{y}{\alpha}\frac{\partial}{\partial x} - \alpha Bx\frac{\partial}{\partial y}, \quad v_2 = \alpha\frac{\partial}{\partial y}, \quad v_3 = \frac{\partial}{\partial x}.$$

The general element $v \in L$ is

$$v = R\left(-\frac{y}{\alpha}\frac{\partial}{\partial x} - \alpha Bx\frac{\partial}{\partial y}\right) + S\alpha\frac{\partial}{\partial y} + T\frac{\partial}{\partial x}; \quad R, S, T \in \mathbf{R}; \tag{6.5}$$

and the local group $G_v$ is given by

$$\frac{\partial f}{\partial t} = -\frac{R}{\alpha}g + T, \quad \frac{\partial \varrho}{\partial t} = -R\alpha Bf + S\alpha.$$

Consider the group

$$f = ax - \frac{1}{\alpha}by + c, \quad g = -aBbx + ay + \alpha d; \tag{6.6}$$

$$a, b, c, d \in \mathbf{R}, \quad a^2 - Bb^2 = 1.$$

We get its identity for $a = 1$, $b = c = 0$. Let $a(t)$, $b(t)$, $c(t)$, $d(t)$ be its one-parametric subgroup $G_1$, let $t = 0$ correspond to its identity. Then

$$a\frac{da}{dt} - Bb\frac{db}{dt} = 0, \quad \frac{da(0)}{dt} = 0.$$

The vector field

$$v = \left(-\frac{1}{\alpha}\frac{\mathrm{d}b(0)}{\mathrm{d}t}y + \frac{\mathrm{d}c(0)}{\mathrm{d}t}\right)\frac{\partial}{\partial x} + \left(-\alpha B\frac{\mathrm{d}b(0)}{\mathrm{d}t}x + \alpha\frac{\mathrm{d}d(0)}{\mathrm{d}t}\right)\frac{\partial}{\partial y}$$

being associated to $G_1$, we see that (6.6) corresponds to (6.5). We have

$$\bar{f} = a\bar{x} - \frac{1}{\bar{\alpha}}b\bar{y} + c, \quad \bar{g} = -\bar{\alpha}Bb\bar{x} + a\bar{y} + \bar{\alpha}d,$$

$$f - \bar{f} = a(x - \bar{x}) - b\left(\frac{y}{\alpha} - \frac{\bar{y}}{\bar{\alpha}}\right), \quad \bar{\alpha}g - \alpha\bar{g} = -\alpha\bar{\alpha}Bb(x - \bar{x}) + a(\bar{\alpha}y - \alpha\bar{y})$$

and

$$B(f - \bar{f})^2 - \left(\frac{g}{\alpha} - \frac{\bar{g}}{\bar{\alpha}}\right)^2 = B(x - \bar{x})^2 - \left(\frac{y}{\alpha} - \frac{\bar{y}}{\bar{\alpha}}\right)^2.$$

Thus we have obtained the type (II).

Now, let $L$ be of the type (3.16) with $A^2 + BC = 0$, $AB \neq 0$, i.e.,

$$[v_1, v_2] = v_3, \quad [v_1, v_3] = Av_1 + Bv_2, \quad [v_2, v_3] = -\frac{A^2}{B}v_1 - Av_2. \tag{6.7}$$

Then $[Av_1 + Bv_2, v_3] = 0$, and there are $\Gamma_s$-coordinates $(u, v)$ such that

$$Av_1 + Bv_2 = \alpha\frac{\partial}{\partial v} \quad (0 \neq \alpha \in \mathbf{C}), \quad v_3 = \frac{\partial}{\partial u},$$

$$v_1 = b(u, v)\frac{\partial}{\partial u} + c(u, v)\frac{\partial}{\partial v}, \quad \frac{\partial b}{\partial u} + \frac{\partial c}{\partial v} = 0.$$

We have

$$v_2 = -\frac{A}{B}b\frac{\partial}{\partial u} + \frac{1}{B}(\alpha - Ac)\frac{\partial}{\partial v};$$

from $(6.7_{1,2})$

$$\frac{\partial b}{\partial v} = -\frac{B}{\alpha}, \quad \frac{\partial c}{\partial v} = 0, \quad \frac{\partial b}{\partial u} = 0, \quad \frac{\partial c}{\partial u} = -\alpha,$$

i.e.,

$$v_1 = \left(-\frac{B}{\alpha}v + b_0\right)\frac{\partial}{\partial u} + (-\alpha u + c_0)\frac{\partial}{\partial v}; \quad b_0, c_0 \in \mathbf{C}.$$

In the $\Gamma_s$-coordinates

$$x = u - \frac{c_0}{\alpha}, \quad y = v - \frac{b_0}{\alpha}B,$$

we get

$$Av_1 + Bv_2 = \alpha\frac{\partial}{\partial y}, \quad v_3 = \frac{\partial}{\partial x}, \quad v_1 = -\frac{B}{\alpha}y\frac{\partial}{\partial x} - \alpha x\frac{\partial}{\partial y}.$$

The general element $v \in L$ being

$$v = R\left(\frac{y}{\alpha}\frac{\partial}{\partial x} + \frac{\alpha}{B}x\frac{\partial}{\partial y}\right) + S\alpha\frac{\partial}{\partial y} + T\frac{\partial}{\partial x}; \; R, S, T \in \mathbf{R}; \tag{6.8}$$

we do not obtain now groups-compare (6.8) with (6.5).

**7.** Above we have considered all possibilities for $L \subset L_s$ with $dim\,[L, L] < 3$. Now, there are exactly two Lie algebras (over $\mathbf{R}$) with $dim\, L = dim\,[L, L] = 3$:

$$[w_1, w_2] = w_3, \quad [w_1, w_3] = -w_2, \quad [w_2, w_3] = w_1 \tag{7.1}$$

and

$$[w_1, w_2] = w_3, \quad [w_1, w_3] = w_2, \quad [w_2, w_3] = w_1. \tag{7.2}$$

First of all, let us consider the Lie algebra $L$ (7.2). The change $v_1 = w_3$, $v_2 = w_2 - w_1$, $v_3 = w_2 + w_1$ of its basis yields

$$[v_1, v_2] = v_2, \quad [v_1, v_3] = -v_3, \quad [v_2, v_3] = -2v_1. \tag{7.3}$$

In $\mathbf{C}^2$, there are $\Gamma_s$-coordinates $(r, s)$ such that

$$v_2 = \frac{\partial}{\partial r}, \quad v_1 = a(r, s)\frac{\partial}{\partial r} + b(r, s)\frac{\partial}{\partial s}, \quad \frac{\partial a}{\partial r} + \frac{\partial b}{\partial s} = 0.$$

From $(7.3_1)$,

$$\frac{\partial a}{\partial r} = -1, \quad \frac{\partial b}{\partial r} = 0,$$

and there exist a function $\alpha(s)$ and $b_0 \in \mathbf{C}$ such that

$$v_1 = (-r + \alpha(s))\frac{\partial}{\partial r} + (s + b_0)\frac{\partial}{\partial s}.$$

Let us choose the $\Gamma_s$-coordinates

$$u = r - (s + b_0)\int \alpha(s)\, ds, \quad v = s + b_0.$$

Then

$$v_2 = \frac{\partial}{\partial u}, \quad v_1 = -u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}.$$

Let

$$v_3 = e(u, v)\frac{\partial}{\partial u} + f(u, v)\frac{\partial}{\partial v}, \quad \frac{\partial e}{\partial u} + \frac{\partial f}{\partial v} = 0.$$

From $(7.3_3)$, we obtain

$$\frac{\partial e}{\partial u} = 2u, \quad \frac{\partial f}{\partial u} = -2v,$$

and there exists a function $\varphi(v)$ and $f_0 \in \mathbf{C}$ such that

$$v_3 = (u^2 + \varphi(v))\frac{\partial}{\partial u} + (-2uv + f_0)\frac{\partial}{\partial v}.$$

From $(7.3_2)$,

$$v\frac{d\varphi(v)}{dv} + 2\varphi(v) = 0,$$

and we obtain the existence of $\varphi_0 \in \mathbf{C}$ such that

$$v_3 = \left(u^2 - \frac{\varphi_0}{v^2}\right)\frac{\partial}{\partial u} + (-2uv + f_0)\frac{\partial}{\partial v}.$$

Finally, introduce the $\Gamma_s$-coordinates

$$x = u + \frac{f_0}{2v}, \quad y = v;$$

we have

$$v_1 = -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad v_2 = \frac{\partial}{\partial x}, \quad v_3 = \left(x^2 - \frac{\alpha^2}{y^2}\right)\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y}. \tag{7.4}$$

Now, it is easy to check that (7.4) are the infinitesimal transformations of (III).

Let the vector fields $w_1, w_2, w_3$ on $\mathbf{C}^2$ generate the algebra (7.1). Then the vector fields $iw_1, iw_2, iw_3$ generate the algebra (7.2), and the vector fields $v_1 = iw_3$, $v_2 = w_2 - iw_1$, $v_3 = w_2 + iw_1$ satisfy (7.3). Thus we obtain the existence of $\Gamma_s$-coordinates $(x, y)$ such that

$$iw_3 = -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad w_2 - iw_1 = \frac{\partial}{\partial x}, \quad w_2 + iw_1 = \left(x^2 - \frac{\alpha^2}{y^2}\right)\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y}.$$

Our result is as follows: Let the vector fields $w_1, w_2, w_3$ satisfy (7.1), then there are (local) $\Gamma_s$-coordinates $(x, y)$ such that

$$\begin{aligned} w_1 &= \frac{1}{2}i\left(1 - x^2 + \frac{\alpha^2}{y^2}\right)\frac{\partial}{\partial x} + ixy\frac{\partial}{\partial y}, \\ w_2 &= \frac{1}{2}\left(1 + x^2 - \frac{\alpha^2}{y^2}\right)\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y}, \\ w_3 &= ix\frac{\partial}{\partial x} - iy\frac{\partial}{\partial y}. \end{aligned} \tag{7.5}$$

**8.** Consider the space $\mathbf{R}^4$ and its decomposition $\mathbf{R}^4 = \mathbf{R}^2_1 \oplus \mathbf{R}^2_2$. Denote by $H$ the group $\{\gamma \in GL(\mathbf{R}^4);\ \gamma(\mathbf{R}^2_1) = \mathbf{R}^2_1,\ \gamma(\mathbf{R}^2_2) = \mathbf{R}^2_2\}$, and let $\Gamma$ be the pseudogroup of local diffeomorphisms $\varphi : U \subset \mathbf{R}^4 \to \mathbf{R}^4$ satisfying $(d\varphi)_x \in H$ for each $x \in U$. We wish to study hypersurfaces $M^3 \subset \mathbf{R}^4$ with respect to $\Gamma$. Let $m \in M^3$, $T_m(M^3)$ the tangent space of $M^3$ at $m$; denote by $S^2_i(m)$; $i = 1,2$; the plane for which $m \in S^2_i(m)$ and $S^2_i(m) \cap R^2_i = \Phi$. In what follows, let us restrict ourselves to the study of hypersurfaces $M^3 \subset \mathbf{R}^4$ satisfying the following conditions: (i) $M^3$ is analytic; (ii) $t_i(m) = T_m(M^3) \cap S^2_i(m)$ is one-dimensional for each $m \in M^3$ and $= 1,2$; (iii) $\tau_m \subset T_m(M^3)$ being the plane spanned by $t_1(m)$ and $t_2(m)$, the field $\tau_m$ is non-integrable. By means of the theory of systems of partial differential equations in involution (see, p.ex., K. Kuranishi, *Lectures on involutive systems of partial differential equations*; Publ. da Soc. Mat. de Sao Paulo, 1967), it is not difficult to prove

**Theorem.** *Let $M^3 \subset \mathbf{R}^4$ be a hypersurface and $\Phi : M^3 \to \mathbf{R}^4$ an analytic mapping such that both $M^3$ and $\tilde{M}^3 = \Phi(M^3)$ are hypersurfaces satisfying the conditions mentioned above. Let $(d\Phi)_m(t_i(m)) = \tilde{t}_i(m)$ for each $m \in M^3$ and $i = 1,2$; let $m_0 \in M^3$ be a fixed point. Then there is a neighbourhood $U \subset M^3$ of $m_0$ and a diffeomorphism $\varphi \in \Gamma$ such that $\varphi$ is defined on $U$ and $\varphi/_U = \Phi$.*

To each hypersurface $M^3 \subset \mathbf{R}^4$, we associate a $G$-structure $B_G(M^3)$ as follows. Let $(v_1, v_2, v_3)$ be a frame in $T_m(M^3)$. Then $(v_1, v_2, v_3) \in B_G(M^3)$ if and only if $v_i$ spans $t_i(m)$ for $i = 1,2$. $(w_1, w_2, w_3) \in B_G(M^3)$ being another frame at $m \in M^3$, we have

$$w_1 = \alpha v_1, \quad w_2 = \beta v_2, \quad w_3 = \gamma v_1 + \delta v_2 + \varphi v_3; \quad \alpha\beta\gamma \neq 0. \tag{8.1}$$

In a neighbourhood $U$ of $m \in M^3$, let us choose an analytic section $(v_1, v_2, v_3)$ of $B_G(M^3)$; $(w_1, w_2, w_3)$ being another section of $B_G(M^3)$, we have (8.1) with $\alpha, \ldots, \varphi$ real-valued functions on $U$. The vector fields $v_1, v_2, [v_1, v_2]$ being $\mathbf{R}$-linearly independent, we may write

$$\begin{aligned} [v_1, [v_1, v_2]] &= a_1 v_1 + a_2 v_2 + a_3 [v_1, v_2], \\ [v_2, [v_1, v_2]] &= b_1 v_1 + b_2 v_2 + b_3 [v_1, v_2] \end{aligned} \tag{8.2}$$

and

$$\begin{aligned} [w_1, [w_1, w_2]] &= A_1 w_1 + A_2 w_2 + A_3 [w_1, w_2], \\ [w_2, [w_1, w_2]] &= B_1 w_1 + B_2 w_2 + B_3 [w_1, w_2]. \end{aligned} \tag{8.3}$$

From the Jacobi identity

$$[v_1, [v_2, [v_1, v_2]]] + [v_2, [[v_1, v_2], v_1]] = 0,$$

we get

$$\begin{aligned} v_1 b_1 - v_2 a_1 + a_1 b_3 - a_3 b_1 &= 0, \\ v_1 b_2 - v_2 a_2 + a_2 b_3 - a_3 b_2 &= 0, \\ v_1 b_3 - v_2 a_3 + b_2 + a_1 &= 0 \end{aligned} \tag{8.4}$$

and analoguous equations for $A_1, \ldots, B_3$. Introduce the functions

$$p = (\alpha\beta^2)^{1/3}, \qquad q = (\alpha^2\beta)^{1/3} \tag{8.5}$$

over $U$ so that the equations $(8.1_{1,2})$ become

$$w_1 = p^{-1} q^2 v_1, \qquad w_2 = p^2 q^{-1} v_2. \tag{8.6}$$

Then

$$\begin{aligned} [w_1, w_2] &= [p^{-1} q^2 v_1, p^2 q^{-1} v_2] = \\ &= (q \,.\, v_2 p - 2p \,.\, v_2 q)\, v_1 + (2q \,.\, v_1 p - p \,.\, v_1 q)\, v_2 + pq[v_1, v_2], \\ [w_1, [w_1, w_2]] &= (.)\, v_1 + (.)\, v_2 + (q^3 a_3 + 3p^{-1} q^3 \,.\, v_1 p)\, [v_1, v_2] = \\ &= (.)\, v_1 + (.)\, v_2 + pqA_3 [v_1, v_2], \\ [w_2, [w_1, w_2]] &= (.)\, v_1 + (.)\, v_2 + (p^3 b_3 + 3p^3 q^{-1} \,.\, v_2 q)\, [v_1, v_2] = \\ &= (.)\, v_1 + (.)\, v_2 + pqB_3 [v_1, v_2], \end{aligned} \tag{8.7}$$

and we have

$$p^{-1} q^2 (a_3 + 3p^{-1} \,.\, v_1 p) = A_3, \qquad p^2 q^{-1} (b_3 + 3q^{-1} \,.\, v_2 q) = B_3. \tag{8.8}$$

The section $(v_1, v_2, v_3)$ of $B_G(M^3)$ being given, there exists (possibly in a small neighbourhood $U_1 \subset U$ of $m \in M^3$) a section $(w_1, w_2, w_3)$ of $B_G(M^3)$ satisfying (8.3) with $A_3 = B_3 = 0$; indeed, it is sufficient to take the section (8.6) where $p$, $q$ are any solutions of the system

$$v_1 p = -\tfrac{1}{3} p a_3, \qquad v_2 q = -\tfrac{1}{3} q b_3. \tag{8.9}$$

In what follows, let us restrict ourselves to the sections $(v_1, v_2, v_3)$, $(w_1, w_2, w_3)$ of $B_G(M^3)$ satisfying

$$a_3 = b_3 = 0 \quad or \quad A_3 = B_3 = 0 \quad resp.; \tag{8.10}$$

we have (8.6) + ($8.1_3$) with

$$v_1 p = 0, \quad v_2 q = 0. \tag{8.11}$$

Now,

$$[w_1, w_2] = q \,.\, v_2 p \,.\, v_1 - p \,.\, v_1 q \,.\, v_2 + pq[v_1, v_2], \tag{8.12}$$

$$\begin{aligned}[w_1, [w_1, w_2]] = {} & (2p^{-1}q^3 \,.\, v_1 v_2 p + 2q^2 \,.\, v_2 v_1 q - 2p^{-1}q^2 \,.\, v_2 p \,.\, v_1 q + \\ & + q^3 a_1)\, v_1 + (- q^2 \,.\, v_1 v_1 q + q^3 a_2)\, v_2 = p^{-1}q^2 A_1 v_1 + p^2 q^{-1} A_2 v_2, \end{aligned} \tag{8.13}$$

$$\begin{aligned}[w_2, [w_1, w_2]] = {} & (p^2 \,.\, v_2 v_2 p + p^3 b_1)\, v_1 + (- 2p^2 \,.\, v_1 v_2 p - 2p^3 q^{-1} \,.\, v_2 v_1 q + \\ & + 2p^2 q^{-1} \,.\, v_2 p \,.\, v_1 q + p^3 b_2)\, v_2 = p^{-1}q^2 B_1 v_1 + p^2 q^{-1} B_2 v_2, \end{aligned}$$

i.e.,

$$\begin{aligned} & - q^3 \,.\, v_1 v_1 q + q^4 a_2 = p^2 A_2, \\ & p^3 \,.\, v_2 v_2 p + p^4 b_1 = q^2 B_1, \\ & 2q \,.\, v_1 v_2 p + 2p \,.\, v_2 v_1 q - 2 v_2 p \,.\, v_1 q + pq a_1 = A_1, \end{aligned} \tag{8.14}$$

$$- 2q \,.\, v_1 v_2 p - 2p \,.\, v_2 v_1 q + 2 v_2 p \,.\, v_1 q + pq b_2 = B_2. \tag{8.15}$$

The equations (8.4) reduce to

$$v_1 b_1 - v_2 a_1 = 0, \quad v_1 b_2 - v_2 a_2 = 0, \quad b_2 + a_1 = 0 \tag{8.16}$$

and analoguous equations for $A_1, \ldots, B_2$; thus, (8.15) is a consequence of ($8.14_3$) and ($8.16_3$).

Let us consider the system (8.11) + (8.14). From (8.11) and ($8.14_{1,2}$), we get

$$\begin{aligned} & v_1 v_1 p = 0, && v_1 v_2 q = 0, \\ & v_2 v_1 p = 0, && v_2 v_2 q = 0, \\ & v_2 v_2 p = p^{-3} q^2 B_1 - p b_1, && v_1 v_1 q = q a_2 - p^2 q^{-3} A_2 \end{aligned} \tag{8.17}$$

and

$$\begin{aligned} & v_1 v_1 v_1 p = v_2 v_1 v_1 p = v_1 v_2 v_1 p = v_2 v_2 v_1 p = 0, \\ & v_1 v_2 v_2 p = 2p^{-3} q B_1 \,.\, v_1 q + p^{-3} q^2 \,.\, v_1 B_1 - p \,.\, v_1 b_1, \\ & v_2 v_2 v_2 p = - 3p^{-4} q^2 B_1 \,.\, v_2 p - b_1 \,.\, v_2 p + p^{-3} q^2 \,.\, v_2 B_1 - p \,.\, v_2 b_1, \\ & v_1 v_1 v_2 q = v_2 v_1 v_2 q = v_1 v_2 v_2 q = v_2 v_2 v_2 q = 0, \\ & v_1 v_1 v_1 q = a_2 \,.\, v_1 q + 3p^2 q^{-4} A_2 \,.\, v_1 q + q \,.\, v_1 a_2 - p^2 q^{-3} \,.\, v_1 A_2, \\ & v_2 v_1 v_1 q = - 2pq^{-3} A_2 \,.\, v_2 p + q \,.\, v_2 a_2 - p^2 q^{-3} \,.\, v_2 A_2. \end{aligned} \tag{8.18}$$

The equations (8.2) may be rewritten as

$$\begin{aligned} & v_1 v_1 v_2 - 2 v_1 v_2 v_1 + v_2 v_1 v_1 - a_1 v_1 - a_2 v_2 = 0, \\ & 2 v_2 v_1 v_2 - v_2 v_2 v_1 - v_1 v_2 v_2 - b_1 v_1 + a_1 v_2 = 0. \end{aligned} \tag{8.19}$$

Applying them to the functions $p$, $q$, we get

$$\begin{aligned} & v_1 v_1 v_2 p = a_2 \,.\, v_2 p, \\ & v_2 v_1 v_2 p = p^{-3} q B_1 \,.\, v_1 q - \tfrac{1}{2} a_1 \,.\, v_2 p + \tfrac{1}{2} p^{-3} q^2 \,.\, v_1 B_1 - \tfrac{1}{2} p \,.\, v_1 b_1, \\ & v_2 v_2 v_1 q = - b_1 \,.\, v_1 q, \\ & v_1 v_2 v_1 q = - pq^{-3} A_2 \,.\, v_2 p - \tfrac{1}{2} a_1 \,.\, v_1 q - \tfrac{1}{2} p^2 q^{-3} \,.\, v_2 A_2 + \tfrac{1}{2} q \,.\, v_2 a_2. \end{aligned} \tag{8.20}$$

Applying $v_1$ and $v_2$ to $(8.14_3)$, we get

$$2q \,.\, v_1v_1v_2p + 2p \,.\, v_1v_2v_1q - 2v_2p \,.\, v_1v_1q + pa_1 \,.\, v_1q + pq \,.\, v_1a_1 = v_1A_1,$$
$$2q \,.\, v_2v_1v_2p + 2p \,.\, v_2v_2v_1q - 2v_1q \,.\, v_2v_2p + qa_1 \,.\, v_2p + pq \,.\, v_2a_1 = v_2A_1,$$

i.e.,

$$q^3(v_1a_1 + v_2a_2) = p^{-1}q^2 \,.\, v_1A_1 + p^2q^{-1} \,.\, v_2A_2 = w_1A_1 + w_2A_2,$$
$$p^3(v_2a_1 - v_1b_1) = p^2q^{-1} \,.\, v_2A_1 - p^{-1}q^2 \,.\, v_1B_1 = w_2A_1 - w_1B_1$$

by means of (8.20). These equations being satisfied because of (8.16), we see that all the differential consequences of $(8.14_3)$ are consequences of the system (8.11) + + $(8.14_{1,2})$.

From (8.2) + (8.10), we get

$$[v_1, [v_1, [v_1, v_2]]] = v_1a_1 \,.\, v_1 + v_1a_2 \,.\, v_2 + a_2[v_1, v_2], \qquad (8.21)$$
$$[v_2, [v_2, [v_1, v_2]]] = v_2b_1 \,.\, v_1 - v_2a_1 \,.\, v_2 - b_1[v_1, v_2],$$

i.e.,

$$L_1 \equiv v_1v_1v_1v_2 - 3v_1v_1v_2v_1 + 3v_1v_2v_1v_1 - v_2v_1v_1v_1 - v_1a_1 \,.\, v_1 - v_1a_2 \,.\, v_2 - - a_2 \,.\, v_1v_2 + a_2 \,.\, v_2v_1 = 0,$$
$$L_2 \equiv v_1v_2v_2v_2 - 3v_2v_1v_2v_2 + 3v_2v_2v_1v_2 - v_2v_2v_2v_1 - v_2b_1 \,.\, v_1 + v_2a_1 \,.\, v_2 + + b_1 \,.\, v_1v_2 - b_1 \,.\, v_2v_1 = 0. \qquad (8.22)$$

Now,

$$v_1v_1v_2v_1q = 3pq^{-4}A_2 \,.\, v_2p \,.\, v_1q - pq^{-3} \,.\, v_1A_1 \,.\, v_2p - pq^{-3}A_2 \,.\, v_1v_2p - \tfrac{1}{2}v_1a_1 \,.\, v_1q - - \tfrac{1}{2}qa_1a_2 + \tfrac{1}{2}p^2q^{-3}a_1A_2 + \tfrac{3}{2}p^2q^{-4} \,.\, v_2A_2 \,.\, v_1q - \tfrac{1}{2}p^2q^{-3} \,.\, v_1v_2A_2 + + \tfrac{1}{2}v_2a_2 \,.\, v_1q + \tfrac{1}{2}q \,.\, v_1v_2a_2,$$
$$v_1v_2v_1v_1q = 6pq^{-4}A_2 \,.\, v_2p \,.\, v_1q - 2pq^{-3} \,.\, v_1A_2 \,.\, v_2p - 2pq^{-3}A_2 \,.\, v_1v_2p + + v_2a_2 \,.\, v_1q + q \,.\, v_1v_2a_2 + 3p^2q^{-4} \,.\, v_2A_2 \,.\, v_1q - p^2q^{-3} \,.\, v_1v_2A_2,$$
$$v_2v_1v_1v_1q = v_2a_2 \,.\, v_1q + a_2 \,.\, v_2v_1q + 6pq^{-4}A_2 \,.\, v_2p \,.\, v_1q + 3p^2q^{-4} \,.\, v_2A_2 \,.\, v_1q + + 3p^2q^{-4}A_2 \,.\, v_2v_1q + q \,.\, v_2v_1a_2 - 2pq^{-3} \,.\, v_1A_2 \,.\, v_2p - p^2q^{-3} \,.\, v_2v_1A_2,$$
$$v_1v_2v_2v_2p = - 6p^{-4}qB_1 \,.\, v_2p \,.\, v_1q - 3p^{-4}q^2 \,.\, v_1B_1 \,.\, v_2p - 3p^{-4}q^2B_1 \,.\, v_1v_2p - - v_1b_1 \,.\, v_2p - b_1 \,.\, v_1v_2p + 2p^{-3}q \,.\, v_2B_1 \,.\, v_1q + p^{-3}q^2 \,.\, v_1v_2B_1 - - p \,.\, v_1v_2b_1,$$
$$v_2v_1v_2v_2p = - 6p^{-4}qB_1 \,.\, v_2p \,.\, v_1q + 2p^{-3}q \,.\, v_2B_1 \,.\, v_1q + 2p^{-3}qB_1 \,.\, v_2v_1q - - 3p^{-4}q^2 \,.\, v_1B_1 \,.\, v_2p + p^{-3}q^2 \,.\, v_2v_1B_1 - v_1b_1 \,.\, v_2p - p \,.\, v_2v_1b_1,$$
$$v_2v_2v_1v_2p = - 3p^{-4}qB_1 \,.\, v_2p \,.\, v_1q + p^{-3}q \,.\, v_2B_1 \,.\, v_1q + + p^{-3}qB_1 \,.\, v_2v_1q - \tfrac{1}{2}v_2a_1 \,.\, v_2p - \tfrac{1}{2}p^{-3}q^2a_1B_1 + + \tfrac{1}{2}pa_1b_1 - \tfrac{3}{2}p^{-4}q^2 \,.\, v_1B_1 \,.\, v_2p + \tfrac{1}{2}p^{-3}q^2 \,.\, v_2v_1B_1 - - \tfrac{1}{2}v_1b_1 \,.\, v_2p - \tfrac{1}{2}p \,.\, v_2v_1b_1.$$

From $L_1q = 0$, $L_2p = 0$, we obtain

$$3pq^{-4}A_2 \,.\, v_2p \,.\, v_1q - 3pq^{-3}A_2 \,.\, v_1v_2p - 3p^2q^{-4}A_2 \,.\, v_2v_1q + + \tfrac{3}{2}p^2q^{-4} \,.\, v_2A_2 \,.\, v_1q - pq^{-3} \,.\, v_1A_2 \,.\, v_2p + \tfrac{3}{2}q \,.\, v_1v_2a_2 + \tfrac{3}{2}qa_1a_2 - - \tfrac{3}{2}p^2q^{-3}a_1A_2 - \tfrac{3}{2}p^2q^{-3} \,.\, v_1v_2A_2 - q \,.\, v_2v_1a_2 + p^2q^{-3} \,.\, v_2v_1A_2 = 0, \qquad (8.23)$$
$$3p^{-4}qB_1 \,.\, v_2p \,.\, v_1q - 3p^{-4}q^2B_1 \,.\, v_1v_2p - 3p^{-3}qB_1 \,.\, v_2v_1q + + \tfrac{3}{2}p^{-4}q^2 \,.\, v_1B_1 \,.\, v_2p - p^{-3}q \,.\, v_2B_1 \,.\, v_1q + p^{-3}q^2 \,.\, v_1v_2B_1 - - \tfrac{3}{2}p^{-3}q^2 \,.\, v_2v_1B_1 - p \,.\, v_1v_2b_1 + \tfrac{3}{2}p \,.\, v_2v_1b_1 - \tfrac{3}{2}p^{-3}q^2a_1B_1 + \tfrac{3}{2}pa_1b_1 = 0.$$

Multiplying (8.14$_3$) by $\frac{3}{2}pq^{-4}A_2$ or $\frac{3}{2}p^{-4}qB_1$ resp. and adding it to (8.23$_1$) or (8.23$_2$) resp., we get

$$\frac{3}{2}p^2q^{-4}\,.\,v_2A_2\,.\,v_1q - pq^{-3}\,.\,v_1A_2\,.\,v_2p + \frac{3}{2}q\,.\,v_1v_2a_2 + \frac{3}{2}qa_1a_2 - \\ - \frac{3}{2}p^2q^{-3}\,.\,v_1v_2A_2 - q\,.\,v_2v_1a_2 + p^2q^{-3}\,.\,v_2v_1A_2 - \frac{3}{2}pq^{-4}A_1A_2 = 0, \\ \frac{3}{2}p^{-4}q^2\,.\,v_1B_1\,.\,v_2p - p^{-3}q\,.\,v_2B_1\,.\,v_1q + p^{-3}q^2\,.\,v_1v_2B_1 - \\ - \frac{3}{2}p^{-3}q^2\,.\,v_2v_1B_1 - p\,.\,v_1v_2b_1 + \frac{3}{2}p\,.\,v_2v_1b_1 + \frac{3}{2}pa_1b_1 - \frac{3}{2}p^{-4}qA_1B_1 = 0. \tag{8.24}$$

From (8.6),

$$v_2v_1 = q^{-2}\,.\,v_2p\,.\,w_1 + p^{-1}q^{-1}\,.\,w_2w_1,\quad v_1v_2 = p^{-2}\,.\,v_1q\,.\,w_2 + p^{-1}q^{-1}\,.\,w_1w_2. \tag{8.25}$$

Finally, we get

$$p^{-1}q^5(3v_1v_2a_2 - 2v_2v_1a_2 + 3a_1a_2) = 3w_1w_2A_2 - 2w_2w_1A_2 + 3A_1A_2, \\ p^5q^{-1}(3v_2v_1b_1 - 2v_1v_2b_1 + 3a_1b_1) = 3w_2w_1B_1 - 2w_1w_2B_1 + 3A_1B_1 \tag{8.26}$$

from (8.6), (8.25) and (8.24).

Let us write

$$j_1 = 3v_1v_2a_2 - 2v_2v_1a_2 + 3a_1a_2,\qquad j_2 = 3v_2v_1b_1 - 2v_1v_2b_1 + 3a_1b_1, \\ J_1 = 3w_1w_2A_2 - 2w_2w_1A_2 + 3A_1A_2,\quad J_2 = 3w_2w_1B_1 - 2w_1w_2B_1 + 3A_1B_1. \tag{8.27}$$

Then

$$\alpha^3\beta j_1 = J_1,\quad \alpha\beta^3 j_2 = J_2. \tag{8.28}$$

Suppose

$$j_1j_2 \neq 0 \tag{8.29}$$

and write

$$k_1 = |j_1{}^{-3}j_2|^{1/8},\qquad k_2 = |j_1j_2{}^{-3}|^{1/8} \tag{8.30}$$

Then

$$k_1 = |\alpha|\,.\,K_1,\qquad k_2 = |\beta|\,.\,K_2 \tag{8.31}$$

and

$$K_1w_1 = \operatorname{sgn}\alpha\,.\,k_1v_1,\qquad K_2w_2 = \operatorname{sgn}\beta\,.\,k_2v_2. \tag{8.32}$$

**Theorem.** *On $M^3$, be given a G-structure $B_G(M^3)$ of the considered type. In a neighbourhood of $m_0 \in M^3$, let us choose its section $(v_1, v_2, v_3)$ in such a way that* (8.2) *and* (8.10) *are satisfied. Suppose that we have* (8.29) *for the functions $j_1, j_2$ defined by* (8.27). *Consider the vector fields*

$$V = k_1v_1.\qquad V_2 = k_2v_2, \tag{8.33}$$

*$k_1$ and $k_2$ being defined by* (8.30). *These vector fields are invariant up to the sign, i.e., choosing another section $(w_1, w_2, w_3)$ satisfying* (8.3) *and* (8.10), *we have $W_1 \equiv K_1w_1 = \pm V_1$, $W_2 \equiv K_2w_2 = \pm V_2$.*

**9.** Consider the space $\mathbf{C}^2$, i.e., the space $\mathbf{R}^4$ endowed with a fixed automorphism $I : \mathbf{R}^4 \to \mathbf{R}^4$ satisfying $I^2 = -id$. Let $H' \subset GL(\mathbf{R}^4)$ be the subgroup of elements $\gamma \in GL(\mathbf{R}^4)$ satisfying $\gamma I = I\gamma$. The local diffeomorphism $\varphi : U \subset \mathbf{R}^4 \to \mathbf{R}^4$ is called holomorphic if $(d\varphi)_x \in H'$ for each $x \in U$. Our task is to study hypersurfaces $M^3 \subset \mathbf{R}^4$ with respect to the pseudogroup $\Gamma'$ of all local holomorphic diffeomorphisms.

Let $m \in M^3$. Write $\tau_m = T_m(M^3) \cap IT_m(M^3)$; $\tau_m$ is always a plane. Let us restrict ourselves to hypersurfaces for which the field of planes $\tau_m$ is non-integrable. To $M^3$, we associate a $G'$-structure $B'_{G'}(M^3)$ as follows. The frame $(u_1, u_2, u_3)$ of $T_m(M^3)$ belongs to $B'_{G'}(M^3)$ if and only if $u_1 \in \tau_m$, $u_2 = Iu_1$. $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ being another frame of $B'_{G'}(M^3)$ over $m$, we have

$$\begin{aligned} \tilde{u}_1 &= \varrho u_1 - \sigma u_2, \\ \tilde{u}_2 &= \sigma u_1 + \varrho u_2, \\ \tilde{u}_3 &= \varkappa_1 u_1 + \varkappa_2 u_2 + \varkappa u_3; \qquad (\varrho^2 + \sigma^2)\varkappa \neq 0. \end{aligned} \tag{9.1}$$

In a neighbourhood of $m \in M^3$, let us choose a section $(u_1, u_2, u_3)$ of $B'_{G'}(M^3)$. We may write

$$\begin{aligned} [u_1, [u_1, u_2]] &= c_1, u_1 + c_2 u_2 + c_3[u_1, u_2], \\ [u_2, [u_1, u_2]] &= d_1 u_1 + d_2 u_2 + d_3[u_1, u_2]. \end{aligned} \tag{9.2}$$

Consider the complexification $T^C(M^3) = T(M^3) \oplus iT(M^3)$ of the tangent bundle $T(M^3)$ and its vector fields

$$v_1 = u_1 + iu_2,\ v_2 = u_1 - iu_2 \text{ or } w_1 = \tilde{u}_1 + i\tilde{u}_2,\ w_2 = \tilde{u}_1 - i\tilde{u}_2 \text{ resp.} \tag{9.3}$$

Then

$$w_1 = \alpha v_1,\ w_2 = \beta v_2, \quad \text{where} \quad \alpha = \varrho + i\sigma,\ \beta = \varrho - i\sigma. \tag{9.4}$$

Further,

$$\begin{aligned} [v_1, [v_1, v_2]] &= \{d_1 - c_2 - i(d_2 + c_1)\} v_1 + \{d_1 + c_2 + i(d_2 - c_1)\} v_2 + \\ &\quad + (c_3 + id_3)[v_1, v_2], \\ [v_2, [v_1, v_2]] &= \{-d_1 - c_2 + i(d_2 - c_1)\} v_1 + \{-d_1 + c_2 - i(d_2 + c_1)\} v_2 + \\ &\quad + (c_3 - id_3)[v_1, v_2]. \end{aligned} \tag{9.5}$$

To obtain invariants of $M^3 \subset \mathbf{C}^2$, we proceed formally in the same way as we have done in the preceding section.