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THE SYMPLECTIC GRAM-SCHMIDT THEOREM AND FUNDAMENTAL GEOMETRIES FOR $\mathscr{A}$-MODULES

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Abstract. Like the classical Gram-Schmidt theorem for symplectic vector spaces, the sheaf-theoretic version (in which the coefficient algebra sheaf $\mathscr{A}$ is appropriately chosen) shows that symplectic $\mathscr{A}$-morphisms on free $\mathscr{A}$-modules of finite rank, defined on a topological space X, induce canonical bases (Theorem 1.1), called *symplectic bases.* Moreover (Theorem 2.1), if $(\mathscr{E}, \varphi)$ is an $\mathscr{A}$-module (with respect to a $\mathbb{C}$-algebra sheaf $\mathscr{A}$ *without zero divisors*) equipped with an orthosymmetric $\mathscr{A}$-morphism, we show, like in the classical situation, that "*componentwise*" φ *is either symmetric* (the (local) geometry is orthogonal) *or skew-symmetric* (the (local) geometry is symplectic). Theorem 2.1 reduces to the classical case for any free $\mathscr{A}$-module of finite rank.

Keywords: symplectic $\mathscr{A}$-modules, symplectic Gram-Schmidt theorem, symplectic basis, orthosymmetric $\mathscr{A}$-bilinear forms, orthogonal/symplectic geometry, strict integral domain algebra sheaf

MSC 2010: 16D90, 18F20

Introduction

Abstract Differential Geometry (ADG) offers a new approach to classical differential geometry (on smooth manifolds). This new approach differs from the classical way of understanding the geometry of smooth manifolds, differential spaces à la Mostow [17], Sikorski [19], and the likes, in the sense that, for instance, differential spaces in general are governed by new classes of "smooth" functions; in ADG the *structural sheaf* of functions is replaced instead by an arbitrary *sheaf of algebras* $\mathscr{A}$, based on an arbitrary topological space X. The same (sheaf of) algebras may in some cases contain a tremendous amount of *singularities*, while still retaining the classical character of a *differential* mechanism, yet without any underlying (smooth) manifold: see e.g. [9], [12]. This results into significant potential applications, even to *quantum gravity* (ibid.). On the side, we may also point out that the *main moral* of ADG is the *functorial mechanism* of (classical) calculus, cf. [11], viz. *Physics is $\mathscr{A}$-invariant regardless of what $\mathscr{A}$ is.* Yet, a particular instance of the foregoing comment that also interests us here is the standard *symplectic differential geometry* (on manifolds), where a special important issue is the so-called *orbifolds theory*; see e.g. [10, Vol. II, Chapt. X; Section 3a] concerning its relation with ADG, or [4] for the classical case. The following constitutes a sheaf-theoretic fundamental prelude with a view towards potential applications of ADG, the whole set-up being in effect a "*Lagrangian perspective*". The aim of the paper is to generalize primarily the *symplectic Gram-Schmidt theorem* (see [14, p. 184, Theorem 3]) and also characterize the *fundamental geometries induced by an orthosymmetric $\mathscr{A}$-morphism on an $\mathscr{A}$-module,* see e.g. [6]. Our main reference, throughout the present account, is [9], which may be useful for the basics of ADG.

This is a continuation of work done by Mallios and Ntumba [14], [15], and [16].

Convention. Throughout the paper, X will denote an arbitrary topological space, the pair $(X, \mathscr{A})$ a fixed $\mathbb{C}$-algebraized space, cf. [9, p. 96] with $\mathscr{A}$ a *unital, commutative* $\mathbb{C}$-algebra sheaf, and all $\mathscr{A}$-modules are understood to be defined on the topological space X.

For easy referencing, we recall a few basic definitions.

A $\mathbb{C}$-*algebraized space* on a topological space X is a pair $(X, \mathscr{A})$, where $\mathscr{A} \equiv (\mathscr{A}, \tau, X)$ is a (preferably unital and commutative) sheaf of $\mathbb{C}$-algebras (or in other words, a $\mathbb{C}$-*algebra sheaf*). A *sheaf of $\mathscr{A}$-modules* (or an *$\mathscr{A}$-module*) on X is a sheaf $\mathscr{E} \equiv (\mathscr{E}, \varrho, X)$ such that the following conditions hold:

(i) $\mathscr{E}$ is a sheaf of abelian groups;
(ii) For every point $x \in X$, the corresponding stalk $\mathscr{E}_x$ of $\mathscr{E}$ is a (left) $\mathscr{A}_x$-module;
(iii) The exterior module multiplication in $\mathscr{E}$, viz. the map $\mathscr{A} \circ \mathscr{E} \longrightarrow \mathscr{E}\colon\ (a, z) \longmapsto a \cdot z \in \mathscr{E}_x \subseteq \mathscr{E}$ with $\tau(a) = \varrho(z) = x \in X$, is continuous.

An $\mathscr{A}$-module $\mathscr{E}$ is called a *free $\mathscr{A}$-module of rank n* ($n \in \mathbb{N}$), provided $\mathscr{E} = \mathscr{A}^n$ within an $\mathscr{A}$-isomorphism. The $\mathscr{A}$-module $\mathscr{A}^n$ is called the *standard free $\mathscr{A}$-module of rank n*. For an open subset $U \subseteq X$, the *canonical (Kronecker) basis* of the $\mathscr{A}(U)$-module $\mathscr{A}^n(U)$ is the set $\{\varepsilon_i^U\}_{1 \leqslant i \leqslant n}$, where $\varepsilon_i^U := \delta_{ij}^U \in \mathscr{A}^n(U) \cong \mathscr{A}(U)^n$ such that $\delta_{ij}^U = 1$ for $i = j$ and $\delta_{ij}^U = 0$ for $i \neq j$. So one gets, for any $x \in X$, $\varepsilon_i^U(x) = (\delta_{ij}^U(x))_{1 \leqslant j \leqslant n} \in \mathscr{A}_x^n$ $(1 \leqslant i \leqslant n)$, where $\delta_{ij}^U(x) = 1_x \in \mathscr{A}_x$, if $i = j$, and $\delta_{ij}^U(x) = 0_x \in \mathscr{A}_x$, if $i \neq j$.

Now suppose there is given a presheaf of unital and commutative $\mathbb{C}$-algebras $A \equiv (A(U), \tau_V^U)$ and a presheaf of abelian groups $E \equiv (E(U), \varrho_V^U)$, both on a topological space X and such that (i) $E(U)$ is a (left) $A(U)$-module, for every open set $U \subseteq X$,

(ii) For any open sets U, V in X, with $V \subseteq U$, $\varrho_V^U(a \cdot s) = \tau_V^U(a) \cdot \varrho_V^U(s)$ for any $a \in A(U)$ and $s \in E(U)$. We call such a presheaf E a *presheaf of* $A(U)$*-modules* on X, or simply an A*-presheaf* on X. $\mathscr{A}$-modules and A-presheaves with their respective morphisms form categories which we denote $\mathscr{A}$-Mod_X and A-$\mathrm{PS}h_X$ respectively. By virtue of the equivalence $\mathrm{S}h_X \cong \mathrm{CoPS}h_X$ (cf.[9, p. 75, (13.18)]), an $\mathscr{A}$-morphism $\varphi = (\varphi_U)_{X \supseteq U, \mathrm{open}}\colon \mathscr{E} \longrightarrow \mathscr{F}$ of $\mathscr{A}$-modules $\mathscr{E}$ and $\mathscr{F}$ may be identified with the A-morphism $\overline{\varphi} = (\overline{\varphi}_U)_{X \supseteq U, \mathrm{open}}\colon E \longrightarrow F$ of the associated A-presheaves. We shall most often denote by just φ the corresponding A-morphism associated with the $\mathscr{A}$-morphism φ. The meaning of φ will always be determined by the situation at hand. Furthermore, to make the paper more self-contained, we also recall some notions, which may be found in our recent papers such as [16], [15], and [14]. Let $\mathscr{E}$ and $\mathscr{F}$ be $\mathscr{A}$-modules and $\varphi\colon \mathscr{E} \oplus \mathscr{F} \longrightarrow \mathscr{A}$ an $\mathscr{A}$-bilinear morphism. The triple $(\mathscr{E}, \mathscr{F}; \mathscr{A}) \equiv ((\mathscr{E}, \mathscr{F}; \varphi); \mathscr{A})$ is said to define a *pairing of* $\mathscr{A}$*-modules*. Now, one defines the *sub-*$\mathscr{A}$*-module* $\mathscr{E}^\perp$ of $\mathscr{F}$, as the *sheaf generated by the presheaf* of sub-$\mathscr{A}(U)$-modules of $\mathscr{F}(U)$, given by

$$\mathscr{E}^\perp(U) := \{t \in \mathscr{F}(U)\colon\ \varphi_V(\mathscr{E}(V), gt|_V) = 0\},$$

for any U, V open in X, with $V \subseteq U$. In the same way, one defines the sub-$\mathscr{A}$-module $\mathscr{F}^\perp$. Thus, for any open $U \subseteq X$,

$$\mathscr{F}^\perp(U) := \{s \in \mathscr{E}(U)\colon\ \varphi_V(s|_V, \mathscr{F}(V)) = 0\},$$

with V open in U. $\mathscr{E}^\perp$ and $\mathscr{F}^\perp$ are called *right kernel* and *left kernel* of the pairing $(\mathscr{E}, \mathscr{F}; \mathscr{A})$, respectively. In this context, in the case of *free* $\mathscr{A}$*-modules* (:= *free* $\mathscr{A}$*-pairings*, for short), one has, for every open subset U of X,

$$\mathscr{F}^\perp(U) = \mathscr{F}(U)^\perp := \{r \in \mathscr{E}(U)\colon\ \varphi_U(r, \mathscr{F}(U)) = 0\},$$

and similarly

$$\mathscr{E}^\perp(U) = \mathscr{E}(U)^\perp := \{r \in \mathscr{F}(U)\colon\ \varphi_U(\mathscr{E}(U), r) = 0\}.$$

Now, let $((\mathscr{E}, \mathscr{E}; \varphi); \mathscr{A})$ be a (self) pairing such that the left kernel, $\mathscr{E}_l^\perp := \mathscr{E}^\perp$, coincides with the right kernel $\mathscr{E}_r^\perp := \mathscr{E}^\top$. Then, we call $\mathscr{E}^\perp$ $(= \mathscr{E}^\top)$ the *radical sheaf* (or *sheaf of* $\mathscr{A}$*-radicals*, or simply $\mathscr{A}$*-radical*) of $\mathscr{E}$, and denote it by $\mathrm{rad}_{\mathscr{A}}\, \mathscr{E} \equiv \mathrm{rad}\, \mathscr{E}$. An $\mathscr{A}$-module $\mathscr{E}$ such that $\mathrm{rad}\, \mathscr{E} \neq 0$ (resp. $\mathrm{rad}\, \mathscr{E} = 0$) is called *isotropic* (resp. *non-isotropic*); $\mathscr{E}$ is *totally isotropic* if φ is identically zero. A *non-zero (local) section* $r \in \mathscr{E}(U)$, U open in X, is called *isotropic*, if $\varphi_U(r, r) = 0$. The $\mathscr{A}$*-radical of a sub-*$\mathscr{A}$*-module* $\mathscr{F}$ *of* $\mathscr{E}$ is defined as $\mathrm{rad}\, \mathscr{F} := \mathscr{F} \cap \mathscr{F}^\perp = \mathscr{F} \cap \mathscr{F}^\top$. If $(\mathscr{E}, \mathscr{F}; \mathscr{A})$

is a *free $\mathscr{A}$-pairing*, then for every open subset U of X, $(\operatorname{rad}\mathscr{E})(U) = \operatorname{rad}\mathscr{E}(U)$ and $(\operatorname{rad}\mathscr{F})(U) = \operatorname{rad}\mathscr{F}(U)$, where $\operatorname{rad}\mathscr{E}(U) = \mathscr{E}(U) \cap \mathscr{E}(U)^{\perp}$ and $\operatorname{rad}\mathscr{F}(U) = \mathscr{F}(U) \cap \mathscr{F}(U)^{\perp}$.

1. Symplectic Gram-Schmidt theorem

For the purpose of Theorem 1.1 below, we assume that the pair $(X, \mathscr{A})$ is a $\mathbb{C}$-*algebraized space*, such that *every nowhere-zero section of $\mathscr{A}$ is invertible*; viz. if $s \in \mathscr{A}(U)$, where U is open in X, is such that $s|_V \neq 0$ for every open $V \subseteq U$, then $s \in \mathscr{A}(U)^{\bullet} \cong \mathscr{A}^{\bullet}(U)$ ($\mathscr{A}^{\bullet}$ denotes the sheaf generated by the complete presheaf $U \longmapsto \mathscr{A}(U)^{\bullet}$, where U runs over the open subsets of X, and $\mathscr{A}(U)^{\bullet} \cong \mathscr{A}^{\bullet}(U)$ consists of the invertible elements of the unital $\mathbb{C}$-algebra $\mathscr{A}(U)$; cf. [9, pp. 282, 283]). For convenience, we call the above the "*inverse-closed section condition*" of $\mathscr{A}$.

For the sake of Definition 1.1 below (see [13]), let us recall the following lemma, whose proof may be found in [13].

Lemma 1.1. *Let $(\mathscr{E}, \mathscr{F}; \varphi)$ be a pairing of $\mathscr{A}$-modules. Then, φ induces an $\mathscr{A}$-morphism, viz.*

$$\varphi^{\mathscr{E}} : \mathscr{F} \longrightarrow \mathscr{E}^{*} := \operatorname{Hom}_{\mathscr{A}}(\mathscr{E}, \mathscr{A}),$$

given by

$$\varphi^{\mathscr{E}}_U(t)(s) := \varphi_V(s, \sigma^U_V(t)) \equiv \varphi_V(s, t|_V),$$

where U is open in X, $t \in \mathscr{F}(U)$, $s \in \mathscr{E}(V)$ and the σ^U_V the restriction maps of the presheaf of sections of $\mathscr{F}$. Likewise, φ gives rise to a similar $\mathscr{A}$-morphism:

$$\varphi^{\mathscr{F}} : \mathscr{E} \longrightarrow \mathscr{F}^{*}.$$

Definition 1.1. Let $(\mathscr{E}, \mathscr{F}; \varphi)$ be an it $\mathscr{A}$-pairing, and $\varphi^{\mathscr{E}}$ and $\varphi^{\mathscr{F}}$ be the induced $\mathscr{A}$-morphisms, according to Lemma 1.1. Then, φ is said to be *non-degenerate* if $\mathscr{E}^{\perp} = \mathscr{F}^{\perp} = 0$, and *degenerate* otherwise.

Now, let us recall that (see e.g. [14]) a *symplectic $\mathscr{A}$-module* is a pair $(\mathscr{E}, \varphi)$, where $\mathscr{E}$ is an $\mathscr{A}$-module, and $\varphi\colon \mathscr{E} \oplus \mathscr{E} \longrightarrow \mathscr{A}$ a *symplectic $\mathscr{A}$-morphism* (or *symplectic $\mathscr{A}$-form*), i.e., a *skew-symmetric* and *non-degenerate* $\mathscr{A}$-form on $\mathscr{E}$. *Skew-symmetry* means that for any open $U \subseteq X$,

$$\varphi_U(r, s) = -\varphi_U(s, r) \quad \text{for any } r, s \in \mathscr{E}(U).$$

We also need for the proof of Theorem 1.1 the following.

Lemma 1.2. *Let $(\mathscr{E}, \varphi)$ be a symplectic free $\mathscr{A}$-module of finite rank n, U an open subset of X and $(r_1, \ldots, r_n) \subseteq \mathscr{E}(U)$ a (local) gauge of $\mathscr{E}$. Then, for any $r \equiv r_i$, $1 \leqslant i \leqslant n$, there exists a nowhere-zero section $s \in \mathscr{E}(U)$ such that $\varphi_U(r, s)$ is nowhere zero.*

P r o o f. Without loss of generality, assume that $r_1 = r$. On the other hand, since the induced $\mathscr{A}$-morphism $\widetilde{\varphi} \in \operatorname{Hom}_{\mathscr{A}}(\mathscr{E}, \mathscr{E}^*)$ is one-to-one and both $\mathscr{E}$ and $\mathscr{E}^*$ have the same finite rank, it follows that the matrix D representing φ_U (see also [1, p. 357, Theorem 2.21, along with p. 356, Definition 2.19] or [5, p. 343, Proposition 20.3]), with respect to the basis $(r_1, \ldots, r_n)$, has a *nowhere-zero determinant*; so since

$$\det D = \sum_{i=1}^{n} (-1)^{1+i} \varphi(r_1, r_i) \det D_{1i} = \varphi\biggl(r_1, \sum_{i=1}^{n} (-1)^{1+i} \det D_{1i} r_i\biggr),$$

where D_{1i} is the minor of the corresponding $\varphi(r_1, r_i)$, and $\det D$ nowhere zero, we thus have a section $s := \sum_{i=1}^{n} (-1)^{1+i} \det D_{1i} r_i \in \mathscr{E}(U)$ such that $\varphi(r, s)$ is nowhere zero. □

Theorem 1.1 below is the analogue of the classical *symplectic Gram-Schmidt theorem*, the latter being an "*important result with many applications*" (cf. [7, p. 12, Theorem 1.15] and [3, p. 10, Proposition 1.13]). It is worth noting that the Gram-Schmidt orthogonalization process is already available for Riemannian $\mathscr{A}$-modules; to this end, see [9, pp. 335–341]. In order to achieve the Riemmanian version of this theorem, Mallios assumes the following conditions:

(1) *Every strictly positive section of the coefficient algebra sheaf $\mathscr{A}$ is invertible*, viz., for any $s \in \mathscr{A}^+(U)$, U open in X, with $s|_V \neq 0$ for any open $V \subseteq U$, one has that $s \in \mathscr{A}(U)^\bullet = \mathscr{A}^\bullet(U)$. Indeed, in the proof of Theorem 1.1 below, we need the "*inverse-closed section condition*" of $\mathscr{A}$, already formulated at the beginning of Section 1.

(2) *Every positive section of $\mathscr{A}$ has a square root*; viz., for every section $s \in \mathscr{A}^+(U)$, with U open in X, there is a (unique) $t \in \mathscr{A}^+(U)$ such that $t^2 = s$.

Based on the previous condition (1), we have the following.

Theorem 1.1. *Let $\mathscr{A}$ be an $\mathbb{R}$-algebra sheaf satisfying the inverse-closed section condition, $(\mathscr{E}, \varphi)$ a free $\mathscr{A}$-module of rank $2n$, $\varphi = (\varphi_U)\colon \mathscr{E} \oplus \mathscr{E} \longrightarrow \mathscr{A}$ a skew-symmetric non-degenerate $\mathscr{A}$-bilinear form, and I and J two (possibly empty) subsets of $\{1, \ldots, n\}$. Moreover, let $A = \{r_i \in \mathscr{E}(U)\colon i \in I\}$ and $B = \{s_j \in \mathscr{E}(U)\colon j \in J\}$ such that*

$$(1) \qquad \varphi_U(r_i, r_j) = \varphi_U(s_i, s_j) = 0, \quad \varphi_U(r_i, s_j) = \delta_{ij}, \ \ (i, j) \in I \times J.$$

Then, there exists a basis $\mathfrak{B}$ of $(\mathscr{E}(U), \varphi_U)$ containing $A \cup B$.

P r o o f. We have three cases. With no loss of generality, we assume that $U = X$.

(1) *Case:* $I = J = \emptyset$. Since $\mathscr{A}^{2n} \neq 0$ (we already assumed that $\mathbb{C} \equiv \mathbb{C}_X \subseteq \mathscr{A}$), there exists an element

$$0 \neq r_1 \in \mathscr{E}(X) \cong \mathscr{A}^{2n}(X) \cong \mathscr{A}(X)^{2n}$$

(take e.g. the image (by the isomorphism $\mathscr{E}(X) \cong \mathscr{A}^{2n}(X)$) of an element in the canonical basis of (sections) of $\mathscr{A}^{2n}(X)$). By virtue of Lemma 1.2, there exists a section $\overline{s}_1 \in \mathscr{E}(X)$ such that $\varphi_V(r_1|_V, \overline{s}_1|_V) \neq 0$ for any open subset V in X. Thus, based on Condition (1), $\varphi_X(r_1, \overline{s}_1)$ is invertible in $\mathscr{A}(X)$. Putting $s_1 := u^{-1}\overline{s}_1$, where $u \equiv \varphi_X(r_1, \overline{s}_1) \in \mathscr{A}(X)$, one gets

$$\varphi_X(r_1, s_1) = 1.$$

Now, let us consider

$$S_1 := [r_1, s_1],$$

that is, the $\mathscr{A}(X)$-*plane*, spanned by r_1 and s_1 in $\mathscr{E}(X)$, along with *its orthogonal complement in* $\mathscr{E}(X)$, i.e.,

$$S_1^{\perp} \equiv T_1 := \{t \in \mathscr{E}(X) \colon\ \varphi_X(t, z) = 0, \text{ for all } z \in S_1\}.$$

The sections are linearly independent, for if $s_1 = ar_1$, with $a \in \mathscr{A}(X)$, then

$$1 = \varphi_X(r_1, s_1) = \varphi_X(r_1, ar_1) = a\varphi_X(r_1, r_1) = 0,$$

a *contradiction*. So, $\{r_1, s_1\}$ is a basis of S_1. Furthermore, we prove that

$$\text{(i)}\ \ S_1 \cap T_1 = 0, \quad \text{(ii)}\ \ S_1 + T_1 = \mathscr{E}(X).$$

Indeed, (i) since $\varphi_X(r_1, s_1) \neq 0$, we have $S_1 \cap T_1 = 0$. On the other hand, (ii) for every $z \in \mathscr{E}(X)$, one has

$$z = (-\varphi_X(z, r_1)s_1 + \varphi_X(z, s_1)r_1) + (z + \varphi_X(z, r_1)s_1 - \varphi_X(z, s_1)r_1),$$

with

$$-\varphi_X(z, r_1)s_1 + \varphi_X(z, s_1)r_1 \in S_1,$$

and

$$z + \varphi_X(z, r_1)s_1 - \varphi_X(z, s_1)r_1 \in T_1.$$

Thus,

$$\mathscr{E}(X) = S_1 \oplus T_1.$$

The restriction $\varphi_1 \equiv \varphi_{1,X}$ of φ_X to T_1 *is non-degenerate*, because if $z_1 \in T_1$ is such that $\varphi_1(z_1, z) = 0$ for all $z \in T_1$, then $z_1 \in T_1^\perp$ and hence $z_1 \in T_1 \cap T_1^\perp = S_1^\perp \cap T_1^\perp = (S_1 + T_1)^\perp = \mathscr{E}(X)^\perp = 0$; so $z_1 = 0$. (T_1, φ_1) *is thus a symplectic free* $\mathscr{A}(X)$*-module of rank* $2(n-1)$. Repeating the construction above $(n-1)$times, we obtain a strictly decreasing sequence

$$(\mathscr{E}(X), \varphi_X) \supseteq (T_1, \varphi_1) \supseteq \ldots \supseteq (T_{n-1}, \varphi_{n-1})$$

of symplectic free $\mathscr{A}(X)$-modules with rank $T_k = 2(n-k)$, $k = 1, \ldots, n-1$, and also an increasing sequence

$$\{r_1, s_1\} \subseteq \{r_1, r_2; s_1, s_2\} \subseteq \ldots \subseteq \{r_1, \ldots, r_n; s_1, \ldots, s_n\}$$

of gauges; each satisfying the relations (2).

(2) *Case* $I = J \neq \emptyset$. We may assume without loss of generality that $I = J = \{1, 2, \ldots, k\}$, and let S be the subspace spanned by $\{r_1, \ldots, r_k; s_1, \ldots, s_k\}$. Clearly, $\varphi_X|_S$ is non-degenerate; by [1, Lemma (2.31), p. 360], it follows that $S \cap S^\perp = 0$. On the other hand, let $z \in \mathscr{E}(X)$. One has

$$z = \left(-\sum_{i=1}^{k} \varphi_X(z, r_i)s_i + \sum_{i=1}^{k} \varphi_X(z, s_i)r_i\right) + \left(z + \sum_{i=1}^{k} \varphi_X(z, r_i)s_i - \sum_{i=1}^{k} \varphi_X(z, s_i)r_i\right),$$

with

$$-\sum_{i=1}^{k} \varphi_X(z, r_i)s_i + \sum_{i=1}^{k} \varphi_X(z, s_i)r_i \in S,$$

and

$$z + \sum_{i=1}^{k} \varphi_X(z, r_i)s_i - \sum_{i=1}^{k} \varphi_X(z, s_i)r_i \in S^\perp.$$

Thus,

$$\mathscr{E}(X) = S \oplus S^\perp.$$

Based on the hypothesis on S_1 the restriction $\varphi_X|_S$ is a symplectic $\mathscr{A}$-bilinear form. It is also easily seen that the restriction $\varphi_{X\,S^\perp}$ is skew-symmetric. Moreover, since $S \oplus S^\perp = \mathscr{E}(X)$ and $\mathscr{E}(X)^\perp = 0$, if there exist $z_1 \in S^\perp$ such that $\varphi_X(z_1, z) = 0$ for all $z \in S^\perp$, then $z_1 \in \mathscr{E}(X)^\perp = 0$, i.e., $z_1 = 0$. Thus, $\varphi_X|_{S^\perp}$ is non-degenerate and hence a symplectic $\mathscr{A}$-form. Applying Case (1), we obtain a symplectic basis of $S^\perp$, which we denote as

$$\{r_{k+1}, \ldots, r_n; s_{k+1}, \ldots, s_n\}.$$

Then,

$$\mathfrak{B} = \{r_1, \ldots, r_n; s_1, \ldots, s_n\}$$

is a symplectic basis of $\mathscr{E}(X)$ with the required property.

(3) *Case* $J \backslash I \neq \emptyset$ *(or* $I \backslash J \neq \emptyset$*)*. Suppose that $k \in J \backslash I$; since φ_X is non-degenerate there exists $\overline{r}_k \in \mathscr{E}(X)$ such that $\varphi_X(\overline{r}_k, s_k) \neq 0$ in the sense that $\varphi_V(\overline{r}_k|_V, s_k|_V) \neq 0$ for any open $V \subseteq X$. In other words, the section $v \equiv \varphi_X(\overline{r}_k, s_k) \in \mathscr{A}(X)$ is nowhere zero, and is therefore *invertible*. So, if $r_k := v^{-1}\overline{r}_k$, we have $\varphi_X(r_k, s_k) = 1$. Next, let us consider the sub-$\mathscr{A}(X)$-module R, spanned by r_k and s_k, viz. $R = [r_k, s_k]$. As in Case (1), we have

$$\mathscr{E}(X) = R \oplus R^{\perp}.$$

Clearly, for every $i \in I$, $r_i \in R^{\perp}$. To show this, fix i in I, and assume that $r_i = ar_k + bs_k + x$, where $a, b \in \mathscr{A}(X)$ and $x \in R^{\perp}$. So, one has

$$0 = \varphi_X(r_i, s_k) = a, \quad 0 = \varphi_X(r_i, r_k) = b,$$

which corroborates the claim that $r_i \in R^{\perp}$ for all $i \in I$. Furthermore, we also clearly have that for every $j \neq k$ in J, $s_j \in R^{\perp}$. Then $A \cup B \cup \{r_k\}$ is a family of linearly independent sections: the equality

$$a_k r_k + \sum_{i \in I} a_i r_i + \sum_{j \in J} b_j s_j = 0$$

implies that $a_k = a_i = b_j = 0$. Repeating this process as many times as necessary, we are led back to Case (2), and the proof is finished. □

Referring to Theorem 1.1, the basis $\mathfrak{B}$ is called a *symplectic* $\mathscr{A}(U)$*-basis of* $(\mathscr{E}(U), \varphi_U)$. The affine Darboux theorem (cf. [14]) is a major application of the symplectic Gram-Schmidt theorem in Abstract Differential Geometry.

Theorem 1.1 helps improve on Lemma 1.2 as we see it in the following.

Corollary 1.1. *Let* $(\mathscr{E}, \varphi)$ *be a symplectic free* $\mathscr{A}$*-module of finite rank. For any nowhere-zero (local) section* $r \in \mathscr{E}(U)$ *(*U *is an open subset of* X*), there exists a nowhere-zero section* $s \in \mathscr{E}(U)$ *such that* $\varphi_U(r, s)$ *is nowhere zero.*

P r o o f. Apply Theorem 1.1 to find a symplectic basis of $(\mathscr{E}(U), \varphi_U)$ containing the given nowhere-zero section r, then apply Lemma 1.2 to find a nowhere-zero section $s \in \mathscr{E}(U)$ such that $\varphi_U(r, s)$ is nowhere zero. □

Corollary 1.2. *If* $(\mathscr{E}, \varphi)$ *is a symplectic free* $\mathscr{A}$*-module of rank* $2n$*, then, for every open* $U \subseteq X$*,*

$$\mathscr{E}(U) = H_1^U \oplus \ldots \oplus H_n^U,$$

where $H_1^U, \ldots, H_n^U$ *are pairwise orthogonal non-isotropic sub-*$\mathscr{A}(U)$*-modules of rank* 2.

P r o o f. The proof is similar *to a good extent* to the first part of the proof of Theorem 1.1. In fact, let U be an open subset of X and $r_1 \in \mathscr{E}(U)$, a *nowhere-zero section.* There exists a section s_1 in $\mathscr{E}(U)$ such that $\varphi_V(r_1|_V, s_1|_V) \neq 0$ for any open $V \subseteq U$. Clearly, r_1, s_1 must be linearly independent, and the sub-$\mathscr{A}(U)$-module $H_1 \equiv H_1^U := [r_1, s_1]$, *spanned* by r_1 and s_1, is *non-isotropic.* As in the proof of Theorem 1.1, Case (1), one has

$$\mathscr{E}(U) = H_1 \oplus H_1^{\perp}.$$

The restriction $\varphi_{H_1^{\perp}} \equiv (\varphi_U)|_{H_1^{\perp}}$ of φ_U to $H_1^{\perp}$ is non-degenerate, because if $t \in H_1^{\perp}$ is such that $\varphi_{H_1^{\perp}}(t, z) = \varphi_U(t, z) = 0$ for all $z \in H_1^{\perp}$, then $t \in H_1^{\perp\perp} \equiv (H_1^{\perp})^{\perp}$ and hence $t \in H_1^{\perp} \cap H_1^{\perp\perp} = (H_1 + H_1^{\perp})^{\perp} = \mathscr{E}(U)^{\perp} = 0$, which implies that $t = 0$. Thus, $(H_1^{\perp}, \varphi_{H_1^{\perp}})$ is a *symplectic free* $\mathscr{A}(U)$*-module of rank* $2(n-1)$. Next, take a nowhere-zero $r_2 \in H_1^{\perp}$; since $\varphi_U(r_2, r_1) = \varphi_U(r_2, s_1) = 0$, there exists a section $s_2 \in H_1^{\perp}$ such that $\varphi_V(r_2|_V, s_2|_V) \neq 0$ for any open $V \subseteq U$. As above, one has

$$H_1^{\perp} = H_2 \oplus H_2^{\perp},$$

where $H_2 := [r_2, s_2]$. The *direct decomposition sum* of $\mathscr{E}(U)$ follows by repeating the construction above $(n-2)$times. □

Each sub-$\mathscr{A}(U)$-module H_i^U in Corollary 1.2 has an ordered basis (r_i, s_i) such that $(\varphi_U(r_i, s_i))|_V \equiv \varphi_V(r_i|_V, s_i|_V) := a_i|_V \neq 0$ for any open subset V of U. Then, based on the hypothesis that every nowhere-zero section of $\mathscr{A}$ is invertible, the restriction of φ_U to H_i^U with respect to the basis $(r_i, a_i^{-1}s_i)$ has matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Hence, we have

Corollary 1.3. *If* $(\mathscr{E}, \varphi)$ *is a symplectic free* $\mathscr{A}$*-module of rank* $2n$*, then for every open subset* U *of* X*, there exists an ordered basis of* $\mathscr{E}(U)$ *with respect to which* φ_U *has matrix*

$$A_{2n}^U = \begin{pmatrix} \begin{array}{cc|} 0 & 1 \\ -1 & 0 \\ \hline \end{array} & & \\ & \ddots & \\ & & \begin{array}{cc|} 0 & 1 \\ -1 & 0 \\ \hline \end{array} \end{pmatrix}.$$

Moreover, symplectic $\mathscr{A}$*-modules of the same rank are isometric.*

2. Orthosymmetric $\mathscr{A}$-bilinear forms

We shall see in this section that *the "geometry of an $\mathscr{A}$-bilinear form"* (see e.g. [2, p. 111]) *is "local"* (par abus de langage) *on arbitrary $\mathscr{A}$-modules*, (Theorem 2.1) but *"global" on free $\mathscr{A}$-modules of finite rank* (Theorem 2.2). We will assume that the $\mathbb{C}$-algebra sheaf *$\mathscr{A}$ has no zero-divisors* (=: *"strict integral domain"*), that is, for any open $U \subseteq X$, *if $r, s \in \mathscr{A}(U)$ are nowhere-zero sections, then their product rs is nowhere zero.*

For convenience, we state hereby the definition of *orthosymmetric $\mathscr{A}$-bilinear forms* (cf. [18] and [8, p. 90, 91]): *An $\mathscr{A}$-bilinear form φ on an $\mathscr{A}$-module $\mathscr{E}$ is called orthosymmetric if the following is true*:

$$\mathscr{E}^{\perp} = \mathscr{E}^{\top}. \tag{2}$$

Equivalently, for every open $U \subseteq X$ and (local) sections $t \in \mathscr{E}(U)$, $s \in \mathscr{E}(V)$, where V is an open subset of U, we have

$$\varphi_V(s, t|_V) = 0 \quad \text{if, and only if,} \quad \varphi_V(t|_V, s) = 0.$$

It is clear that if φ is orthosymmetric, then $\perp(\varphi) \equiv \perp = \top \equiv \top(\varphi)$, i.e. $\mathscr{F}^{\perp} = \mathscr{F}^{\top}$ for any sub-$\mathscr{A}$-module $\mathscr{F}$ of $\mathscr{E}$, which entails that *orthosymmetry is hereditary, with respect to sub-$\mathscr{A}$-modules.* Of course, *if φ is symmetric or skew-symmetric, then φ is orthosymmetric.* We will show (Theorem 2.2) that *the converse of the preceding statement holds in the special case of free $\mathscr{A}$-modules of finite rank, and $\mathscr{A}$ has no zero divisors.* However, for *arbitrary $\mathscr{A}$-modules*, we have the following.

Theorem 2.1. *Let $\mathscr{A}$ be an strict integral domain $\mathbb{C}$-algebra sheaf, $\mathscr{E}$ an $\mathscr{A}$-module and $\varphi \equiv (\varphi_U)\colon \mathscr{E} \oplus \mathscr{E} \longrightarrow \mathscr{A}$ an orthosymmetric $\mathscr{A}$-bilinear form. Then, "componentwise" (i.e., for every φ_U), φ is either symmetric or skew-symmetric.*

Proof. Let U be an open subset of X, and $r, s, t \in \mathscr{E}(U)$. Clearly, we have

$$\varphi_U(r, \varphi_U(r,t)s) - \varphi_U(r, \varphi_U(r,s)t) = \varphi_U(r,t)\varphi_U(r,s) - \varphi_U(r,s)\varphi_U(r,t) = 0,$$

but

$$\varphi_U(r, \varphi_U(r,t)s - \varphi_U(r,s)t) = 0$$

is equivalent to

$$\varphi_U(\varphi_U(r,t)s - \varphi_U(r,s)t, r) = 0;$$

thus we obtain

$$\varphi_U(r,t)\varphi_U(s,r) = \varphi_U(r,s)\varphi_U(t,r). \tag{3}$$

For $t = r$, $\varphi_U(r,r)\varphi_U(s,r) = \varphi_U(r,s)\varphi_U(r,r)$. If

$$\varphi_V(r|_V, s|_V) \neq \varphi_V(s|_V, r|_V), \quad \text{for any open } V \subseteq U, \tag{4}$$

then ($\mathscr{A}$ is an *strict integral domain algebra sheaf*)

$$\varphi_U(r,r) = 0.$$

(We note in passing that (4) suggests that both $\varphi_V(r|_V, s|_V)$ and $\varphi_V(s|_V, r|_V)$ are nowhere zero on V, because if, for instance, $\varphi_V(r|_V, s|_V)(x) = 0$ for some $x \in V$ then $\varphi_V(r|_V, s|_V) = 0$ on some open neighborhood $R \subseteq V$ of x (cf. [9, (3.7), p. 13]), i.e., assuming that (ϱ^U_V) and (σ^U_V) are the restriction maps for the presheaves of sections of $\mathscr{E}$ and $\mathscr{A}$, respectively, we have

$$\sigma^U_R(\varphi_U(s,r)) = \varphi_R(\varrho^U_R(s), \varrho^U_R(r)) \equiv \varphi_R(s|_R, r|_R) = 0,$$

which, by hypothesis, is equivalent to $\varphi_R(r|_R, s|_R) = 0$. That is a contradiction to (4)).

Similarly, as

$$\varphi_U(s, \varphi_U(s,t)r) - \varphi_U(s, \varphi_U(s,r)t) = 0,$$

which, obviously, leads to

$$\varphi_U(s,t)\varphi_U(r,s) = \varphi_U(s,r)\varphi_U(t,s), \tag{5}$$

one has, for $t = s$,

$$\varphi_U(s,s)\varphi_U(r,s) = \varphi_U(s,r)\varphi_U(s,s).$$

Using (4), we have

$$\varphi_U(s,s) = 0.$$

We actually have *more* than just what we have obtained so far. Indeed, if (4) holds, then $\varphi_U(t,t) = 0$ for all $t \in \mathscr{E}(U)$. We prove this statement as follows.

(A) Let $\varphi_V(r|_V, t|_V) \neq \varphi_V(t|_V, r|_V)$ for any open $V \subseteq U$. Since

$$\varphi_U(t,r)\varphi_U(s,t) = \varphi_U(t,s)\varphi_U(r,t), \tag{6}$$

by putting $s = t$, we have $\varphi_U(t,t) = 0$.

(B) Suppose that there exists an open $W \subseteq U$ such that $\varphi_W(r|_W, t|_W) = \varphi_W(t|_W, r|_W)$. Then, by virtue of (3) and since $\varphi_W(r|_W, s|_W) \neq \varphi_W(s|_W, r|_W)$ everywhere on W, it follows that

$$\varphi_W(r|_W, t|_W) = \varphi_W(t|_W, r|_W) = 0.$$

On the other hand, suppose that $\varphi_V(s|_V, t|_V) \neq \varphi_V(t|_V, s|_V)$ for any open $V \subseteq U$. Putting $r = t$ in (6), one gets $\varphi_U(t,t) = 0$. Now, assume that there exists an open $T \subseteq U$ such that $\varphi_T(s|_T, t|_T) = \varphi_T(t|_T, s|_T)$ and for any open subset $V \subseteq U \setminus \overline{T}$, where $\overline{T}$ is the closure of T in X, $\varphi_V(s|_V, t|_V) \neq \varphi_V(t|_V, s|_V)$. By virtue of (5) and of

$$\varphi_T(s|_T, r|_T) \neq \varphi_T(r|_T, s|_T),$$

it follows that

$$\varphi_T(s|_T, t|_T) = \varphi_T(t|_T, s|_T) = 0.$$

Hence,

$$\varphi_T(r|_T + t|_T, s|_T) = \varphi_T(r|_T, s|_T) \neq \varphi_T(s|_T, r|_T) = \varphi_T(s|_T, r|_T + t|_T),$$

and if we substitute $r|_T + t|_T$ and $s|_T$ for $t|_V$ and $r|_V$ respectively in (A), we get

$$\varphi_T(r|_T + t|_T, r|_T + t|_T) = 0.$$

But $\varphi_T(r|_T, r|_T) = 0$ (since $\varphi_U(r,r) = 0$ and $T \subseteq U$ is open), then if $\varphi_T(r|_T, t|_T) = \varphi_T(t|_T, r|_T) = 0$, one has

$$\varphi_T(t|_T, t|_T) = 0. \tag{7}$$

If both $\varphi_T(r|_T, t|_T)$ and $\varphi_T(t|_T, r|_T)$ are nowhere zero on T, and $\varphi_T(r|_T, t|_T) \neq \varphi_T(t|_T, r|_T)$, we deduce from (6), by putting $s = t$, $\varphi_T(t|_T, t|_T) = 0$. If instead we have $\varphi_T(r|_T, t|_T) = \varphi_T(t|_T, r|_T)$, we will end up with

$$\varphi_T(r|_T, t|_T) = \varphi_T(t|_T, r|_T) = 0,$$

which leads to (7) as previously shown. If there exists an open subset $L \subseteq T$ such that $\varphi_L(r|_L, t|_L) = \varphi_L(t|_L, r|_L) = 0$ and $\varphi_V(r|_V, t|_V) \neq \varphi_V(t|_V, r|_V)$ for every $V \subseteq T \setminus \overline{L}$, where $\overline{L}$ is the closure of L in X, then $\varphi_L(t|_L, t|_L) = 0$ and $\varphi_V(t|_V, t|_V) = 0$ for every open $V \subseteq T \setminus \overline{L}$. Hence, by the fact that sections are continuous $\varphi_T(t|_T, t|_T) = 0$. Next, $\varphi_V(s|_V, t|_V) \neq \varphi_V(t|_V, s|_V)$ for every open $V \subseteq U \setminus \overline{T}$, so $\varphi_V(t|_V, t|_V) = 0$ for every such V; coupling the latter observation with (7) and continuity of sections, one gets in this case too that $\varphi_U(t,t) = 0$.

We have shown that there are only two cases: either $\varphi_U(r,r) = 0$ for all $r \in \mathscr{E}(U)$, or for some $r \in \mathscr{E}(U)$, $\varphi_U(r,r) \neq 0$, from which we deduce that $\varphi_U(s,t) = \varphi_U(t,s)$ for all $s, t \in \mathscr{E}(U)$.

Finally, we notice in ending the proof that if $\varphi_U(r,r) = 0$ for all $r \in \mathscr{E}(U)$, then

$$\varphi_U(r,s) = -\varphi_U(s,r)$$

for all $r, s \in \mathscr{E}(U)$. □

Referring still to Theorem 2.1, if φ_U is symmetric, the geometry is called *orthogonal.* If φ_U is skew-symmetric, the geometry is called *symplectic.* No other case can occur if φ must be orthosymmetric. A *pairing* $(\mathscr{E}, \varphi)$ is called *symmetric*, respectively, *skew-symmetric* if φ is such componentwise.

The classical case (cf. [6, p. 4, Proposition 1.1.3]) turns out to be a particular case of Theorem 2.1, as we see it in the following.

Corollary 2.1. *Let $\mathscr{E}$ be a free $\mathscr{A}$-module of finite rank (with respect to a $\mathbb{C}$-algebra sheaf $\mathscr{A}$ that has no zero divisor sections) and $\varphi\colon \mathscr{E}\oplus\mathscr{E} \longrightarrow \mathscr{A}$ an orthosymmetric $\mathscr{A}$-bilinear form. Then, φ is either symmetric or skew-symmetric.*

Proof. Let us assume that $\{s_1, \dots, s_n\}$ is a basis of $\mathscr{E}(X)$. According to Theorem 2.1, φ_X is either symmetric or skew-symmetric, and since for any open U in X, $\{s_1|_U, \dots, s_n|_U\}$ is a basis of $\mathscr{E}(U)$, it follows that φ_U is symmetric (resp. skew-symmetric) if φ_X is symmetric (resp. skew-symmetric). □

The above discussion can be summarized in the following.

Theorem 2.2. *Let $\mathscr{A}$ be a $\mathbb{C}$-algebra sheaf with no zero divisors, $\mathscr{E}$ a free $\mathscr{A}$-module of finite rank and an $\mathscr{A}$-bilinear form $\varphi\colon \mathscr{E}\oplus\mathscr{E} \longrightarrow \mathscr{A}$. Then, φ is orthosymmetric if, and only if, it is either symmetric or skew-symmetric.*

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