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HOMOGENIZED DOUBLE POROSITY MODELS FOR PORO-ELASTIC MEDIA WITH INTERFACIAL FLOW BARRIER

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Abstract. In the paper a Barenblatt-Biot consolidation model for flows in periodic porous elastic media is derived by means of the two-scale convergence technique. Starting with the fluid flow of a slightly compressible viscous fluid through a two-component poro-elastic medium separated by a periodic interfacial barrier, described by the Biot model of consolidation with the Deresiewicz-Skalak interface boundary condition and assuming that the period is too small compared with the size of the medium, the limiting behavior of the coupled deformation-pressure is studied.

Keywords: homogenization, porelasticity, two-scale convergence

MSC 2010: 35B27, 74Q05, 76M50

1. Introduction

The concept of double porosity was introduced by Aifantis to consider diffusion or infiltration processes in porous deformable media which are characterized by two distinct families of diffusion or flow paths (usually pores and fissures). The derived equations read as follows (see [10]):

$$-\mu\Delta\mathbf{u} - (\lambda+\mu)\nabla(\operatorname{div}\mathbf{u}) + \alpha^{(1)}\nabla p^{(1)} + \alpha^{(2)}\nabla p^{(2)} = \mathbf{f}, \tag{1.1}$$

$$c^{(1)}\partial_t p^{(1)} + \alpha^{(1)}\operatorname{div}(\partial_t\mathbf{u}) - k^{(1)}\Delta p^{(1)} + h(p^{(1)} - p^{(2)}) = g^{(1)}, \tag{1.2}$$

$$c^{(2)}\partial_t p^{(2)} + \alpha^{(2)}\operatorname{div}(\partial_t\mathbf{u}) - k^{(2)}\Delta p^{(2)} - h(p^{(1)} - p^{(2)}) = g^{(2)}. \tag{1.3}$$

Here $\mathbf{u}$ is the displacement of the medium; the elastic constants λ and μ are referred to as the dilation and shear moduli of elasticity, respectively; $p^{(1)}$ and $p^{(2)}$ are the pressure of the fluid in the pores and fissures, respectively; $c^{(m)}$ $(m=1,2)$ is the compressibility, $k^{(m)}$ is the permeability and $\alpha^{(m)}$ is the pressure-deformation;

they are well-known in literature as the Biot-Willis parameters [6]. It is a measure of changes of porosities in each phase $m = 1, 2$ due to applied volumetric strains. The dilation $\alpha^{(m)} \operatorname{div}(\partial_t \mathbf{u})$ accounts then for the *additional* pore fluid content while the term $\alpha^{(m)} \nabla p^{(m)}$ for the pressure stress of the pore fluid on the structure.

We observe that if we let the volume of fissures shrink to zero so that $c^{(2)}$, $\alpha^{(2)}$, $k^{(2)}$, h become negligible then the system (1.1)–(1.3) reduces to the Biot system with single porosity [5]

$$-\mu \Delta \mathbf{u} - (\lambda + \mu) \nabla (\operatorname{div} \mathbf{u}) + \alpha^{(1)} \nabla p^{(1)} = \mathbf{f}, \tag{1.4}$$

$$c^{(1)} \partial_t p^{(1)} + \alpha^{(1)} \operatorname{div}(\partial_t \mathbf{u}) - k^{(1)} \Delta p^{(1)} = g^{(1)}. \tag{1.5}$$

On the other hand, by neglecting the deformation effects λ, μ and $\alpha^{(m)}$ the system (1.1)–(1.3) reduces to the Barenblatt-Zheltov-Kochina model with double porosity [4]

$$c^{(1)} \partial_t p^{(1)} - k^{(1)} \Delta p^{(1)} + h(p^{(1)} - p^{(2)}) = g^{(1)}, \tag{1.6}$$

$$c^{(2)} \partial_t p^{(2)} - k^{(2)} \Delta p^{(2)} - h(p^{(1)} - p^{(2)}) = g^{(2)}. \tag{1.7}$$

Aifantis' theory of consolidation with the concept of double porosity unifies then the proposed models (1.4)–(1.5) of Biot for consolidation of deformable porous media with single porosity and (1.6)–(1.7) of Barenblatt for fluid flow through undeformable porous media with double porosity.

It is the aim of this paper to derive a *more general* system in which, at the microscale, the inhomogeneities are taken into account. More precisely, we consider porous elastic inclusions periodically distributed and embedded in an extra porous elastic matrix. The micro-model is based on Biot's system for consolidation processes with interfacial barrier formulation. The macro-model is then derived by means of the two scale convergence technique and it reads as follows:

$$-\operatorname{div} \sigma(\mathbf{u}) + \alpha^{(1)} \nabla p^{(1)} + \alpha^{(2)} \nabla p^{(2)} = \mathbf{f}, \tag{1.8}$$

$$(\widetilde{c^{(m)}} p^{(m)} + \operatorname{tr}(\beta^{(m)} e(\mathbf{u}))_t - \operatorname{div}[K^{(m)} \nabla p^{(m)}] - (-1)^m \widetilde{h}(p^{(1)} - p^{(2)}) = g^{(m)} \tag{1.9}$$

where σ, $\alpha^{(m)}$, $\beta^{(m)}$ and $K^{(m)}$ are some effective tensors (see (3.10)–(3.11) for their definition). It is then worth pointing out that the Aifantis model (1.1)–(1.3) can be seen as a special case of the homogenized model (1.8)–(1.9) ($\beta^{(m)} = \alpha^{(m)} = \alpha^{(m)} \mathrm{I}_n$, I_n is the identity matrix).

The outline of the paper is as follows: Section 2 is devoted to the problem setting of the governing equations at the microscale for double-diffusion model in heterogeneous media. Section 3 is aimed towards deriving, via the two scale convergence technique, the Barenblatt-Biot model which is stated in the main result of the paper, Theorem 3.1.

Throughout this paper, integration symbols $\mathrm{d}x, \mathrm{d}y, \mathrm{d}t, \ldots$ will be omitted.

2. The micro-model

We consider a bounded domain $\Omega \subset \mathbb{R}^n$ with a smooth boundary Γ. Let $Y = [0,1]^n$ ($n \geqslant 3$) denote the generic cell of periodicity divided as $Y := Y^{(1)} \cup Y^{(2)} \cup \Sigma$ where $Y^{(1)}$, $Y^{(2)}$ are two open subsets and $\Sigma := \partial Y^{(1)} \cap \partial Y^{(2)}$ is the interface that separates them. We assume that the Y-periodic continuation of $Y^{(m)}$ defined as $\bigcup_{k \in \mathbb{Z}^n} (k + Y^{(m)})$ is open and connected.

Let $\varepsilon > 0$ be a sufficiently small positive number. We define the *inclusions*

$$\Omega_\varepsilon^{(2)} := \bigcup_{k \in K_\varepsilon} (\varepsilon k + \varepsilon Y^{(2)})$$

where $K_\varepsilon = \{k \in \mathbb{Z}^n \colon \ \varepsilon k + \varepsilon \overline{Y^{(2)}} \subset \Omega\}$. Let

$$\Omega_\varepsilon^{(1)} := \Omega \setminus \overline{\Omega_\varepsilon^{(2)}}$$

be the *matrix* part and

$$\Sigma_\varepsilon := \partial \Omega_\varepsilon^{(1)} \cap \partial \Omega_\varepsilon^{(2)}$$

the periodic *interface* between these two materials assumed to be sufficiently smooth. We have then $\partial \Omega_\varepsilon^{(2)} \cap \partial \Omega = \emptyset$. Note that other geometrical settings can be considered without affecting the main result of the paper, see for example [8], [1].

We assume that the material $\Omega_\varepsilon^{(m)}$ ($m = 1, 2$) is saturated with a slightly compressible and viscous fluid with pore-pressure denoted by $p_\varepsilon^{(m)}$ and let

$$c_\varepsilon^{(m)}(x) = c^{(m)}\Big(\frac{x}{\varepsilon}\Big), \quad K_\varepsilon^{(m)}(x) = \Big(k_{ij}^{(m)}\Big(\frac{x}{\varepsilon}\Big)\Big)_{1 \leqslant i,j \leqslant n}$$

denote respectively the combined compressibility-porosity and the permeability. We assume that $c^{(m)}$ is a smooth and Y-periodic function such that $c^{(m)}(y) \geqslant C > 0$ where (here and throughout this paper) C is any positive constant independent of ε. We also assume that $(k_{ij}^{(m)}(y))_{1 \leqslant i,j \leqslant n}$ is smooth, Y-periodic and satisfies the following symmetry and ellipticity conditions:

$$k_{ij}^{(m)}(y) = k_{ij}^{(m)}(y),$$

$$\sum_{i,j=1}^{n} k_{ij}^{(m)}(y) \eta_j \eta_i \geqslant C \sum_{i=1}^{n} \eta_i \eta_i, \quad \forall y \in Y^{(m)}, \ \forall \eta = (\eta_i) \in \mathbb{R}^n. \tag{2.1}$$

Let the total stress tensor at any point in $\Omega_\varepsilon^{(m)}$ $(m = 1, 2)$ be given by

$$\sigma_\varepsilon^{(m)} = \mathcal{A}_\varepsilon e(\mathbf{u}_\varepsilon) - \alpha^{(m)} p_\varepsilon^{(m)} \mathrm{I}_n \quad \text{in } \Omega_\varepsilon^{(m)}$$

where $\mathbf{u}_\varepsilon$ denotes the displacement in Ω, $e(\cdot)$ is the linearized stain tensor and $\mathcal{A}_\varepsilon = \big(a_{ijkl}(x/\varepsilon)\big)_{1\leqslant i,j,k,l\leqslant n}$ is the elasticity tensor of Ω where $a_{ijkl}(y)$ are smooth, Y-periodic and satisfy

$$\begin{gathered} a_{ijkl}(y) = a_{jikl}(y) = a_{ijlk}(y) = a_{klij}(y), \\ \sum_{i,j,k,l=1}^{n} a_{ijkl}(y)\eta_{kl}\eta_{ij} \geqslant C \sum_{i,j=1}^{n} \eta_{ij}\eta_{ij},\ \forall y \in Y,\ \forall \eta = (\eta_{ij}) \in \mathbb{R}^{n\times n},\ \eta_{ij} = \eta_{ji}. \end{gathered} \tag{2.2}$$

We shall include here the situation when the inertia effects are negligible both in the matrix and inclusions materials. Let $T > 0$ and let $t \in [0, T]$ denote the time variable. The micro-model for the Biot system with interfacial barrier formulation reads as follows: for $m = 1, 2$,

$$-\operatorname{div} \sigma_\varepsilon^{(m)} = \mathbf{f} \quad \text{in } (0,T)\times\Omega_\varepsilon^{(m)}, \tag{2.3}$$

$$(c_\varepsilon^{(m)} p_\varepsilon^{(m)} + \alpha_\varepsilon^{(m)} \operatorname{div} \mathbf{u}_\varepsilon)_t - \operatorname{div}(K_\varepsilon^{(m)} \nabla p_\varepsilon^{(m)}) = 0 \quad \text{in } (0,T)\times\Omega_\varepsilon^{(m)}, \tag{2.4}$$

$$[\mathbf{u}_\varepsilon]_{\Sigma_\varepsilon} = \mathbf{0}, [\mathcal{A}_\varepsilon e(\mathbf{u}_\varepsilon)]_{\Sigma_\varepsilon} \cdot \mathbf{n}_\varepsilon = \mathbf{0} \quad \text{on } (0,T)\times\Sigma_\varepsilon, \tag{2.5}$$

$$(K_\varepsilon^{(1)} \nabla p_\varepsilon^{(1)}) \cdot \mathbf{n}_\varepsilon = -\varepsilon h(\frac{x}{\varepsilon})[p_\varepsilon^{(m)}]_{\Sigma_\varepsilon} \quad \text{on } (0,T)\times\Sigma_\varepsilon, \tag{2.6}$$

$$[K_\varepsilon^{(m)} \nabla p_\varepsilon^{(m)}]_{\Sigma_\varepsilon} \cdot \mathbf{n}_\varepsilon = \mathbf{0} \quad \text{on } (0,T)\times\Sigma_\varepsilon, \tag{2.7}$$

$$\mathbf{u}_\varepsilon = \mathbf{0} \quad \text{and } p_\varepsilon^{(1)} = 0 \quad \text{on } (0,T)\times\Gamma, \tag{2.8}$$

$$\mathbf{u}_\varepsilon(0,\cdot) = \mathbf{0}, p_\varepsilon^{(m)}(0,\cdot) = 0 \quad \text{in } \Omega_\varepsilon^{(m)} \tag{2.9}$$

where $\mathbf{f} \in \mathbf{L}^2(\Omega)$ is the volume-distributed force in Ω, the subscript $(\cdot)_t$ represents the time derivative, $[\cdot]_{\Sigma_\varepsilon}$ denotes the jump across Σ_ε, $\mathbf{n}_\varepsilon$ stands for the unit normal to Σ_ε pointing out into $\Omega_\varepsilon^{(2)}$ and $h(y)$ is the rescaled interface hydraulic permeability function assumed to be smooth and Y-periodic on $\mathbb{R}^n$ such that $h(y) \geqslant C > 0$. The interfacial barrier exchange formulation (2.6) is well-known in literature as the Deresiewicz-Skalak boundary condition [7].

Let

$$\begin{gathered} \mathbf{H}_\varepsilon := [H_0^1(\Omega)]^n, \mathbf{L}_\varepsilon := L^2(\Omega_\varepsilon^{(1)}) \times L^2(\Omega_\varepsilon^{(2)}), \\ V_\varepsilon^{(1)} := \{v \in H^1(\Omega_\varepsilon^{(1)});\ v_{|\Gamma} = 0\},\ V_\varepsilon^{(2)} := H^1(\Omega_\varepsilon^{(2)}) \end{gathered}$$

and let us introduce the Hilbert space $\mathbf{V}_\varepsilon := V_\varepsilon^{(1)} \times V_\varepsilon^{(2)}$ equipped with the norm

$$\|\mathbf{p}\|_{V_\varepsilon}^2 = \|\nabla p^{(1)}\|_{L^2(\Omega_\varepsilon^{(1)})}^2 + \|\nabla p^{(2)}\|_{L^2(\Omega_\varepsilon^{(2)})}^2 + \varepsilon \| \,[\mathbf{p}]_{\Sigma_\varepsilon} \|_{L^2(\Sigma_\varepsilon)}^2,\ \mathbf{p} = (p^{(1)}, p^{(2)}).$$

The variational formulation of (2.3)–(2.9) can be read as follows: Find $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon) \in L^\infty(0,T;\mathbf{H}_\varepsilon) \times L^2(0,T;\mathbf{V}_\varepsilon)$, $\mathbf{p}_\varepsilon = (p_\varepsilon^{(1)}, p_\varepsilon^{(2)})$ such that

$$(2.10)\qquad (c_\varepsilon^{(m)} p_\varepsilon^{(m)} + \alpha_\varepsilon^{(m)} \operatorname{div} \mathbf{u}_\varepsilon)_t \in L^2(0,T;(V_\varepsilon^{(m)})^*),$$

$$(2.11)\qquad \int_\Omega \mathcal{A}_\varepsilon e(\mathbf{u}_\varepsilon) e(\mathbf{v}) - \sum_m \int_{\Omega_\varepsilon^{(m)}} \alpha_\varepsilon^{(m)} p_\varepsilon^{(m)} \operatorname{div} \mathbf{v} = \int_\Omega \mathbf{f}\mathbf{v} \quad \text{for all } v \in \mathbf{H}_\varepsilon,$$

(2.12)

$$\sum_m \left(\langle (c_\varepsilon^{(m)} p_\varepsilon^{(m)} + \alpha_\varepsilon^{(m)} \operatorname{div} \mathbf{u}_\varepsilon)_t, q^{(m)} \rangle_{(V_\varepsilon^{(m)})^*, V_\varepsilon^{(m)}} + \int_{\Omega_\varepsilon^{(m)}} K_\varepsilon^{(m)} \nabla p_\varepsilon^{(m)} \nabla q^{(m)} \right)$$
$$+\varepsilon \int_{\Sigma_\varepsilon} h_\varepsilon \, [\mathbf{p}_\varepsilon]_{\Sigma_\varepsilon} \, [\mathbf{q}]_{\Sigma_\varepsilon} = 0 \quad \text{for all } \mathbf{q} = (q^{(1)}, q^{(2)}) \in V_\varepsilon^{(1)} \times V_\varepsilon^{(2)},$$

$$(2.13)\qquad \mathbf{u}_\varepsilon(0,\cdot) = \mathbf{0}, p_\varepsilon^{(m)}(0,\cdot) = 0 \quad \text{in } \Omega_\varepsilon^{(m)} \ (m = 1,2).$$

Theorem 2.1 (see [9]). *For any sufficiently small $\varepsilon > 0$ there exists a solution $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon) \in L^\infty(0,T;\mathbf{H}_\varepsilon) \times L^2(0,T;\mathbf{V}_\varepsilon)$ of the system* (2.10)–(2.13).

Next, we shall give uniform a priori estimates. By taking $\mathbf{v} = (\mathbf{u}_\varepsilon)_t$ in (2.11) and $q^{(m)}(\cdot) = p_\varepsilon^{(m)}(t,\cdot)$ ($t \in [0,T]$ and $m = 1,2$) in (2.12), adding these two equations and integrating over $(0,t)$, we find

$$(2.14)\qquad \int_\Omega \mathcal{A}_\varepsilon e(\mathbf{u}_\varepsilon) e(\mathbf{u}_\varepsilon) + \frac{1}{2} \sum_m \int_{\Omega_\varepsilon^{(m)}} \alpha_\varepsilon^{(m)} (p_\varepsilon^{(m)})^2$$
$$+ \sum_m \int_0^t \int_{\Omega_\varepsilon^{(m)}} K_\varepsilon^{(m)} \nabla p_\varepsilon^{(m)} \nabla p_\varepsilon^{(m)} + \varepsilon \int_0^t \int_{\Sigma_\varepsilon} h_\varepsilon ([\mathbf{p}_\varepsilon]_{\Sigma_\varepsilon})^2 = \int_\Omega \mathbf{f}\mathbf{u}_\varepsilon.$$

Now, using Korn's and Poincaré's inequalities on the right hand sides of (2.14) and taking into account (2.1), (2.2) we get

$$(2.15)\qquad \|\mathbf{u}_\varepsilon\|_{L^\infty(0,T;\mathbf{H}_\varepsilon)} + \|\mathbf{p}_\varepsilon\|_{L^2(0,T;\mathbf{V}_\varepsilon)} + \|\mathbf{p}_\varepsilon\|_{L^\infty(0,T;\mathbf{L}_\varepsilon)} \leqslant C.$$

In view of the estimate (2.15), one is led to study the limiting behaviour as $\varepsilon \to 0$ of the sequence $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon)$. This is the scope of the next section.

3. Homogenization procedure

The study of the limiting behaviour of $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon)$ is performed by the two scale convergence technique. For more details on this method, we refer the reader for instance to [2], [3].

In view of the a priori estimate (2.15) and owing to [2, Theorem 2.9], [3, Proposition 2.6] there exist a subsequence still denoted $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon)$ and $\mathbf{u} \in L^\infty(0,T;\mathbf{H}^1_0(\Omega))$, $\mathbf{u}_1 \in L^\infty(0,T;\mathbf{L}^2(\Omega;\mathbf{H}^1_\#(Y)/\mathbb{R}^n))$, $p^{(m)} \in L^\infty(0,T;H^1_0(\Omega))$, $p_1^{(m)} \in L^2((0,T)\times\Omega;H^1_\#(Y^{(m)})/\mathbb{R})$ $(m=1,2)$ such that for a.e. $t \in (0,T)$, for all $\mathbf{\Phi} = (\varphi_{ij})_{1\leqslant i,j\leqslant n}$, $\varphi_{ij} = \varphi_{ji} \in \mathcal{D}(\Omega;\mathcal{C}^\infty(Y))$, $\psi \in \mathcal{D}((0,T)\times\Omega;\mathcal{C}^\infty(Y))$ and $\varphi \in \mathcal{D}((0,T)\times\Omega;\mathcal{C}^\infty(Y))^n$, the following two scale convergences hold:

$$\lim_{\varepsilon\to 0}\int_\Omega e(\mathbf{u}_\varepsilon)\mathbf{\Phi}_\varepsilon = \int_{\Omega\times Y}(e(\mathbf{u}) + e_y(\mathbf{u}_1))\mathbf{\Phi}, \tag{3.1}$$

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^{(m)}} p_\varepsilon^{(m)}\psi_\varepsilon = \int_{\Omega\times Y^{(m)}} p^{(m)}\psi, \tag{3.2}$$

$$\lim_{\varepsilon\to 0}\int_0^t\int_{\Omega_\varepsilon^{(m)}} \nabla p_\varepsilon^{(m)}\varphi_\varepsilon = \int_0^t\int_{\Omega\times Y^{(m)}} (\nabla p^{(m)} + \nabla_y p_1^{(m)})\varphi, \tag{3.3}$$

$$\lim_{\varepsilon\to 0}\int_0^t\int_{\Sigma_\varepsilon} \varepsilon(p_\varepsilon^{(1)} - p_\varepsilon^{(2)})\psi_\varepsilon = \int_0^t\int_{\Omega\times\Sigma}(p^{(1)} - p^{(2)})\psi \tag{3.4}$$

where $\mathbf{\Phi}_\varepsilon(x) = \mathbf{\Phi}(x, x/\varepsilon)$, $\psi_\varepsilon(t,x) = \psi(t,x,x/\varepsilon)$ and $\varphi_\varepsilon(t,x) = \varphi(t,x,x/\varepsilon)$.

Next we introduce the test functions: let

$$\Psi_\varepsilon(x) = \Psi(x) + \varepsilon\Psi_1\Big(x, \frac{x}{\varepsilon}\Big)$$

where $\Psi \in \mathcal{D}(\Omega)^n$ and $\Psi_1 \in \mathcal{D}(\Omega;\mathcal{C}^\infty_\#(Y))^n$ and let also

$$\psi_\varepsilon^{(m)}(t,x) = \psi^{(m)}(t,x) + \varepsilon\psi_1^{(m)}\Big(t,x,\frac{x}{\varepsilon}\Big)$$

$(m=1,2)$ where $\psi^{(m)} \in \mathcal{D}((0,T)\times\Omega)$ and $\psi_1^{(m)} \in \mathcal{D}((0,T)\times\Omega;\mathcal{C}^\infty_\#(Y))$. Taking $\mathbf{v} = \Psi_\varepsilon$ in (2.11) we have

$$\begin{aligned}\int_\Omega \mathcal{A}_\varepsilon e(\mathbf{u}_\varepsilon)(e(\Psi) + \varepsilon e(\Psi_1) + e_y(\Psi_1)) \\ -\sum_m\int_{\Omega_\varepsilon^{(m)}} \alpha_\varepsilon^{(m)} p_\varepsilon^{(m)}(\operatorname{div}\Psi + \varepsilon\operatorname{div}\Psi_1 + \operatorname{div}_y\Psi_1) = \int_\Omega \mathbf{f}(\Psi + \varepsilon\Psi_1).\end{aligned} \tag{3.5}$$

According to (3.1), (3.2), letting $\varepsilon \to 0$ in (3.5) we obtain

$$\begin{aligned}\int_{\Omega\times Y}\mathcal{A}(e(\mathbf{u}) + e_y(\mathbf{u}_1))(e(\Psi) + e_y(\Psi_1)) \\ -\sum_m\int_{\Omega\times Y^{(m)}} \alpha^{(m)} p_\varepsilon^{(m)}(\operatorname{div}\Psi + \operatorname{div}_y\Psi_1) = \int_\Omega \mathbf{f}\Psi.\end{aligned} \tag{3.6}$$

Similarly, we pass to the limit in (2.12) with $t = T$, $q^{(m)} = \psi_\varepsilon^{(m)}$ and taking into account (3.1), (3.2) and (3.4) we find

$$
\begin{aligned}
(3.7) \qquad -\sum_m \int_0^T \int_{\Omega \times Y^{(m)}} &[(c^{(m)} p^{(m)} + \alpha^{(m)} (\operatorname{div} \mathbf{u} + \operatorname{div}_y \mathbf{u}_1)) \psi_t^{(m)} \\
&+ K^{(m)} (\nabla p^{(m)} + \nabla_y p_1^{(m)})(\nabla \psi^{(m)} + \nabla_y \psi_1^{(m)})] \\
&+ \int_0^T \int_{\Omega \times \Sigma} h(p^{(1)} - p^{(2)})(\psi^{(1)} - \psi^{(2)}) = 0.
\end{aligned}
$$

In view of the linearity of the equation (3.6) we can write

$$
(3.8) \qquad \mathbf{u}_1(t,x,y) = -\sum_{k,h=1}^{n} \frac{\partial \mathbf{u}_h}{\partial x_k}(t,x) \lambda^{kh}(y) + c(x)
$$

where for $1 \leqslant k,h \leqslant n$, $\lambda^{kh} = (\lambda_i^{kh})_{1 \leqslant i \leqslant n} \in \mathbf{H}^1_\#(Y)/\mathbb{R}^n$ is the solution of the microscopic problem

$$
\mathfrak{a}(\lambda^{kh} - P^{kh}, \mathbf{w}) = 0, \quad \forall \mathbf{w} \in \mathbf{H}^1_\#(Y)/\mathbb{R}^n,
$$

where $P^{kh} = (y_k \delta_{hj})_{1 \leqslant j \leqslant n}$, δ_{hj} is the Krönecker symbol and

$$
\mathfrak{a}(\mathbf{z}, \mathbf{w}) = \int_Y \mathcal{A} e_y(\mathbf{z}) e_y(\mathbf{w}), \quad \mathbf{z}, \mathbf{w} \in \mathbf{H}^1_\#(Y)/\mathbb{R}^n.
$$

Similarly, we seek $p_1^{(m)}$ $(m = 1, 2)$ in the form

$$
(3.9) \qquad p_1^{(m)}(t,x,y) = -\sum_{i=1}^{n} \frac{\partial p^{(m)}}{\partial x_i}(t,x) \zeta_i^{(m)}(y) + c_1(x),
$$

where $\zeta_i^{(m)} \in (H^1(Y^{(m)}))/\mathbb{R}$ is a solution of the micro-pressure equation

$$
\begin{gathered}
-\operatorname{div}_y(K^{(m)}(y)(\nabla \zeta_i^{(m)} + e_i)) = 0 \quad \text{in } Y^{(m)}, \\
(\nabla \zeta_i^{(m)} + e_i) \cdot \mathbf{n} = 0 \quad \text{on } \Sigma, \\
y \longmapsto \zeta_i^{(m)}(y) \colon \ Y\text{-periodic}
\end{gathered}
$$

where e_i is the ith vector of the canonical basis of $\mathbb{R}^n$ and $\mathbf{n}$ is the unit normal to Σ pointing out into $Y^{(2)}$.

Let

$$(3.10)\qquad \mathfrak{a}_{ijkl} = \mathfrak{a}(\lambda^{kh} - P^{kh}, \lambda^{ij} - P^{ij}), \quad \sigma_{ij}(\mathbf{u}) = \sum_{i,j=1}^{n} \mathfrak{a}_{ijkl} e_{kl}(\mathbf{u}),$$

(3.11)

$$K_{ij}^{(m)} = \int_{Y^{(m)}} K_{il}^{(m)}(y)\Big(\delta_{lj} + \frac{\partial \zeta_j^{(m)}}{\partial y_l}\Big), \quad \alpha_{kh}^{(m)} = \int_{Y^{(m)}} \Big(\alpha^{(m)} \sum_{i=1}^{n} \Big(\delta_{ik}\delta_{ih} - \frac{\partial \lambda_i^{kh}}{\partial y_i}\Big)\Big),$$

$$(3.12)\qquad \beta_{ij}^{(m)} = \int_{Y^{(m)}} \alpha^{(m)} K^{(m)}(y)(\nabla\zeta_i - e_i)(\nabla\zeta_j - e_j),$$

$$(3.13)\qquad \widetilde{c^{(m)}} = \int_{Y^{(m)}} c^{(m)}(y), \quad \widetilde{h} = \int_{\Sigma} h(s).$$

Finally, inserting (3.8), (3.9) into (3.6), (3.7) together with an integration by parts we obtain the following homogenization result:

Theorem 3.1. *The two scale limits* $(\mathbf{u}, p^{(1)}, p^{(2)})$ $(m = 1, 2)$ *satisfy the homogenized model*

$$\begin{gathered}
-\operatorname{div}\sigma(\mathbf{u}) + \alpha^{(1)}\nabla p^{(1)} + \alpha^{(2)}\nabla p^{(2)} = \mathbf{f}, \quad \text{in } (0,T)\times\Omega,\\
(\widetilde{c^{(1)}} p^{(1)} + \operatorname{tr}(\beta^{(1)} e(\mathbf{u}))_t - \operatorname{div}[K^{(1)}\nabla p^{(1)}] + \widetilde{h}(p^{(1)} - p^{(2)}) = 0 \quad \text{in } (0,T)\times\Omega,\\
(\widetilde{c^{(2)}} p^{(m)} + \operatorname{tr}(\beta^{(2)} e(\mathbf{u}))_t - \operatorname{div}[K^{(2)}\nabla p^{(2)}] - \widetilde{h}(p^{(1)} - p^{(2)}) = 0 \quad \text{in } (0,T)\times\Omega\\
\mathbf{u} = 0, \quad p^{(m)} = 0 \quad \text{on } (0,T)\times\Gamma,\\
u(0,x) = 0, \quad p^{(m)}(0,x) = 0 \quad \text{in } \Omega.
\end{gathered}$$

We have thus shown that the "general" Aifantis model can be obtained with help of the multiscale homogenization technique starting with a Biot micro-model for a two component heterogeneous media with interfacial exchange barrier. An interesting problem is to investigate the limiting behaviour of such media when the flow potential in the inclusions is rescaled by ε^2. This occurs especially when the flow in the inclusions presents very high frequency spatial variations as a result of a relatively very low permeability, for instance $\varepsilon^2 K^{(2)}$.

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