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ON THE EXISTENCE OF A HAAR MEASURE IN TOPOLOGICAL IP-LOOPS

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In this paper, we give conditions ensuring the existence of a Haar measure in topological IP-loops.

Keywords: quasigroup, topological IP-loop, Haar measure, content, uniform, space, left-invariant uniformity

Classification: 28C10, 20N05

1. INTRODUCTION

The Haar measure was introduced by Alfréd Haar, a Hungarian mathematician, in about 1932. Haar measures are used in many parts of analysis and number theory, and also in the estimation theory. It is known (see, e. g., [5, 6, 19]) that in every locally compact topological group there exists at least one Haar measure. This is not true in the case of topological quasigroups, what we illustrate by presented example. We give conditions ensuring the existence of a Haar measure in locally compact topological IP-loops. Our proof of the existence of a Haar measure in topological IP-loops follows the ideas of the classical proof for locally compact topological groups which can be found in Halmos's book [6]. It is based analogously as in [6] on the construction of a nonzero left-invariant content.

Note that some results on the existence of invariant measures on certain types of bicompact semigroups can be found in [17]. But as far as we know, the existence of invariant measures has not been studied for non-associative structures yet.

2. TOPOLOGICAL QUASIGROUPS AND TOPOLOGICAL IP-LOOPS

First, we give the definitions of some algebraic notions and some facts which will be used in the following.

A non-empty set G is said to be a groupoid relative to a binary operation denoted by $\cdot$, if for every ordered pair a, b of elements in G a unique element $ab \in G$ is defined. Instead of $a \cdot b$ we write ab. A quasigroup is a groupoid $(G, \cdot)$ in which for every two elements $a, b \in G$ every of the equations $ax = b$ and $ya = b$ has a unique solution in G. If a quasigroup G contains an element e such that $ex = xe = x$ for all x in G, then e is called an identity element of G and G is called a loop. It is easy to verify that every associative loop is a group.

A quasigroup $(G,\cdot)$ is called an IP-quasigroup (or a quasigroup with the invertibility property), if there exist mappings $f_P : G \to G$ and $f_L : G \to G$ such that, for any $x, y \in G$, it holds

(*i*) $(xy)\, f_P\,(y) = x,$

(*ii*) $f_L\,(x)\,(xy) = y.$

An IP-quasigroup with an identity element is called an IP-loop (or a loop with the invertibility property).

Let $(G,\cdot)$ be a loop with an identity element e and let $a \in G$. Then every of the equations $ax = e$ and $ya = e$ has a unique solution in G. The element x is called a right inverse element to the element a and we denote it by a^{-1}. Analogously, the element y is called a left inverse element to the element a and we denote it by ${}^{-1}a$. Let $(G,\cdot)$ be an IP-loop. If we put $x = e$ in (i) and $y = e$ in (ii), we see that $f_P\,(y) = y^{-1}$ and $f_L\,(x) ={}^{-1} x$. For every elements $x, y \in G$ there holds ${}^{-1}x\,(xy) = = y$. If we put $y = x^{-1}$, we get ${}^{-1}x ={}^{-1} x\left(xx^{-1}\right) = x^{-1}$. This means that every element in G has an inverse. It is easy to see that an IP-loop is a groupoid $(G,\cdot)$ with an identity element and with the following property:

(*iii*) for each $x \in G$ there exists an element $x^{-1} \in G$ such that $(yx)\,x^{-1} = y$ and $x^{-1}\,(xy) = y$ for every $y \in G$.

We will use throughout this paper the following notations. If E is any subset of X, then E^{-1} is the set of all elements of the form x^{-1}, where $x \in E$. If E and F are any two subsets of X, then EF is the set of all elements of the form xy, where $x \in E$ and $y \in F$. If $x \in X$, it is customary to write xE and Ex in place of $\{x\}E$ and $E\{x\}$ respectively. The set xE (or Ex) is called a left translation (or right translation) of E.

The notions of an IP-quasigroup and an IP-loop were introduced by Bruck ([3, 4]), see also [2, 9, 15, 18]. Moufang loops ([10, 12]) are a very important case of IP-loops. The above described structures play a fundamental role in many areas of mathematics.

The octonions (see, e.g., [1]) are another interesting example of IP-loops. The octonions were discovered in 1843 by John T. Graves, inspired by his friend William Hamilton's discovery of quaternions. They were discovered independently by Arthur Cayley (1845). They are sometimes referred to as Cayley numbers or the Cayley algebra. Octonions have applications in fields such as string theory, special relativity, and quantum logic. The octonion algebra is usually represented by the capital letter O. Because the octonions will be important for us also in the next, we deal with them in more detail. Every octonion is a real linear combination of the unit octonions e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7, where e_0 is the scalar element. That is, every octonion x can be written in the form

$$x = x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7$$

with real coefficients x_i. Addition of octonions is accomplished by adding corresponding coefficients, as with the complex numbers and quaternions. By linearity, multiplication of octonions is completely determined once given a multiplication table for the unit octonions (see, e. g., [1]). A more systematic way of defining the octonions is via the Cayley–Dickson construction. Just as quaternions can be defined as pairs of complex numbers, the octonions can be defined as pairs of quaternions.

The conjugate of an octonion

$$x = x_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7$$

is given by

$$x^* = x_0 - x_1e_1 - x_2e_2 - x_3e_3 - x_4e_4 - x_5e_5 - x_6e_6 - x_7e_7.$$

Conjugation is an involution of O and satisfies $(xy)^* = y^*x^*$. The norm of the octonion x is defined as

$$||x|| = \sqrt{x^*x}.$$

The square root is well-defined here as $x^*x = xx^*$ is always a nonnegative real number:

$$||x||^2 = x^*x = x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2.$$

The norm on O satisfies

$$||xy|| = ||x||\ ||y||.$$

This norm agrees with the standard Euclidean norm on R^8. The existence of a norm on O implies the existence of inverses for every nonzero element of O. The inverse of $x \neq 0$ is given by

$$x^{-1} = \frac{x^*}{||x||^2}.$$

It satisfies $xx^{-1} = x^{-1}x = 1$.

Octonionic multiplication is neither commutative nor associative. The octonions do satisfy a weaker form of associativity: they are alternative. This means that the subalgebra generated by any two elements is associative. Not being associative, the nonzero elements of O do not form a group. They do, however, form an IP-loop, indeed a Moufang loop.

Example 2.1. Let us consider the couple $(O^n,\ \cdot)$ where the operation $\cdot$ is defined by

$$(o_1^1, o_2^1, \dots, o_n^1) \cdot (o_1^2, o_2^2, \dots, o_n^2) = (o_1^1 o_1^2, o_2^1 o_2^2, \dots, o_n^1 o_n^2).$$

It is easy to see that the couple $(O^n,\ \cdot)$ is also an IP-loop.

There are properties required for the Haar measure, which a measure has not, in generally. These are properties associated with the measure of open and compact sets. Then the Haar measure is built in such algebraic structures, on which some topology is given. Let us introduce therefore the notions of a topological groupoid, a topological quasigroup and a topological IP-loop.

Let $(G, \cdot)$ be a groupoid. It is natural to require that if an element x is "located near" the element a (we write $x \approx a$) and while $y \approx b$, then $xy \approx ab$. This is a motivation for the definition of a topological groupoid.

A topological groupoid is a set G with a Hausdorff topology and a continuous operation $\cdot : G \times G \to G$ (i.e., if $a, b \in G$, then for every neighborhood O_{ab} there exist neighborhoods O_a, O_b such that $O_a O_b \subset O_{ab}$).

Remark 2.2. Let $(G, \cdot)$ be any topological groupoid. It is evident that the mapping $L_a : G \to G$ defined, for each $a \in G$, by the equality $L_a(x) = ax$, $x \in G$, is continuous. The mapping L_a is called a left translation. A right translation is defined analogously.

But we will deal in the following with quasigroups. Our requirement is as follows: if $a \approx a'$ and $b \approx b'$ then the solutions of the equations $ax = b$ and $a'x' = b'$ are "close together", i.e. $x \approx x'$. A topological quasigroup is a quasigroup $(G, \cdot)$ which is a topological groupoid and the following property holds: if $a_t \to a$ and $b_t \to b$ and while $a_t x_t = b_t$ and $ax = b$, then $x_t \to x$, where $\{a_t; t \in T\}$, $\{b_t; t \in T\}$, $\{x_t; t \in T\}$ are nets in G.

Remark 2.3. Translations L_a are in topological quasigroups continuous. It is easy to see that they are also bijective.

Let two topological spaces X, Y be given. A mapping $f : X \to Y$ is called continuous if the inverse image of every open set in the space Y is open in the space X. A homeomorphism is a bijective, continuous transformation of X onto Y whose inverse is also continuous. If $f : X \to Y$ is a homeomorphism, then the image of every open set in the space X is open in the space Y.

Proposition 2.4. Let $(G, \cdot)$ be a topological quasigroup. Then the left and right translations of G are homeomorphisms.

P r o o f. Let $a \in G$. Define the mapping $K_a : G \to G$ as follows: if $z \in G$, then $K_a(z)$ is a solution of the equation $ax = z$. Thus it holds $aK_a(z) = z$. From the definition of a topological quasigroup it follows that the mapping K_a is continuous. Since $aK_a(ax) = ax$, we get $x = K_a(ax) = K_a(L_a(x)) = (K_a \circ L_a)(x)$ and therefore $K_a = L_a^{-1}$. The mappings L_a, L_a^{-1} are continuous and L_a is bijective, so that L_a is a homeomorphism. The proof for the right translation is analogous. □

Corollary 2.5. Let $(G, \cdot)$ be a topological quasigroup. If U is an open subset of G, then, for every $a \in G$, the sets aU and Ua are open, too.

Proposition 2.6. Let $(G, \cdot)$ be a topological loop such that for every $x \in G$ there exists the inverse element $x^{-1} \in G$. Then the mapping $I : G \to G$, defined by $I(x) = x^{-1}$ for every $x \in G$, is a homeomorphism.

P r o o f. Let e be the identity element of $(G, \cdot)$ and $x_t \to x$. Since $x_t x_t^{-1} = e$ and $xx^{-1} = e$, from the definition of a topological quasigroup it follows that $x_t^{-1} \to x^{-1}$. This means that the mapping I is continuous. Since I is bijective and $I = I^{-1}$, the mapping I is a homeomorphism. □

Definition 2.7. A topological IP-loop is an IP-loop $(G, \cdot)$ with a Hausdorff topology such that the following two conditions are satisfied: the binary operation $\cdot$ is continuous function with respect to the topology and the inverse function $G \to G : x \mapsto x^{-1}$ is continuous function with respect to the topology.

Remark 2.8. Let $(G, \cdot)$ be a topological IP-loop. It is easy to see that the two conditions given in the definition above are equivalent to the condition that the transformation (from $G \times G$ onto G) $(x, y) \mapsto x^{-1}y$ is continuous. As a corollary of the preceding proposition we obtain the following result: the mapping $I : G \to G$, defined by $I(x) = x^{-1}$ for every $x \in G$, is a homeomorphism also in the case that $(G, \cdot)$ is a topological IP-loop. Therefore, if U is an open set in a topological IP-loop, then the set U^{-1} is open, too. A topological IP-loop is said to be connected, totally disconnected, compact, locally compact, etc., if the corresponding property holds for its underlying topological space. The above described topological structures are studied, e. g., in [7, 8, 13, 14].

Definition 2.9. (Halmos [6]) The σ-algebra generated by compact sets in a topological space X is called a Borel σ-algebra on X. A set from the Borel σ-algebra is called Borel.

It is known that in a topological space X homeomorphisms preserve Borel sets. Let $(G, \cdot)$ be a topological quasigroup. If B is a Borel set of the topological space G and $a \in G$, then the sets aB and Ba are Borel, too.

3. HAAR MEASURE

Let $(G, \cdot)$ be a locally compact topological quasigroup. A non-negative measure m defined on the Borel σ-algebra on G is called Borel, if $m(K) < \infty$ for every compact set $K \subset G$.

Definition 3.1. (Halmos [6]) A Haar measure is a Borel measure m such that $m(U) > 0$ for every non-empty Borel open set U and $m(aB) = m(B)$ for every Borel set B and every $a \in G$.

Remark 3.2. (Halmos [6]) The Haar measure from the preceding definition is called more precisely a left Haar measure. Analogously is defined a right Haar measure. It is easy to verify that if m is a left Haar measure in a topological IP-loop, and if the set function μ is defined, for every Borel set E, by $\mu(E) = m\left(E^{-1}\right)$, then μ is a right Haar measure, and conversely. The second defining property of a Haar measure is called left invariance (or invariance under left translation). We observe that the first property is equivalent to the assertion that m is not identically zero. Indeed, if $m(U) = 0$ for some non-empty Borel open set U and if C is compact, then the class $\{xU;\ x \in C\}$ is an open covering of C. Since C is compact, there exists a finite subset $\{x_1, \ldots, x_n\}$ of C such that $C \subset \bigcup_{i=1}^{n} x_i U$. The monotonicity, the subaditivity and the left invariance of m imply that

$$m(C) \leq m\left(\bigcup_{i=1}^{n} x_i U\right) \leq \sum_{i=1}^{n} m(x_i U) = \sum_{i=1}^{n} m(U) = nm(U) = 0.$$

Since the vanishing of m on the class of all compact sets implies its vanishing on the class of all Borel sets, we obtain the desired result: a Haar measure is a left-invariant Borel measure which is not identically zero.

It is known that in every locally compact topological group there exists at least one left Haar measure. In the case of quasigroups this is not so, as is illustrated by the following example.

Example 3.3. Let R be the set of all real numbers with the simple topology, $\circ$ be a binary operation on R defined by the prescription $a \circ b = \frac{a+b}{2}$, $a,\ b \in R$. The couple $(R, \circ)$ is a locally compact topological quasigroup. Let m be a left-invariant Borel measure in $(R, \circ)$. Let us consider the compact set $\langle 0,\ h\rangle$, where $h \in R^+$. Since m is left-invariant, for any $c \in R$ it holds

$$\begin{aligned} m\left(\langle 0, h\rangle\right) &= m\left(c \circ \langle 0, h\rangle\right) = m\left(\left\langle \tfrac{c}{2}, \tfrac{h}{2} + \tfrac{c}{2}\right\rangle\right) \\ &= m\left(c \circ \left\langle \tfrac{c}{2}, \tfrac{h}{2} + \tfrac{c}{2}\right\rangle\right) = m\left(\left\langle \tfrac{c}{2} + \tfrac{c}{2^2}, \tfrac{h}{2^2} + \tfrac{c}{2} + \tfrac{c}{2^2}\right\rangle\right) = \dots \\ &= m\left(\left\langle c\left(\tfrac{1}{2} + \tfrac{1}{2^2} + \dots + \tfrac{1}{2^n}\right), \tfrac{h}{2^n} + c\left(\tfrac{1}{2} + \tfrac{1}{2^2} + \dots + \tfrac{1}{2^n}\right)\right\rangle\right) = \dots = m\left(\{c\}\right). \end{aligned}$$

The arbitrariness of c implies that the measure of every one-point set is constant. At the same time it is less than ∞, because the set $\langle 0,\ h\rangle$ is compact and m is a Borel measure. Moreover, from the additivity and the monotonicity of measure m we obtain

$$m(\{c\}) + m(\{c\}) = m(\{0\}) + m(\{c\}) = m(\{0,\ c\}) \leq m\left(\langle 0,\ c\rangle\right) = m(\{c\}).$$

Hence $m(\{c\}) = 0$. Then also $m(\langle 0,\ c\rangle) = m(\{c\}) = 0$. From the monotonicity of measure m it follows that $m\left((0, c)\right) \leq m\left(\langle 0, c\rangle\right) = 0$. Since m is non-negative, $m((0,\ c)) = 0$. Thus the measure of an open set is zero. But this means that in $(R, \circ)$ there exists no Haar measure.

4. THE CONSTRUCTION OF A HAAR MEASURE IN A TOPOLOGICAL IP-LOOP

Let $(G, \cdot)$ be a topological IP-loop. Our aim is to prove the existence of at least one left Haar measure in $(G, \cdot)$. In the following, we show that, based on results of Halmos ([6]), it is sufficient to this to construct a left-invariant content in $(G, \cdot)$ which is not identically zero.

Definition 4.1. (Halmos [6]) Let X be a topological space. A content is a non-negative, finite, monotone, subadditive and additive set function defined on the class of all compact subsets of X.

Definition 4.2. (Halmos [6]) The inner content induced by a content λ is the set function λ_* defined for every Borel open set U by

$$\lambda_*(U) = \sup\{\lambda(C);\ U \supset C,\ C \,\text{is compact}\}.$$

In the following, we assume that X is a locally compact topological space. By means of the set function λ_* an outer measure is defined on the system of all σ-bounded subsets of X.

Definition 4.3. (Halmos [6]) Let λ be a content in a locally compact topological space X and let λ_* be the inner content induced by λ. Define a set function m^* on the σ-ring of all σ-bounded sets of X by

$$m^*(E) = \inf\{\lambda_*(U);\ E \subset U,\ U \,\text{is open Borel}\}.$$

The set function m^* is an outer measure, it is called an outer measure induced by λ. The following lemma gives an answer to the question of the relationship between the set functions m^*, λ_* and λ.

Lemma 4.4. (Halmos [6]) Let λ be a content in a locally compact topological space X. Let λ_* be the inner content and let m^* be the outer measure induced by λ. Then $m^*(U) = \lambda_*(U)$ for every open Borel set U and $m^*(C^\circ) \leq \lambda(C) \leq m^*(C)$ for every compact set C, where C° denotes the interior of the set C.

Lemma 4.5. (Halmos [6]) Let λ be a content in a locally compact topological space X. If m^* is an outer measure induced by λ, then the set function m, defined for every Borel set E by $m(E) = m^*(E)$, is a regular Borel measure.

The Borel measure m from the preceding lemma is called a Borel measure induced by λ.

Lemma 4.6. (Halmos [6]) Suppose that T is a homeomorphism of a locally compact topological space X onto itself and that λ is a content in X. Put $\hat{\lambda}(C) = \lambda(T(C))$ for every compact set C. If m and $\hat{m}$ are the Borel measures induced by λ and $\hat{\lambda}$ respectively, then $\hat{m}(E) = m(T(E))$ for every Borel set E. If, in particular, the content λ is invariant under T, then the measure m is invariant under T, too.

A left translation L_a on topological quasigroups is a homeomorphism. When a content λ is left-invariant (i. e. invariant under $T = L_a$), then the Borel measure induced by λ is also left-invariant. Therefore, based on Lemma 4.5 and Lemma 4.6, to the construction of a left Haar measure in a locally compact topological IP-loop $(G, \cdot)$ it is sufficient to construct a left-invariant content in $(G, \cdot)$ which is not identically zero. Lemma 4.4 implies that the induced measure is not identically zero and hence (in accordance with Remark 3.2) is a regular Haar measure.

Let $(G, \cdot)$ be a locally compact topological IP-loop. Denote by $\mathbf{U}_e$ the system of all open neighborhoods of the identity element $e \in G$. Let $U \in \mathbf{U}_e$. If $C \subset G$, then $C \subset \bigcup_{x \in C} xU$. Since the set xU is for every $x \in G$ open, the system $\{xU; x \in C\}$ is an open covering of the set C. If C is compact, then it holds the property

$(*)$ there exists a finite set $\{x_1, x_2, \ldots, x_m\} \subset C$ such that $C \subset \bigcup\limits_{i=1}^{m} x_i U$.

Definition 4.7. Let A be a fixed compact set with a non-empty interior. For each set $U \in \mathbf{U}_e$ we construct a set function λ_U defined on the class of all compact sets in the following way: if C is a compact set, then

$$\lambda_U(C) = \frac{C:U}{A:U},$$

where $C:U$ is defined as the least non-negative integer with the property $(*)$.

It is easy to prove the following lemma.

Lemma 4.8. The set function λ_U is non-negative, finite, subadditive, monotone and not identically zero.

Lemma 4.9. Let C, D be any compact subsets of G and $U \in \mathbf{U}_e$. If $CU^{-1} \cap \cap DU^{-1} = \emptyset$, then $\lambda_U(C \cup D) = \lambda_U(C) + \lambda_U(D)$.

Proof. Let U be an open set such that $CU^{-1} \cap DU^{-1} = \emptyset$. Let $x \in G$. Suppose that $xU \cap C \neq \emptyset$, i.e. there exists $c \in C$ such that $c = xy$, where $y \in U$. Multiply this equality by the element y^{-1}. We obtain $(xy)\,y^{-1} = cy^{-1}$, whence it follows that $x = cy^{-1}$. Thus $x \in Cy^{-1}$, what means that $x \in CU^{-1}$.

Let $xU \cap D \neq \emptyset$. Hence there exists $d \in D$ such that $d = xy$, where $y \in U$. Then $(xy)\,y^{-1} = dy^{-1}$. We get that $x = dy^{-1}$, i.e. $x \in DU^{-1}$. But this is a contradiction with the assumption $CU^{-1} \cap DU^{-1} = \emptyset$. Hence, for any $x \in G$, it holds $xU \cap C = \emptyset$ or $xU \cap D = \emptyset$.

Let further $m_1 = C:U$ and $m_2 = D:U$, i.e. m_1, m_2 be the least integers such that $C \subset \bigcup_{i=1}^{m_1} x_iU$ and $D \subset \bigcup_{i=1}^{m_2} y_iU$, where $x_i \in C$ for $i = 1, 2, \dots, m_1$ and $y_i \in D$ for $i = 1, 2, \dots, m_2$. Then $C \cup D \subset (\bigcup_{i=1}^{m_1} x_iU) \cup (\bigcup_{i=1}^{m_2} y_iU)$. Since $x_iU \cap D = \emptyset$ for $i = 1, 2, \dots, m_1$ and $y_iU \cap C = \emptyset$ for $i = 1, 2, \dots, m_2$, we have $\bigcup_{i=1}^{m_1} x_iU \cap D = \emptyset$ and $\bigcup_{i=1}^{m_2} y_iU \cap C = \emptyset$. Therefore

$$(C \cup D) : U = m_1 + m_2 = (C:U) + (D:U)$$

and consequently

$$\lambda_U(C \cup D) = \lambda_U(C) + \lambda_U(D).$$

□

Remark 4.10. Note that while the associativity plays its role in the standard Halmos's proof of the above property of the set function λ_U, it is replaced in our proof by the property (iii) of an IP-loop.

We will prove in the following that in every locally compact topological IP-loop, whose topology is induced by a left-invariant uniformity, there exists at least one left Haar measure. First, we will recall the definition of a uniform topology and remind the facts which will be further used.

A uniformity of a set X is a non-empty system W of subsets of the Cartesian product $X \times X$ which satisfies the following conditions:

(*i*) Every element of W contains the diagonal $\Delta = \{(x, x);\ x \in X\}$.

(*ii*) If $U \in W$, then $\{(y,x);\ (x,y) \in U\} \in W$.

(*iii*) If $U \in W$, then there exists $V \in W$ such that, whenever (x,y) and (y,z) are in V, then (x,z) is in U.

(*iv*) If $U, V \in W$, then $U \cap V \in W$.

(*v*) If $U \in W$ and $U \subset V \subset X \times X$ then $V \in W$.

The elements of the uniformity are called entourages and the above described couple (X, W) is called a uniform space. Let $x \in X$. Put $U[x] = \{y \in X;\ (x, y) \in U\}$ for any $U \in W$. Every uniform space X becomes a topological space by defining a subset U of X to be open if and only if for every $x \in U$ there exists an entourage V such that $V[x]$ is a subset of U. The topology defined by a uniform structure is said to be induced by the uniformity.

A base of a uniformity W is any system $\mathbf{B}$ of entourages of W such that every entourage of W contains a set belonging to $\mathbf{B}$. Thus, by property (v) above, a base $\mathbf{B}$ is enough to specify the uniformity W unambiguously: W is the set of subsets of $X \times X$ that contain a set of $\mathbf{B}$. Every uniform space has a base of entourages consisting of symmetric entourages. A uniformity of a groupoid $(X, \cdot)$ is called left-invariant, if it has a left-invariant base $\mathbf{B}$, i. e. $(a,a)\mathbf{B} = \mathbf{B}$ for every $a \in X$, where $(a,a)(x,y) = (ax, ay)$. A right-invariant uniformity is defined analogously.

A uniform topology is a generalization of a metric topology because if (X, ρ) is a metric space, then the system $\mathbf{B} = \{U_\varepsilon;\ \varepsilon > 0\}$, where $U_\varepsilon = \{(x,y);\ \rho(x,y) < \varepsilon\}$, is a base of some uniformity of X. This uniformity is called a uniformity induced by the metric ρ. If a metric ρ of a quasigroup $(X, \cdot)$ is left-invariant (i. e., for every $a, x, y \in X,\ \rho(ax, ay) = \rho(x, y)$), then the uniformity induced by the metric ρ is left-invariant, too. Indeed, for every $a \in X$ and every $U_\varepsilon \in \mathbf{B}$, we have

$$\begin{aligned} U_\varepsilon = \{(x,y);\ \rho(x,y) < \varepsilon\} &= \{(at, av);\ \rho(at, av) < \varepsilon\} = \{(at, av);\ \rho(t,v) < \varepsilon\} \\ &= (a,a)\{(t,v);\ \rho(t,v) < \varepsilon\} = (a,a)U_\varepsilon. \end{aligned}$$

Analogously, if a metric ρ of a quasigroup $(X, \cdot)$ is right-invariant, then the uniformity induced by the metric ρ is also right-invariant.

Proposition 4.11. Let O be the set of all octonions with a unit norm. Then in the IP-loop $(O,\ \cdot)$ there exists a left-invariant metric.

Proof. Let x, y, a be octonions with a unit norm,

$$x = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_5 e_5 + x_6 e_6 + x_7 e_7,$$

$$y = y_0 + y_1 e_1 + y_2 e_2 + y_3 e_3 + y_4 e_4 + y_5 e_5 + y_6 e_6 + y_7 e_7,$$

$$a = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 + a_7 e_7.$$

Put $\rho(x,y)=\sqrt{\sum_{k=0}^{7}(x_k-y_k)^2}$. Then ρ is a metric. Since

$$\rho(ax,ay)=\sqrt{\sum_{i=0}^{7}(a_i^2\sum_{k=0}^{7}(x_k-y_k)^2)}=\sqrt{\sum_{i=0}^{7}a_i^2\cdot\sum_{k=0}^{7}(x_k-y_k)^2}$$

$$=\sqrt{\sum_{i=0}^{7}a_i^2}\cdot\sqrt{\sum_{k=0}^{7}(x_k-y_k)^2}=\sqrt{\sum_{k=0}^{7}(x_k-y_k)^2}=\rho(x,y),$$

it is left-invariant. □

Proposition 4.12. Let O be the set of all octonions with a unit norm. Then in the IP-loop $(O^n,\ \cdot)$ there exists a left-invariant metric.

P r o o f. Let O be the set of all octonions with a unit norm and $x,y,a\in O^n$, $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)$, $a=(a_1,\dots,a_n)$.

Put $\bar\rho(x,y)=\sqrt{\sum_{i=1}^{n}\rho(x_i,y_i)}=\sqrt{\sum_{i=1}^{n}\sum_{k=0}^{7}(x_{i_k}-y_{i_k})^2}$. ρ is the metric from the preceding proposition. It is easy to verify that $\bar\rho$ is a metric. Since ρ is left-invariant, we obtain

$$\bar\rho(ax,ay)=\sqrt{\sum_{i=1}^{n}\rho(a_ix_i,a_iy_i)}=\sqrt{\sum_{i=1}^{n}\rho(x_i,y_i)}=\bar\rho(x,y).$$

The proof is complete. □

Remark 4.13. The structures from the preceding propositions are examples of topological IP-loops with a left-invariant uniformity, and they are not groups.

Proposition 4.14. Let $(G,\cdot)$ be a topological IP-loop. Its topology is induced by a left-invariant uniformity W of G if and only if there exists a base $\mathbf{B}_e$ of neighborhoods of the identity element e of G such that for any neighborhood $U\in\mathbf{B}_e$ and any elements $x,y\in G$ it holds $x\,(yU)=(xy)\,U$.

P r o o f. Let the topology of $(G,\cdot)$ be induced by a left-invariant uniformity W of G. Let $\mathbf{B}$ be a left-invariant base of the uniformity W. Then the system $V[e]$, $V\in\mathbf{B}$, is a base of neighborhoods of the identity element e of G. We have to prove that for every $x,y\in G$ and every neighborhood $V[e]$, $V\in\mathbf{B}$, it holds $x(yV[e])=(xy)V[e]$. First, we will prove some auxiliary equalities.

$$xV[e]=\{xz;\ z\in V[e]\}=\{xz;\ (e,\,z)\in V\}=\{xz;\ (x,\,xz)=(x,\,x)(e,\,z)\in V\},$$

because $\mathbf{B}$ is a left-invariant base, and thus $(x,x)\,V=V$. So that

$$xV[e]=\{xz;\ (x,\,xz)\in V\}=\{t;\ (x,\,t)\in V\}=V[x],$$

$$xV[y] = \{xz;\ z \in V[y]\} = \{xz;\ (y,\, z) \in V\} = \{xz;\ (xy,\, xz) = (x,\, x)(y,\, z) \in V\}$$
$$= \{t;\ (xy,\, t) \in V\} = V[xy].$$

Our results above imply the desired equality:

$$x(yV[e]) = xV[y] = V[xy] = (xy)V[e].$$

Let us prove the reverse implication. Put $W_U = \{(x,\, y);\ x^{-1}y \in U\}$. Then the system $\{W_U; U \in \mathbf{B}_e\}$ is a base of some uniformity of the set G. We will prove that this base is left-invariant, i.e. that, for every $a \in G$, it holds $(a, a)\, W_U = W_U$. Let $(x, y) \in W_U$, i.e. $x^{-1}y \in U$, what means that $x^{-1}y = z$, where $z \in U$. Multiply this equality from the left by the element x. We obtain $x\left(x^{-1}y\right) = xz$. Since G is an IP-loop, it holds $x\left(x^{-1}y\right) = y$. Hence $y = xz$, where $z \in U$. Then, for every $a \in G$, it holds $ay = a\,(xz)$, where $z \in U$. This means that $ay \in a\,(xU)$. Since by the assumption $a\,(xU) = (ax)\,U$, we obtain that $ay = (ax)\,z$, where $z \in U$. When multiplied this equality from the left by the element $(ax)^{-1}$, we get $(ax)^{-1}(ay) = (ax)^{-1}((ax)z) = z$, where $z \in U$, what means that $(ax,\ ay) \in W_U$. Indeed, for every $a \in G$,

$$(a,\, a)W_U = W_U.$$

□

In the following, we prove that the set function λ_U is left-invariant. Unlike Halmos's proof where the associativity plays its role, we use an assumption of the existence of a left uniformity and the property (iii) of any IP-loop in our proof.

Lemma 4.15. Let the topology of a topological IP-loop $(G, \cdot)$ be induced by a left-invariant uniformity. Then the function λ_U is left-invariant.

Proof. Let $C \subset G$ be a compact set and $n = C : U$. Thus it holds $C \subset \bigcup_{i=1}^{n} x_i U$, where $x_i \in C$ for $i = 1, 2, \ldots, n$. This means that for any $y \in C$ there exists $i \in \{1, 2, \ldots, n\}$ such that $y \in x_i U$, i.e. $y = x_i z$, where $z \in U$. Multiply this equality from the left by any element $x \in G$. We obtain $xy = x\,(x_i z)$, where $z \in U$, i.e. $xy \in x\,(x_i U)$. By the preceding proposition $x(x_i U) = (xx_i)U$. Therefore, for every $y \in C$, $xy \in \bigcup_{i=1}^{n} (xx_i)U$, where $xx_i \in xC$ for $i = 1, 2, \ldots, n$. This means that $xC \subset \bigcup_{i=1}^{n} (xx_i)U$. From this it follows that $xC : U \leq n$, i.e. $xC : U \leq C : U$. We have thereby proved that $\lambda_U(xC) \leq \lambda_U(C)$ for every $x \in G$ and every compact set $C \subset G$.

We will prove the reverse inequality. Let $C \subset G$ be a compact set and $m = xC : U$. So that $xC \subset \bigcup_{j=1}^{m} y_j U$. From this it follows that

$$x^{-1}(xC) \subset \bigcup_{j=1}^{m} x^{-1}(y_j U) = \bigcup_{j=1}^{m} x^{-1}((xx_j)U) = \bigcup_{j=1}^{m} x^{-1}(x(x_j U)) = \bigcup_{j=1}^{m} x_j U,$$

where $x_j \in C$ for $j = 1, 2, \ldots, m$.

Since $(G, \cdot)$ is an IP-loop, for any $z \in G$ it holds $x^{-1}(xz) = z$, so that $x^{-1}(xC) = C$. We have proved that $C \subset \bigcup_{j=1}^{m} x_j U$, where $x_j \in C$ for $j = 1, 2, \ldots, m$. From this we obtain the inequality $C : U \leq m$, i.e. $C : U \leq xC : U$.

Consequently $\lambda_U(C) \leq \lambda_U(xC)$ for every $x \in G$ and every compact set $C \subset G$. $\square$

The set function λ_U differs from a content only in that it may not be additive, in generally. By means of the function λ_U we will define a function λ that has all properties of a content, thus also the additivity. Our construction of a content λ is analogous to the Halmos construction provided in [6], 58.B.

Lemma 4.16. Let $(G, \cdot)$ be a topological IP-loop with the topology induced by a left-invariant uniformity. Then on the system of all compact subsets of G there exists a left-invariant content λ that is not identically zero.

P r o o f. If E is any bounded set and F is any set with a non-empty interior, then the "ratio" $E : F$ is defined as the least non-negative integer n with the following property: there exists a set $\{x_1, x_2, \ldots, x_n\} \subset G$ such that $E \subset \bigcup_{i=1}^{n} x_i F$. It is easy to verify that $E : F$ is always finite, and that, if A is a bounded set with a non-empty interior, then $E : F \leq (E : A)(A : F)$. Let $A \subset G$ be a fixed compact set with a non-empty interior. Assign to each compact set $C \subset G$ the closed interval $\langle 0,\ C : A\rangle$. Denote by Φ the Cartesian product (in the topological sense) of all these intervals. By Tychonoff's theorem the space Φ is a compact Hausdorff topological space. Its elements are real functions φ defined on the class of all compact subsets of the set G. For each compact set $C \subset G$ it holds $\varphi(C) \in \langle 0,\ C : A\rangle$, i. e. $0 \leq \varphi(C) \leq C : A$. Since for every $U \in \mathbf{U}_e, C : U \leq (C : A)(A : U)$, it follows that $0 \leq \lambda_U(C) \leq C : A$, and therefore the set function λ_U is a point in the space Φ for every fixed neighborhood $U \in \mathbf{U}_e$. Define for every $U \in \mathbf{U}_e$ the set $\Lambda(U) = \{\lambda_V;\ U \supset V \in \mathbf{U}_e\}$.

If $\{U_1, \ldots, U_n\}$ is any system of neighborhoods of the identity element e, i. e. any subsystem of $\mathbf{U}_e$, then $\bigcap_{i=1}^{n} U_i$ is also a neighborhood of the identity element e and, moreover, $\bigcap_{i=1}^{n} U_i \subset U_j$ for $j = 1, 2, \ldots, n$. Hence $\lambda_{\bigcap_{i=1}^{n} U_i} \in \Lambda(U_j) = \{\lambda_V;\ U_j \supset V \in \mathbf{U}_e\}$ for $j = 1, 2, \ldots, n$. From this it follows that $\lambda_{\bigcap_{i=1}^{n} U_i} \in \bigcap_{j=1}^{n} \Lambda(U_j) = \bigcap_{i=1}^{n} \Lambda(U_i)$, i. e. $\bigcap_{i=1}^{n} \Lambda(U_i)$ is non-empty. This means that the system $\{\Lambda(U);\ U \in \mathbf{U}_e\}$ has the finite intersection property. Because the space Φ is compact, there exists a point λ in the intersection of the closures of all $\Lambda(U) : \lambda \in \cap \left\{\overline{\Lambda(U)}; U \in \mathbf{U}_e\right\}$.

We shall prove that the set function λ is the desired content. It is clear that $0 \leq \lambda(C) \leq C : A < \infty$ for every compact set $C \subset G$. Thus λ is a non-negative finite set function. We show that it is monotone. The projection ξ_C defined for each fixed compact set C by $\xi_C(\psi) = \psi(C)$, $\psi \in \Phi$, is a continuous function on Φ. Hence, for any two compact sets $C, D \subset G$, the set $\Delta = \{\psi;\ \psi(C) \leq \psi(D)\}$ is closed. If $C \subset D$ and $U \in \mathbf{U}_e$, then according to Lemma 4.8 $\lambda_U \in \Delta$, and therefore $\Lambda(U) \subset \Delta$. Because Δ is closed, it holds $\overline{\Lambda(U)} \subset \Delta$. Hence $\lambda \in \Delta$, and thus λ is monotone. Analogously we shall prove that it is subadditive. The set $\Delta' = \{\psi;\ \psi(C \cup D) \leq \psi(C) + \psi(D)\} \subset \Phi$ is closed for any two compact sets $C, D \subset G$. By Lemma 4.8 $\lambda_U \in \Delta'$ for all $U \in \mathbf{U}_e$, and therefore $\Lambda(U) \subset \Delta'$. Since Δ' is closed, it holds $\overline{\Lambda(U)} \subset \Delta'$. Hence we get that $\lambda \in \Delta'$, i. e. λ is a subadditive set function. According to preceding lemma the function λ_U is left-invariant, whence it follows by continuity that the function λ is left-invariant, too. It remains to prove that λ is additive. Let $C,\ D \subset G$ be any disjoint compact sets. Then there exists a neighborhood U of the identity element e such that $CU^{-1} \cap DU^{-1} = \emptyset$. If $V \in \mathbf{U}_e$ and $V \subset U$ then $CV^{-1} \cap DV^{-1} = \emptyset$ and hence by Lemma 4.9 $\lambda_V(C \cup D) = \lambda_V(C) + \lambda_V(D)$. But this means that, whenever $V \subset U$, the function λ_V belongs to the closed set $\Delta'' = \{\psi;\ \psi(C \cup D) = \psi(C) + \psi(D)\} \subset \Phi$. Hence $\Lambda(U) \subset \Delta''$. Because Δ'' is closed, we have $\overline{\Lambda(U)} \subset \Delta''$. From this it follows that $\lambda \in \Delta''$, and therefore λ is additive. We have thereby constructed a left-invariant content λ.

Finally, we show that this constructed content λ is not identically zero. Put $\Delta''' = \{\psi;\ \psi(A) = 1\} \subset \Phi$. Since $\lambda_U(A) = 1$ for every $U \in \mathbf{U}_e$, $\lambda_U \in \Delta'''$ for every $U \in \mathbf{U}_e$, and therefore $\Lambda(U) \subset \Delta'''$. The set Δ''' is closed, and hence $\overline{\Lambda(U)} \subset \Delta'''$. But this means that $\lambda \in \Delta'''$, i. e. $\lambda(A) = 1$ and hence λ is not identically zero. The proof is complete. $\square$

In view of the preceding results we obtain immediately the final theorem of this paper.

Theorem 4.17. In every locally compact topological IP-loop whose topology is induced by a left-invariant uniformity there exists at least one regular left Haar measure.

Corollary 4.18. In every locally compact topological IP-loop whose topology is induced by a left-invariant metric there exists at least one regular left Haar measure.

Remark 4.19. From Proposition 4.11 it follows that in the topological IP-loop of all octonions with a unit norm there exists at least one regular left Haar measure. Analogously, from Proposition 4.12 it follows that in the topological IP-loop $(O^n,\ \cdot)$ where O is the set of all octonions with a unit norm, there exists at least one regular left Haar measure. Since every associative IP-loop is a group, the associativity of the operation $\cdot$ ensures the existence of a regular left Haar measure in every locally compact topological IP-loop $(G, \cdot)$. The previous conditions imposed on quasigroups are sufficient to the existence of a left Haar measure. They are not, however, necessary. This follows from the fact that in every finite or countable quasigroup with discrete topology (i. e. all singletons are open) there exists a Haar measure. It is defined as the number of elements of the set.

If $(G, \cdot)$ is any topological IP-loop, one can consider the topological IP-loop $\left(\hat{G}, \circ\right)$ dual to G. The topological IP-loop $\hat{G}$ has, by definition, the same elements and the same topology as G, the product $\circ$ in $\hat{G}$ is defined by $x \circ y = y \cdot x$ for every $x, y \in \hat{G}$. Since, for every $x, y \in \hat{G}$, $x^{-1} \circ (x \circ y) = x^{-1} \circ (y \cdot x) = (y \cdot x) \cdot x^{-1} = y$ and $(x \circ y) \circ y^{-1} = (y \cdot x) \circ y^{-1} = y^{-1} \cdot (y \cdot x) = x$, we see that $\left(\hat{G}, \circ\right)$ is in fact an IP-loop.

Let $(G, \cdot)$ be any locally compact topological IP-loop with a topology induced by a right-invariant uniformity W. Consider the topological IP-loop $\left(\hat{G}, \circ\right)$ dual to G.

If $\mathbf{B}$ is a right-invariant base of the uniformity W of $(G, \cdot)$ then $\mathbf{B}$ is a left-invariant base of a uniformity W of the groupoid $\left(\hat{G}, \circ\right)$. Indeed, if U is any element of $\mathbf{B}$, then for every $a \in \hat{G}$ and every $(x, y) \in U$ we have

$$(a, a) \circ (x, y) = (a \circ x, a \circ y) = (x \cdot a, y \cdot a) = (x, y) \cdot (a, a) = (x, y) \,.$$

Thus the locally compact topological IP-loop $\left(\hat{G}, \circ\right)$ has a topology induced by a left-invariant uniformity. Therefore in $\left(\hat{G}, \circ\right)$ there exists a regular left Haar measure m. Since, for every $x \in G$ and every Borel set E, it holds

$$m\,(E \cdot x) = m\,(x \circ E) = m\,(E)\,,$$

m is a right Haar measure in $(G, \cdot)$.

5. CONCLUSION

In this paper we have proved that in every locally compact topological IP-loop, whose topology is induced by a left-invariant uniformity, there exists at least one left Haar measure. We have showed that the existence of a right Haar measure in every locally compact topological IP-loop, whose topology is induced by a right-invariant uniformity, follows from the existence of a regular left Haar measure by consideration of the topological IP-loop $\hat{G}$ dual to G. Haar measure is obviously not unique, since, for any Haar measure m and any positive number c, the product cm is also a Haar measure. In [11], we have proved that the left Haar measure in a locally compact topological IP-loop is unique up to multiplication by a positive constant. Note that the existence of an invariant uniformity is not necessary in the proof of the uniqueness of Haar measure.

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