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## ON A CLASS OF ESTIMATORS IN A MULTIVARIATE RCA(1) MODEL

ZUZANA PRÁŠKOVÁ AND PAVEL VANĚČEK

This work deals with a multivariate random coefficient autoregressive model (RCA) of the first order. A class of modified least-squares estimators of the parameters of the model, originally proposed by Schick for univariate first-order RCA models, is studied under more general conditions. Asymptotic behavior of such estimators is explored, and a lower bound for the asymptotic variance matrix of the estimator of the mean of random coefficient is established. Finite sample properties are demonstrated in a small simulation study.

*Keywords:* multivariate RCA models, parameter estimation, asymptotic variance matrix

*Classification:* 60F05, 60G10, 60G46, 62M10

### 1. INTRODUCTION

Random coefficient autoregressive models belong to a broad class of conditional heteroscedastic time series models because of their varying conditional variance and as such may be used in various applications.

We say that a process of random vectors  $\mathbf{X}_t = (X_t^1, \dots, X_t^m)' \in \mathbb{R}^m, t \in \mathbb{Z}$ , follows the multivariate first-order random coefficient autoregressive model, abbreviated as RCA(1), if  $\mathbf{X}_t$  for each  $t \in \mathbb{Z}$  satisfies

$$\mathbf{X}_t = (\boldsymbol{\beta} + \mathbf{B}_t)\mathbf{X}_{t-1} + \mathbf{Y}_t, \quad (1)$$

where  $\boldsymbol{\beta}$  is an  $m \times m$  matrix of (unknown) parameters,  $\{\mathbf{B}_t, t \in \mathbb{Z}\}$  is a sequence of  $m \times m$  random matrices and  $\{\mathbf{Y}_t, t \in \mathbb{Z}\}$  is an  $m \times 1$  random error process.

Equation (1) can be rewritten into

$$\mathbf{X}_t = \boldsymbol{\beta}\mathbf{X}_{t-1} + \mathbf{u}_t \quad (2)$$

with the new error process  $\mathbf{u}_t = \mathbf{B}_t\mathbf{X}_{t-1} + \mathbf{Y}_t = (\mathbf{X}_{t-1}' \otimes \mathbf{I}) \cdot \text{vec}(\mathbf{B}_t) + \mathbf{Y}_t$ , where  $\mathbf{I}$  is the  $m \times m$  identity matrix,  $\otimes$  denotes the Kronecker product and  $\text{vec}$  is the matrix column operator (for definitions and properties of matrices operators see the Appendix and Lemmas A.2–A.4 there).

To specify model (1) in detail, we introduce the following assumptions.

**A1:** The random coefficient matrix process  $\{\mathbf{B}_t, t \in \mathbb{Z}\}$  is a centered *iid* sequence with a finite positive definite matrix  $\mathbf{\Sigma} = \mathbb{E}[\text{vec}(\mathbf{B}_0) \cdot \text{vec}'(\mathbf{B}_0)]$ .

**A2:** All the eigenvalues of the matrix  $\mathbb{E}(\mathbf{B}_0 \otimes \mathbf{B}_0) + (\boldsymbol{\beta} \otimes \boldsymbol{\beta})$  are less than unity in modulus.

**A3:** The error process  $\{\mathbf{Y}_t\}$  is an ergodic and strictly stationary martingale difference sequence with respect to the filtration  $\mathcal{F}_t = \sigma(\mathbf{B}_s, \mathbf{Y}_s; s \leq t)$ , such that  $\mathbb{E}[\mathbf{Y}_t \mathbf{Y}_t' | \mathcal{F}_{t-1}] = \mathbf{G}$  *a.s.* for all  $t$ , where  $\mathbf{G}$  is a finite positive definite matrix.

**A4:**  $\{\mathbf{B}_t, t \in \mathbb{Z}\}$  and  $\{\mathbf{Y}_t, t \in \mathbb{Z}\}$  are mutually independent.

The following theorem is of the fundamental importance.

**Theorem 1.1.** Under the assumptions A1 to A4, there exists a unique solution to the stochastic difference equation (1) that is  $\mathcal{F}_t$ -measurable, strictly stationary, ergodic, and is of the form

$$\mathbf{X}_t = \mathbf{Y}_t + \sum_{j=1}^{+\infty} \left[ \prod_{i=0}^{j-1} (\boldsymbol{\beta} + \mathbf{B}_{t-i}) \right] \cdot \mathbf{Y}_{t-j}, \tag{3}$$

where the sum in (3) converges (component-wise) in the quadratic mean and also absolutely with probability one. Further, for all  $t \in \mathbb{Z}$ ,

$$\mathbb{E}\mathbf{X}_t = \mathbf{0}, \quad \mathbb{E}\mathbf{X}_t \mathbf{X}_t' = \mathbf{M},$$

where

$$\begin{aligned} \text{vec}(\mathbf{M}) &= \sum_{j=0}^{\infty} [\mathbb{E}(\mathbf{B}_0 \otimes \mathbf{B}_0) + (\boldsymbol{\beta} \otimes \boldsymbol{\beta})]^j \text{vec}(\mathbf{G}) \\ &= [\mathbf{I} - (\mathbb{E}(\mathbf{B}_0 \otimes \mathbf{B}_0) + (\boldsymbol{\beta} \otimes \boldsymbol{\beta}))]^{-1} \text{vec}(\mathbf{G}). \end{aligned}$$

*Proof.* The assertion summarizes results that were proved in [11], Chapter 2 (see Theorem 2, Corollaries 2.2.1 and 2.2.2, and Theorem 2.7 there) with errors  $\mathbf{Y}_t$  being *iid* random vectors. However, it can be shown that all the crucial steps in the proofs remain valid under the assumptions A1 to A4. See also [16] for some details.  $\square$

**Remark 1.2.** A strictly stationary solution of (1) can be obtained under weaker or modified assumptions than those considered in Theorem 1.1. Under the assumptions that  $\{(\mathbf{B}_t, \mathbf{Y}_t), t \in \mathbb{Z}\}$  is strictly stationary and ergodic, that both  $\mathbb{E} \log^+(\|\boldsymbol{\beta} + \mathbf{B}_0\|)$  and  $\mathbb{E} \log^+(\|\mathbf{Y}_0\|_m)$  are finite and  $\mathbb{E} \log(\|\boldsymbol{\beta} + \mathbf{B}_0\|) < 0$ , where  $x^+ = \max(x, 0)$ ,  $\|\cdot\|_m$  is any norm in  $\mathbb{R}^m$ , and  $\|\cdot\|$  denotes an operator norm defined for an  $m \times m$  matrix  $\mathbf{A}$  by

$$\|\mathbf{A}\| = \sup\{\|\mathbf{A}\mathbf{x}\|_m / \|\mathbf{x}\|_m, \mathbf{x} \in \mathbb{R}^m, \mathbf{x} \neq \mathbf{0}\},$$

we can prove that  $\mathbf{X}_t$  in (3) converges absolutely almost surely and  $\{\mathbf{X}_t, t \in \mathbb{Z}\}$  is the unique strictly stationary solution to (1), see [5] and [6]. In [6], necessary and sufficient conditions for the existence of a non-anticipative, strictly stationary solution of (1) were stated provided that  $(\mathbf{B}_t, \mathbf{Y}_t)$  are *iid* pairs. Considering a univariate RCA(1) process and the same assumptions as in [6], the authors of [1] established minimal conditions to obtain finite moments of  $\{\mathbf{X}_t\}$  of order  $\nu \geq 1$ . For  $\nu = 2$ , their conditions coincide with the moment assumptions in A1–A3. In a multivariate case, sufficient conditions for the existence of finite moments of  $\{\mathbf{X}_t\}$  of any even order  $\nu$  were established in [8]. Conditions for the existence of finite moments of order  $\nu \geq 1$  can be established analogously as in [1], but will not be considered here, see also Remark 2.2 below.

Usually,  $\beta$ ,  $\Sigma$ , and  $\mathbf{G}$  are unknown parameters of the model. Estimators of parameters in univariate RCA models have been considered by many authors. The least-squares estimators, their weighted versions, maximum and quasi-maximum likelihood estimators were studied, e. g., in [1, 2, 3, 9, 11] under various model assumptions. Adaptive estimators of  $\beta$  in a univariate RCA(1) were studied in [10]. Schick in [12] proposed a class of modified least-squares estimators in a univariate RCA(1) model, indexed by a family of bounded measurable functions. The best estimator in that class minimizing the asymptotic variance is asymptotically equivalent to the conditionally weighted least-squares estimator, which coincides with the quasi-maximum likelihood estimator. Statistical properties of the Schick-type estimators in univariate RCA models were further developed and extended in [14] and [16]. These estimators are computationally simpler than the conditionally weighted least-squares estimators, do not require any prior knowledge of additional parameters, and, as numerical studies performed in [16] show, they behave well under more general conditions and better than the least-squares estimators.

Estimation in multivariate RCA models is more complicated due to increasing number of unknown parameters. Asymptotic properties of the least-squares estimators in multivariate RCA models with *iid* errors were studied in [11]. Maximum likelihood estimating procedures were only briefly mentioned there without further details.

In this paper, we study properties of the Schick-type estimators of parameters in a multivariate RCA(1) model that satisfies assumptions A1 to A4 and generalize and extend results published recently in [15].

The paper is further organized as follows. In Section 2 we introduce Schick-type estimators and prove their strong consistency and asymptotic normality. In Section 3 we will continue with consistent variance matrices estimators of the parameter  $\beta$  and find the lower bound for the asymptotic variance matrices. A small simulation study is included in Section 4. Some necessary results from matrix theory and some auxiliary assertions are given in the Appendix.

## 2. MODIFIED LEAST-SQUARES ESTIMATOR

Let  $\mathbf{X}_0, \dots, \mathbf{X}_n$  be observations of process (1) that satisfy assumptions A1 to A4. Then we have

$$\mathbb{E}[\mathbf{X}_t | \mathcal{F}_{t-1}] = \boldsymbol{\beta} \mathbf{X}_{t-1} = (\mathbf{X}'_{t-1} \otimes \mathbf{I}) \text{vec}(\mathbf{B}_t),$$

$$\begin{aligned} \text{var}[\mathbf{X}_t | \mathcal{F}_{t-1}] &= \mathbb{E} \left[ (\mathbf{X}_t - \mathbb{E}[\mathbf{X}_t | \mathcal{F}_{t-1}]) (\mathbf{X}_t - \mathbb{E}[\mathbf{X}_t | \mathcal{F}_{t-1}])' | \mathcal{F}_{t-1} \right] = \mathbb{E}[\mathbf{u}_t \mathbf{u}'_t | \mathcal{F}_{t-1}] \\ &= \mathbb{E} \left[ ((\mathbf{X}'_{t-1} \otimes \mathbf{I}) \text{vec}(\mathbf{B}_t) + \mathbf{Y}_t) ((\mathbf{X}'_{t-1} \otimes \mathbf{I}) \text{vec}(\mathbf{B}_t) + \mathbf{Y}_t)' | \mathcal{F}_{t-1} \right] \\ &= \mathbb{E}[(\mathbf{X}'_{t-1} \otimes \mathbf{I}) \text{vec}(\mathbf{B}_t) \text{vec}'(\mathbf{B}_t) (\mathbf{X}_{t-1} \otimes \mathbf{I}) | \mathcal{F}_{t-1}] + \mathbb{E}[\mathbf{Y}_t \mathbf{Y}'_t | \mathcal{F}_{t-1}] \\ &= (\mathbf{X}'_{t-1} \otimes \mathbf{I}) \cdot \boldsymbol{\Sigma} \cdot (\mathbf{X}_{t-1} \otimes \mathbf{I}) + \mathbf{G}. \end{aligned} \quad (4)$$

Using operators  $\text{vec}$  and  $\text{vech}$  and their properties as given in Lemmas A.3 and A.4, we further get after some computations

$$\text{vech}(\text{var}[\mathbf{X}_t | \mathcal{F}_{t-1}]) = \text{vech}(\mathbb{E}[\mathbf{u}_t \mathbf{u}'_t | \mathcal{F}_{t-1}]) = \mathbf{A}'_{t-1} \cdot \text{vech}(\boldsymbol{\Sigma}) + \text{vech}(\mathbf{G}) \quad (5)$$

where  $\mathbf{A}'_{t-1} = \mathbf{H}_m (\mathbf{X}'_{t-1} \otimes \mathbf{I}) \otimes (\mathbf{X}'_{t-1} \otimes \mathbf{I}) \mathbf{K}'_{m^2}$  and  $\mathbf{H}_m$  and  $\mathbf{K}_{m^2}$  are the elimination and duplication matrices from Lemma A.4. The least-squares estimator of the parameter  $\text{vec}(\boldsymbol{\beta})$  (and  $\boldsymbol{\beta}$ , respectively) can be obtained by minimizing

$$\sum_{t=1}^n \mathbf{u}'_t \mathbf{u}_t = \sum_{t=1}^n [\mathbf{X}_t - (\mathbf{X}'_{t-1} \otimes \mathbf{I}) \text{vec}(\boldsymbol{\beta})]' [\mathbf{X}_t - (\mathbf{X}'_{t-1} \otimes \mathbf{I}) \text{vec}(\boldsymbol{\beta})]$$

with respect to  $\text{vec}(\boldsymbol{\beta})$ , which leads to the normal equation

$$\sum_{t=1}^n (\mathbf{X}_{t-1} \otimes \mathbf{I}) [\mathbf{X}_t - (\mathbf{X}'_{t-1} \otimes \mathbf{I}) \text{vec}(\boldsymbol{\beta})] = \mathbf{0}$$

and to the estimator

$$\text{vec}(\hat{\boldsymbol{\beta}}_{LS}) = \left( \sum_{t=1}^n [\mathbf{X}_{t-1} \mathbf{X}'_{t-1} \otimes \mathbf{I}] \right)^{-1} \sum_{t=1}^n (\mathbf{X}_{t-1} \otimes \mathbf{I}) \mathbf{X}_t, \quad (6)$$

respectively,

$$\hat{\boldsymbol{\beta}}_{LS} = \left( \sum_{t=1}^n \mathbf{X}_t \mathbf{X}'_{t-1} \right) \cdot \left( \sum_{t=1}^n \mathbf{X}_{t-1} \mathbf{X}'_{t-1} \right)^{-1}. \quad (7)$$

In [11], Section 7.2, the strong consistency and asymptotic normality of  $\text{vec}(\hat{\boldsymbol{\beta}}_{LS})$  are proved under assumptions that the 4<sup>th</sup> moments of the components of the vector  $\mathbf{X}_t$  are finite.

The other parameters of the model, variance matrices  $\boldsymbol{\Sigma}$  and  $\mathbf{G}$ , can be estimated from the regression equation (5), when we use estimated residuals  $\hat{\mathbf{u}}_t =$

$\mathbf{X}_t - \widehat{\beta}_{LS} \mathbf{X}_{t-1}$ . The least-squares estimators are then as follows:

$$\begin{aligned} \text{vech}(\widehat{\Sigma}_{LS}) &= \left( \sum_{t=1}^n (\mathbf{A}_{t-1} - \overline{\mathbf{A}}) (\mathbf{A}_{t-1} - \overline{\mathbf{A}})' \right)^{-1} \left( \sum_{t=1}^n (\mathbf{A}_{t-1} - \overline{\mathbf{A}}) \cdot \text{vech}(\widehat{\mathbf{u}}_t \widehat{\mathbf{u}}_t') \right), \\ \text{vech}(\widehat{\mathbf{G}}_{LS}) &= \frac{1}{n} \sum_{t=1}^n \text{vech}(\widehat{\mathbf{u}}_t \widehat{\mathbf{u}}_t') - \overline{\mathbf{A}}' \cdot \text{vech}(\widehat{\Sigma}_{LS}), \end{aligned} \tag{8}$$

where  $\overline{\mathbf{A}} = \frac{1}{n} \sum_{t=1}^n \mathbf{A}_{t-1}$ . Under higher moment conditions on the process  $\{\mathbf{X}_t\}$  such estimators are strongly consistent (one requires finite  $4^{th}$  moments of the components of the vector  $\mathbf{X}_t$ ) and asymptotically normal (finite  $8^{th}$  moments), see [11], Theorem 7.2.

In [12] Schick considered estimators of parameters in a univariate RCA(1) model with *iid* errors as a solution of modified normal equations and studied asymptotic properties of these estimators without additional moment assumptions. We generalize the Schick method and propose an extension of the least-squares estimators of  $\beta$  into a class of estimators that solve the equation

$$\sum_{t=1}^n (\phi(\mathbf{X}_{t-1}) \otimes \mathbf{I}) [\mathbf{X}_t - (\mathbf{X}'_{t-1} \otimes \mathbf{I}) \text{vec}(\beta)] = \mathbf{0} \tag{9}$$

where  $\phi$  is a measurable function  $\phi : \mathbb{R}^m \rightarrow \mathbb{R}^m$ . Thus, we define estimator

$$\text{vec}(\widehat{\beta}_\phi) = \left( \sum_{t=1}^n \phi(\mathbf{X}_{t-1}) \mathbf{X}'_{t-1} \otimes \mathbf{I} \right)^{-1} \cdot \sum_{t=1}^n (\phi(\mathbf{X}_{t-1}) \otimes \mathbf{I}) \mathbf{X}_t, \tag{10}$$

respectively,

$$\widehat{\beta}_\phi = \left( \sum_{t=1}^n \mathbf{X}_t \phi(\mathbf{X}_{t-1})' \right) \cdot \left( \sum_{t=1}^n \mathbf{X}_{t-1} \phi(\mathbf{X}_{t-1})' \right)^{-1}. \tag{11}$$

The following theorem reveals basic properties of this estimator.

**Theorem 2.1.** Consider a multivariate RCA(1) model as in (1) that satisfies assumptions A1 to A4. Let  $\phi : \mathbb{R}^m \rightarrow \mathbb{R}^m$  be a measurable function. Denote  $\mathbf{P} = \phi(\mathbf{X}_0) \mathbf{X}'_0 \otimes \mathbf{I}$  and  $\mathbf{Q} = \phi(\mathbf{X}_0) \otimes \mathbf{I}$ , and assume that the matrix  $\text{E}(\phi(\mathbf{X}_0) \mathbf{X}'_0)$  is finite and invertible, and  $\text{E}(\mathbf{P} \Sigma \mathbf{P}' + \mathbf{Q} \mathbf{G} \mathbf{Q}')$  is finite. Then  $\text{vec}(\widehat{\beta}_\phi)$  defined by (10) is a strongly consistent and asymptotically normal estimator of the parameter  $\text{vec}(\beta)$ . The asymptotic variance matrix of  $\sqrt{n} \cdot \text{vec}(\widehat{\beta}_\phi - \beta)$  is given by

$$\mathbf{V}(\phi) = (\mathbf{E} \mathbf{P})^{-1} \cdot \text{E}(\mathbf{Q} \mathbf{G} \mathbf{Q}' + \mathbf{P} \Sigma \mathbf{P}') \cdot (\mathbf{E} \mathbf{P}')^{-1}. \tag{12}$$

**Remark 2.2.** The choice  $\phi(\mathbf{x}) = \mathbf{x}$  leads to the least-squares estimator of  $\beta$  and fulfills the finite matrices assumptions provided that  $\{\mathbf{X}_t\}$  has finite fourth moments. If  $\phi$  is bounded, this assumption reduces to the finiteness of the second moments of  $\{\mathbf{X}_t\}$ .

Proof. According to definition (10) and using Lemma A.3, we have

$$\text{vec}(\widehat{\beta}_\phi - \beta) = \left[ \left( \frac{1}{n} \sum_{t=1}^n \phi(\mathbf{X}_{t-1}) \mathbf{X}'_{t-1} \right)^{-1} \otimes \mathbf{I} \right] \left[ \frac{1}{n} \sum_{t=1}^n \left( \phi(\mathbf{X}_{t-1}) \otimes \mathbf{I} \right) \mathbf{u}_t \right]. \quad (13)$$

Strict stationarity and ergodicity of  $\mathbf{X}_t$  guarantee strict stationarity and ergodicity of both sequences  $\{\phi(\mathbf{X}_{t-1}) \mathbf{X}'_{t-1} \otimes \mathbf{I}, t \in \mathbb{Z}\}$  and  $\{(\phi(\mathbf{X}_{t-1}) \otimes \mathbf{I}) \mathbf{u}_t, t \in \mathbb{Z}\}$ . Moreover, the components of the latter sequence form a martingale difference sequence with zero mean value, which can be seen by choosing an arbitrary  $\alpha \in \mathbb{R}^{m^2}$  and noticing that

$$\mathbf{E}[\alpha'(\phi(\mathbf{X}_{t-1}) \otimes \mathbf{I}) \mathbf{u}_t | \mathcal{F}_{t-1}] = \alpha'(\phi(\mathbf{X}_{t-1}) \otimes \mathbf{I}) \mathbf{E}[\mathbf{u}_t | \mathcal{F}_{t-1}] = 0.$$

The ergodic theorem (see, e.g., [7], Theorem 13.12) tells us that, almost surely, the first term on the r.h.s. in (13) converges to  $(\mathbf{E}(\phi(\mathbf{X}_0) \mathbf{X}'_0))^{-1} \otimes \mathbf{I} = (\mathbf{E}\mathbf{P})^{-1}$  and the second term converges to zero, which implies that  $\text{vec}(\widehat{\beta}_\phi - \beta) \rightarrow \mathbf{0}$  almost surely as  $n \rightarrow +\infty$ .

Further notice that

$$\begin{aligned} \text{var} \left( \alpha'(\phi(\mathbf{X}_{t-1}) \otimes \mathbf{I}) \mathbf{u}_t \right) &= \mathbf{E} \left( \mathbf{E} \left[ (\alpha'(\phi(\mathbf{X}_{t-1}) \otimes \mathbf{I}) \mathbf{u}_t)^2 | \mathcal{F}_{t-1} \right] \right) \\ &= \mathbf{E} \left( \mathbf{E} \left[ \alpha'(\phi(\mathbf{X}_{t-1}) \otimes \mathbf{I}) \mathbf{u}_t \mathbf{u}'_t (\phi(\mathbf{X}_{t-1})' \otimes \mathbf{I}) \alpha | \mathcal{F}_{t-1} \right] \right) \\ &= \mathbf{E} \left( \alpha'(\phi(\mathbf{X}_{t-1}) \otimes \mathbf{I}) \cdot \mathbf{E}[\mathbf{u}_t \mathbf{u}'_t | \mathcal{F}_{t-1}] \cdot (\phi(\mathbf{X}_{t-1})' \otimes \mathbf{I}) \alpha \right) \\ &= \mathbf{E} \left( \alpha'(\phi(\mathbf{X}_{t-1}) \otimes \mathbf{I}) \cdot [(\mathbf{X}'_{t-1} \otimes \mathbf{I}) \Sigma (\mathbf{X}_{t-1} \otimes \mathbf{I}) + \mathbf{G}] \cdot (\phi(\mathbf{X}_{t-1})' \otimes \mathbf{I}) \alpha \right) \\ &= \alpha' \mathbf{E} \left( (\phi(\mathbf{X}_{t-1}) \mathbf{X}'_{t-1} \otimes \mathbf{I}) \Sigma (\mathbf{X}_{t-1} \phi(\mathbf{X}_{t-1})' \otimes \mathbf{I}) \right) \alpha \\ &\quad + \alpha' \mathbf{E} \left( (\phi(\mathbf{X}_{t-1}) \otimes \mathbf{I}) \mathbf{G} (\phi(\mathbf{X}_{t-1})' \otimes \mathbf{I}) \right) \alpha \\ &= \alpha' \mathbf{E}(\mathbf{P} \Sigma \mathbf{P}' + \mathbf{Q} \mathbf{G} \mathbf{Q}') \alpha. \end{aligned} \quad (14)$$

We know that, due to Lindeberg–Lévy theorem for martingales, see [4], for any  $\alpha \in \mathbb{R}^{m^2}$ ,  $\frac{1}{\sqrt{n}} \sum_{t=1}^n \alpha'(\phi(\mathbf{X}_{t-1}) \otimes \mathbf{I}) \mathbf{u}_t$  has an asymptotic normal distribution with zero mean and variance (14). Since

$$\sqrt{n} \cdot \text{vec}(\widehat{\beta}_\phi - \beta) = \left[ \left( \frac{1}{n} \sum_{t=1}^n \phi(\mathbf{X}_{t-1}) \mathbf{X}'_{t-1} \right)^{-1} \otimes \mathbf{I} \right] \cdot \left[ \frac{1}{\sqrt{n}} \sum_{t=1}^n \left( \phi(\mathbf{X}_{t-1}) \otimes \mathbf{I} \right) \mathbf{u}_t \right],$$

the rest of the proof easily follows by using the previous considerations and the Cramér–Wold device.  $\square$

To obtain estimators of the remaining parameters  $\Sigma$  and  $\mathbf{G}$ , we can use the regression (5) again, now with residuals  $\widehat{\mathbf{u}}_t = \mathbf{X}_t - \widehat{\beta}_\phi \mathbf{X}_{t-1}$ , and define modified

least-squares estimators

$$\begin{aligned} \text{vech}(\widehat{\Sigma}_h) &= \left( \sum_{t=1}^n (h(\mathbf{A}_{t-1}) - \bar{h})(\mathbf{A}_{t-1} - \bar{\mathbf{A}})' \right)^{-1} \left( \sum_{t=1}^n (h(\mathbf{A}_{t-1}) - \bar{h}) \cdot \text{vech}(\widehat{\mathbf{u}}_t \widehat{\mathbf{u}}_t') \right), \\ \text{vech}(\widehat{\mathbf{G}}_h) &= \frac{1}{n} \sum_{t=1}^n \text{vech}(\widehat{\mathbf{u}}_t \widehat{\mathbf{u}}_t') - \bar{\mathbf{A}}' \cdot \text{vech}(\widehat{\Sigma}_h), \end{aligned} \tag{15}$$

where

$$h(\mathbf{A}_{t-1}) = [\mathbf{H}_m(h(\mathbf{X}_{t-1})' \otimes \mathbf{I}) \otimes (h(\mathbf{X}_{t-1})' \otimes \mathbf{I}) \mathbf{K}'_{m^2}]',$$

$\bar{h} = \frac{1}{n} \sum_{t=1}^n h(\mathbf{A}_{t-1})$ , and  $h$  is a measurable function  $h : \mathbb{R}^m \rightarrow \mathbb{R}^m$ . The choice  $h(\mathbf{x}) = \mathbf{x}$  leads to the least-squares estimators (8).

**Theorem 2.3.** Consider a multivariate RCA(1) model as in (1) that satisfies assumptions A1 to A4. Let  $\phi, h$  be bounded measurable functions from  $\mathbb{R}^m$  to  $\mathbb{R}^m$  and let the matrix  $E[(h(\mathbf{A}_0) - Eh(\mathbf{A}_0))(\mathbf{A}_0 - E\mathbf{A}_0)']$  be invertible. Then the estimators  $\text{vech}(\widehat{\Sigma}_h)$  and  $\text{vech}(\widehat{\mathbf{G}}_h)$  defined in (15) are strongly consistent estimators of  $\text{vech}(\Sigma)$  and  $\text{vech}(\mathbf{G})$ , respectively.

*Proof.* We give only the main steps of the proof. Obviously, both  $\{h(\mathbf{A}_t)\}$  and  $\{\mathbf{A}_t\}$  are strictly stationary and ergodic and thus, as  $n \rightarrow \infty$ ,

$$\frac{1}{n} \sum_{t=1}^n (h(\mathbf{A}_{t-1}) - \bar{h})(\mathbf{A}_{t-1} - \bar{\mathbf{A}})' \rightarrow Eh(\mathbf{A}_0)\mathbf{A}'_0 - Eh(\mathbf{A}_0)(E\mathbf{A}_0)'$$

almost surely. Further,

$$\begin{aligned} \text{vech}(\widehat{\mathbf{u}}_t \widehat{\mathbf{u}}_t') &= \text{vech}(\mathbf{u}_t \mathbf{u}_t') + \text{vech}[(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}_\phi) \mathbf{X}_{t-1} \mathbf{X}'_{t-1} (\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}_\phi)'] \\ &\quad + \text{vech}((\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}_\phi) \mathbf{X}_{t-1} \mathbf{u}_t') + \text{vech}[(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}_\phi) \mathbf{X}_{t-1} \mathbf{u}_t']' \end{aligned}$$

and

$$\text{vech}[(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}_\phi) \mathbf{X}_{t-1} \mathbf{X}'_{t-1} (\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}_\phi)'] = \mathbf{A}'_{t-1} \text{vech}[\text{vec}(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}_\phi)(\text{vec}(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}_\phi))'].$$

Then, from the strict stationarity and ergodicity, strong consistency of  $\widehat{\boldsymbol{\beta}}_\phi$ , and the martingale difference properties of  $\{\mathbf{X}_{t-1} \mathbf{u}_t'\}$  we conclude that

$$\frac{1}{n} \sum_{t=1}^n \text{vech}(\widehat{\mathbf{u}}_t \widehat{\mathbf{u}}_t') \rightarrow E\text{vech}(\mathbf{u}_0 \mathbf{u}_0') = E\mathbf{A}_0 \text{vech}(\Sigma) + \text{vech}(\mathbf{G})$$

almost surely. In the same way, we get

$$\frac{1}{n} \sum_{t=1}^n h(\mathbf{A}_{t-1}) \text{vech}(\widehat{\mathbf{u}}_t \widehat{\mathbf{u}}_t') \rightarrow Eh(\mathbf{A}_0)\mathbf{A}'_0 \text{vech}(\Sigma) + Eh(\mathbf{A}_0) \text{vech}(\mathbf{G})$$

almost surely. Combining all these results we complete the proof.  $\square$



### 3. ASYMPTOTIC VARIANCE MATRIX

In this section we deal with the asymptotic variance matrix of the estimator  $\text{vec}(\widehat{\beta}_\phi)$ . First, we suggest a consistent estimator of  $\mathbf{V}(\phi)$ .

**Theorem 3.1.** Consider a multivariate RCA(1) model as in (1) that satisfies assumptions A1 to A4. Let  $\phi : \mathbb{R}^m \rightarrow \mathbb{R}^m$  be a measurable function such that the assumptions of Theorem 2.1 are satisfied. Let  $\widehat{\mathbf{G}}_n$  and  $\widehat{\Sigma}_n$  be strongly consistent estimators of  $\mathbf{G}$  and  $\Sigma$ , respectively. Denote  $\mathbf{P}_t = \phi(\mathbf{X}_{t-1})\mathbf{X}'_{t-1} \otimes \mathbf{I}$  and  $\mathbf{Q}_t = \phi(\mathbf{X}_{t-1}) \otimes \mathbf{I}$ .

Then

$$\begin{aligned} \widehat{\mathbf{V}}_n(\phi) &= n \left( \sum_{t=1}^n \mathbf{P}_t \right)^{-1} \cdot \sum_{t=1}^n (\mathbf{Q}_t \widehat{\mathbf{G}}_n \mathbf{Q}'_t) \cdot \left( \sum_{t=1}^n \mathbf{P}'_t \right)^{-1} \\ &\quad + n \left( \sum_{t=1}^n \mathbf{P}_t \right)^{-1} \cdot \sum_{t=1}^n (\mathbf{P}_t \widehat{\Sigma}_n \mathbf{P}'_t) \cdot \left( \sum_{t=1}^n \mathbf{P}'_t \right)^{-1} \end{aligned}$$

is a strongly consistent estimator of the asymptotic variance matrix  $\mathbf{V}(\phi)$  given by (12).

*Proof.* The process  $\{\mathbf{X}_t, t \in \mathbb{Z}\}$  is strictly stationary and ergodic which implies that  $\{\mathbf{P}_t, t \in \mathbb{Z}\}$ ,  $\{\mathbf{Q}_t, t \in \mathbb{Z}\}$ ,  $\{\mathbf{P}_t \Sigma \mathbf{P}'_t, t \in \mathbb{Z}\}$ , and  $\{\mathbf{Q}_t \mathbf{G} \mathbf{Q}'_t, t \in \mathbb{Z}\}$  are strictly stationary and ergodic. According to the ergodic theorem,  $\frac{1}{n} \sum_{t=1}^n \mathbf{P}_t \xrightarrow{\text{a.s.}} \mathbf{E} \mathbf{P}$  and  $\frac{1}{n} \sum_{t=1}^n \mathbf{Q}_t \xrightarrow{\text{a.s.}} \mathbf{E} \mathbf{Q}$  as  $n \rightarrow +\infty$ , where  $\mathbf{P}$  and  $\mathbf{Q}$  are defined in Theorem 2.1. Consistency of the estimators  $\widehat{\mathbf{G}}_n$ ,  $\widehat{\Sigma}_n$ , and Lemma A.5 complete the proof.  $\square$

We are interested in an optimal choice of the generating function  $\phi$  for the estimator  $\widehat{\beta}_\phi$ . Optimality of an estimator could be defined using its asymptotic variance matrix. We say that the estimator  $\widehat{\beta}(\psi)$  defined by equation (10) is optimal if its asymptotic variance matrix  $\mathbf{V}(\psi)$  defined by (12) satisfies  $\mathbf{V}(\phi) - \mathbf{V}(\psi) \geq \mathbf{0}$  for any estimator  $\widehat{\beta}_\psi$  with variance matrix  $\mathbf{V}(\psi)$ , i. e., the difference of the variance matrices is a positively semi-definite matrix.

We have performed an analysis of the asymptotic variance matrix for the univariate higher-order RCA models, see [16]. In that case, the optimal choice of the function  $\phi$  leads to the estimator of  $\beta$  that is formally equivalent to the conditionally weighted least-squares estimator when the remaining parameters are known. In the multivariate first-order case, we can only establish a lower bound for the asymptotic variance matrix of the estimators defined by (10).

**Theorem 3.2.** Let  $\{\mathbf{X}_t, t \in \mathbb{Z}\}$  be a multivariate RCA(1) model as in (1) that satisfies assumptions A1 to A4 such that the matrix

$$\mathbf{J} = \mathbf{E} \left( (\mathbf{X}_0 \otimes \mathbf{I}) \cdot [\mathbf{w}(\mathbf{X}_0)]^{-1} \cdot (\mathbf{X}'_0 \otimes \mathbf{I}) \right) \tag{16}$$

is nonsingular, where for any  $\mathbf{z} \in \mathbb{R}^m$ ,

$$\mathbf{w}(\mathbf{z}) = (\mathbf{z}' \otimes \mathbf{I}) \cdot \Sigma \cdot (\mathbf{z} \otimes \mathbf{I}) + \mathbf{G}. \tag{17}$$

Then the matrix  $\mathbf{J}^{-1}$  is a lower bound of the asymptotic variance matrix for all estimators  $\hat{\beta}_\phi$  such that  $\mathbb{E}(\phi(\mathbf{X}_0)\mathbf{X}'_0)$  is finite and invertible, and  $\mathbb{E}(\mathbf{P}\Sigma\mathbf{P}' + \mathbf{Q}\mathbf{G}\mathbf{Q}')$  is finite.

*Proof.* Consider the  $m^2$ -dimensional random vectors

$$\begin{aligned} \mathbf{T}_1 &= \sum_{t=1}^n \left( \phi(\mathbf{X}_{t-1}) \otimes \mathbf{I} \right) \cdot (\mathbf{X}_t - \beta' \mathbf{X}_{t-1}) = \sum_{t=1}^n \left( \phi(\mathbf{X}_{t-1}) \otimes \mathbf{I} \right) \cdot \mathbf{u}_t, \\ \mathbf{T}_2 &= \sum_{t=1}^n \left( \mathbf{X}_{t-1} \otimes \mathbf{I} \right) \cdot [\mathbf{w}(\mathbf{X}_{t-1})]^{-1} \cdot (\mathbf{X}_t - \beta' \mathbf{X}_{t-1}) \\ &= \sum_{t=1}^n \left( \mathbf{X}_{t-1} \otimes \mathbf{I} \right) \cdot [\mathbf{w}(\mathbf{X}_{t-1})]^{-1} \cdot \mathbf{u}_t, \end{aligned}$$

where  $\mathbf{w}(z)$  is defined by (17).

Since both sequences  $\{(\phi(\mathbf{X}_{t-1}) \otimes \mathbf{I}) \cdot \mathbf{u}_t\}$  and  $\{(\mathbf{X}_{t-1} \otimes \mathbf{I}) \cdot \mathbf{w}(\mathbf{X}_{t-1})^{-1} \cdot \mathbf{u}_t\}$  are martingale differences w.r.t.  $\mathcal{F}_t$ , it immediately follows that  $\mathbb{E}\mathbf{T}_1 = \mathbf{0}$ ,  $\mathbb{E}\mathbf{T}_2 = \mathbf{0}$  and the variance matrix of the vector  $\mathbf{T}_1$  equals

$$\begin{aligned} \mathbb{E}\mathbf{T}_1\mathbf{T}'_1 &= \mathbb{E} \left( \sum_{t=1}^n \left( \phi(\mathbf{X}_{t-1}) \otimes \mathbf{I} \right) \mathbf{u}_t \right) \cdot \left( \sum_{s=1}^n \mathbf{u}'_s \left( \phi(\mathbf{X}_{t-1})' \otimes \mathbf{I} \right) \right) \\ &= \sum_{t=1}^n \mathbb{E} \left( \left( \phi(\mathbf{X}_{t-1}) \otimes \mathbf{I} \right) \mathbf{u}_t \mathbf{u}'_t \left( \phi(\mathbf{X}_{t-1})' \otimes \mathbf{I} \right) \right) \\ &= \sum_{t=1}^n \mathbb{E} \left( \left( \phi(\mathbf{X}_{t-1}) \otimes \mathbf{I} \right) \cdot \mathbb{E}[\mathbf{u}_t \mathbf{u}'_t | \mathcal{F}_{t-1}] \cdot \left( \phi(\mathbf{X}_{t-1})' \otimes \mathbf{I} \right) \right) \\ &= n \cdot \mathbb{E} \left( \left( \phi(\mathbf{X}_0) \otimes \mathbf{I} \right) \cdot \mathbf{w}(\mathbf{X}_0) \cdot \left( \phi(\mathbf{X}_0)' \otimes \mathbf{I} \right) \right) \\ &= n \cdot \mathbb{E} \left( \left( \phi(\mathbf{X}_0) \otimes \mathbf{I} \right) \cdot \left( (\mathbf{X}'_0 \otimes \mathbf{I}) \cdot \Sigma \cdot (\mathbf{X}_0 \otimes \mathbf{I}) + \mathbf{G} \right) \cdot \left( \phi(\mathbf{X}_0)' \otimes \mathbf{I} \right) \right) \\ &= n \cdot \mathbb{E} \left( \left( \phi(\mathbf{X}_0) \mathbf{X}'_0 \otimes \mathbf{I} \right) \cdot \Sigma \cdot \left( \mathbf{X}_0 \phi(\mathbf{X}_0)' \otimes \mathbf{I} \right) \right) \\ &\quad + n \cdot \mathbb{E} \left( \left( \phi(\mathbf{X}_0) \otimes \mathbf{I} \right) \cdot \mathbf{G} \cdot \left( \phi(\mathbf{X}_0)' \otimes \mathbf{I} \right) \right) \\ &= n \cdot \mathbb{E}(\mathbf{P}\Sigma\mathbf{P}' + \mathbf{Q}\mathbf{G}\mathbf{Q}'), \end{aligned} \tag{18}$$

as follows from Lemma A.2 and the notation defined in Theorem 2.1. As a direct analogue we can infer that

$$\mathbb{E}\mathbf{T}_2\mathbf{T}'_2 = n \cdot \mathbb{E} \left( \left( \mathbf{X}_0 \otimes \mathbf{I} \right) \cdot [\mathbf{w}(\mathbf{X}_0)]^{-1} \cdot \left( \mathbf{X}'_0 \otimes \mathbf{I} \right) \right). \tag{19}$$

The cross-covariance matrix of  $\mathbf{T}_1$  and  $\mathbf{T}_2$  can be computed as follows:

$$\begin{aligned}
\mathbf{E}\mathbf{T}_1\mathbf{T}'_2 &= \mathbf{E} \left( \left( \sum_{t=1}^n \left( \phi(\mathbf{X}_{t-1}) \otimes \mathbf{I} \right) \cdot \mathbf{u}_t \right) \cdot \left( \sum_{s=1}^n \mathbf{u}'_s \cdot [\mathbf{w}(\mathbf{X}_{s-1})]^{-1} \cdot (\mathbf{X}'_{s-1} \otimes \mathbf{I}) \right) \right) \\
&= \sum_{t=1}^n \mathbf{E} \left( \left( \phi(\mathbf{X}_{t-1}) \otimes \mathbf{I} \right) \cdot \mathbf{E}[\mathbf{u}_t \mathbf{u}'_t | \mathcal{F}_{t-1}] \cdot [\mathbf{w}(\mathbf{X}_{t-1})]^{-1} \cdot (\mathbf{X}'_{t-1} \otimes \mathbf{I}) \right) \\
&= \sum_{t=1}^n \mathbf{E} \left( \left( \phi(\mathbf{X}_{t-1}) \otimes \mathbf{I} \right) \cdot \mathbf{w}(\mathbf{X}_{t-1}) \cdot [\mathbf{w}(\mathbf{X}_{t-1})]^{-1} \cdot (\mathbf{X}'_{t-1} \otimes \mathbf{I}) \right) \\
&= n \cdot \mathbf{E} \left( \left( \phi(\mathbf{X}_0) \otimes \mathbf{I} \right) \cdot (\mathbf{X}'_0 \otimes \mathbf{I}) \right) = n \cdot \mathbf{E} \left( \phi(\mathbf{X}_0) \mathbf{X}'_0 \otimes \mathbf{I} \right) = n \cdot \mathbf{E}\mathbf{P}.
\end{aligned} \tag{20}$$

The variance matrix of the  $2m^2$ -dimensional random vector  $(\mathbf{T}'_1, \mathbf{T}'_2)'$  is equal to

$$\begin{pmatrix} \mathbf{E}\mathbf{T}_1\mathbf{T}'_1 & \mathbf{E}\mathbf{T}_1\mathbf{T}'_2 \\ \mathbf{E}\mathbf{T}_2\mathbf{T}'_1 & \mathbf{E}\mathbf{T}_2\mathbf{T}'_2 \end{pmatrix}$$

where the block elements were computed in (18) – (20). Lemma A.6 tells us that, if the block elements  $\mathbf{E}\mathbf{T}_1\mathbf{T}'_2$  and  $\mathbf{E}\mathbf{T}_2\mathbf{T}'_2$  are invertible matrices, then

$$\begin{aligned}
& \left( \mathbf{E}\mathbf{T}_1\mathbf{T}'_2 \right)^{-1} \cdot \left( \mathbf{E}\mathbf{T}_1\mathbf{T}'_1 \right) \cdot \left( \mathbf{E}\mathbf{T}_2\mathbf{T}'_1 \right)^{-1} - \left( \mathbf{E}\mathbf{T}_2\mathbf{T}'_2 \right)^{-1} \geq \mathbf{0} \\
& \iff \left( \mathbf{E}\mathbf{P} \right)^{-1} \cdot \mathbf{E} \left( \mathbf{P}\Sigma\mathbf{P}' + \mathbf{Q}\mathbf{G}\mathbf{Q}' \right) \cdot \left( \mathbf{E}\mathbf{P}' \right)^{-1} - \left( \mathbf{E}\mathbf{T}_2\mathbf{T}'_2 \right)^{-1} \geq \mathbf{0} \\
& \iff \mathbf{V}(\phi) - \left( \mathbf{E} \left( (\mathbf{X}_0 \otimes \mathbf{I}) \cdot [\mathbf{w}(\mathbf{X}_0)]^{-1} \cdot (\mathbf{X}'_0 \otimes \mathbf{I}) \right) \right)^{-1} \geq \mathbf{0},
\end{aligned}$$

where  $\mathbf{V}(\phi)$  is the asymptotic variance matrix of the general estimator  $\widehat{\beta}_\phi$ . Thus

$$\left( \mathbf{E} \left( (\mathbf{X}_0 \otimes \mathbf{I}) \cdot [\mathbf{w}(\mathbf{X}_0)]^{-1} \cdot (\mathbf{X}'_0 \otimes \mathbf{I}) \right) \right)^{-1} = \mathbf{J}^{-1}$$

is a lower bound of the asymptotic variance matrix for all functional  $\widehat{\beta}_\phi$  such that  $\mathbf{E} \left( \phi(\mathbf{X}_0) \mathbf{X}'_0 \otimes \mathbf{I} \right)$  and  $\mathbf{J}$  are invertible.  $\square$

There are a few special cases when we can compute the optimal estimator explicitly. If  $\Sigma = \mathbf{0}$  for instance, which corresponds to the classical AR model, and  $\mathbf{G} = \sigma^2 \mathbf{I}$  for some  $\sigma^2 > 0$ , we have  $\mathbf{w}(\mathbf{z}) = \sigma^2 \mathbf{I}$ . Then the lower bound equals

$$\mathbf{J}^{-1} = \left( \mathbf{E} \left( (\mathbf{X}_0 \otimes \mathbf{I}) \cdot [\mathbf{w}(\mathbf{X}_0)]^{-1} \cdot (\mathbf{X}'_0 \otimes \mathbf{I}) \right) \right)^{-1} = \sigma^2 \left( \mathbf{E} \left( \mathbf{X}_0 \mathbf{X}'_0 \otimes \mathbf{I} \right) \right)^{-1},$$

and the asymptotic variance matrix of the estimator  $\widehat{\beta}_\phi$  with  $\phi(\mathbf{z}) = \mathbf{z}$  attains this lower bound, whereas such estimator corresponds to the least-squares estimator. In a general RCA(1) model, however, the lower bound for the asymptotic variance of the least-squares estimator is not attained.

If  $\Sigma = \tilde{\Sigma} \otimes I$ , where  $\tilde{\Sigma}$  is an  $m \times m$  positive definite matrix, and  $G = \sigma^2 I$ ,  $\sigma^2 > 0$ , then  $w(z) = (z' \tilde{\Sigma} z + \sigma^2) I$ ,

$$J^{-1} = \left[ E \left( \frac{X_0 X_0'}{X_0' \tilde{\Sigma} X_0 + \sigma^2} \right) \right]^{-1} \otimes I$$

and this lower bound is attained with the function  $\phi(z) = z(z' \tilde{\Sigma} z + \sigma^2)^{-1} = w(z)^{-1} z$  which corresponds to the conditionally weighted estimator. Like in the univariate case, such choice, of course, depends heavily on the matrix  $\tilde{\Sigma}$  and the parameter  $\sigma^2$  that are usually unknown.

If we assume that  $\Sigma = I, G = I$ , then the function  $\phi(z) = z(1 + z'z)^{-1}$  leads to the optimal estimator with the asymptotic variance

$$V(\phi) = J^{-1} = (E((X_0 X_0' \otimes I)(1 + X_0' X_0)^{-1}))^{-1}.$$

In a simulation study we show that this choice of  $\phi$  provides a reasonable estimator even with other values of variance matrices  $G$  and  $\Sigma$ . The advantage of this estimator is that it does not depend on nuisance parameters and seems to be more stable than the least-squares estimator.

**Remark 3.3.** Notice that any one-dimensional RCA process of order  $p$ ,

$$X_t = \sum_{i=1}^p (\beta_i + B_{ti}) X_{t-i} + Y_t, \tag{21}$$

where  $\beta_i$  and  $B_{ti}, i = 1, \dots, p$ , are constant and random components of the vector random coefficient, can be written as a multivariate RCA(1) model of form (1) with  $X_t = (X_t, X_{t-1}, \dots, X_{t-p+1})', Y_t = (Y_t, 0, \dots, 0)'$ , and

$$\beta = \begin{pmatrix} \beta_1 & \beta_2 & \dots & \beta_{p-1} & \beta_p \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \quad B_t = \begin{pmatrix} B_{t1} & \dots & B_{tp} \\ 0 & \dots & 0 \\ \vdots & & \\ 0 & \dots & 0 \end{pmatrix}. \tag{22}$$

Then the role of  $\Sigma$  and  $G$  is played by  $E(B_t B_t')$  and  $E[Y_t^2 | \mathcal{F}_{t-1}]$ , respectively, where  $B_t = (B_{t1}, \dots, B_{tp})'$ . The least-squares estimator of the parameter  $\beta = (\beta_1, \dots, \beta_p)'$  is

$$\hat{\beta}_{LS} = \left( \sum_{t=1}^n X_{t-1} X_{t-1}' \right)^{-1} \sum_{t=1}^n X_{t-1} X_t$$

and the Schick-type estimator is of the form

$$\hat{\beta}_\phi = \left( \sum_{t=1}^n \phi(X_{t-1}) X_{t-1}' \right)^{-1} \sum_{t=1}^n \phi(X_{t-1}) X_t$$

where  $\phi$  denotes a measurable function  $\phi : \mathbb{R}^p \rightarrow \mathbb{R}^p$ . The asymptotic behavior of  $\hat{\beta}_\phi$  was studied in detail in [16].

## 4. SIMULATION STUDY

In this short study we compare the least-squares estimator to a particular choice of the Schick-type estimator. We simulated 100 observations from a 2-dimensional RCA(1) model given by

$$\begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} = \left( \begin{pmatrix} 0.2 & 0.1 \\ 0.3 & 0.4 \end{pmatrix} + \begin{pmatrix} B_t^{11} & B_t^{12} \\ B_t^{21} & B_t^{22} \end{pmatrix} \right) \cdot \begin{pmatrix} X_{t-1}^1 \\ X_{t-1}^2 \end{pmatrix} + \begin{pmatrix} Y_t^1 \\ Y_t^2 \end{pmatrix} \quad (23)$$

where both random coefficients  $\mathbf{B}_t$  and error process  $\mathbf{Y}_t$  are mutually independent and identically normally distributed with  $\boldsymbol{\Sigma} = \text{var}(\text{vec } \mathbf{B}_t) = 0.2 \cdot \mathbf{I}$  and  $\mathbf{G} = \text{var} \mathbf{Y}_t = \mathbf{I}$ . Notice that assumption A2 is fulfilled, because the matrix  $\text{E}(\mathbf{B}_0 \otimes \mathbf{B}_0) + (\boldsymbol{\beta} \otimes \boldsymbol{\beta})$  has eigenvalues 0.583,  $0.064 \pm 0.03i$ , 0.050 that are less than one in the modulus.

Then we estimated the parameter  $\boldsymbol{\beta}$  using both  $\widehat{\boldsymbol{\beta}}_{LS}$  given by equation (7) and  $\widehat{\boldsymbol{\beta}}_\phi$  given by (10) with  $\phi(\mathbf{z}) = \frac{\mathbf{z}}{1+\mathbf{z}'\mathbf{z}}$ . We ran the simulation 1000 times. The results of the simulation study are displayed in Figure 1 and in Table below. The estimators are compared using sample means and density estimations of the 1000 estimated values of the true parameters  $\beta_{11}$ ,  $\beta_{12}$ ,  $\beta_{21}$  and  $\beta_{22}$  (we used the default density estimation procedure in the R programming language). We can see that the least-squares estimator  $\widehat{\boldsymbol{\beta}}_{LS}$  always underestimates the true value, especially for  $\beta_{11}$  and  $\beta_{21}$  whereas the estimator  $\widehat{\boldsymbol{\beta}}_\phi$  is closer to the true values. The density estimation also reveals bias for the least-squares estimator. These results are in accordance with previous simulations made for univariate RCA processes, see [16].

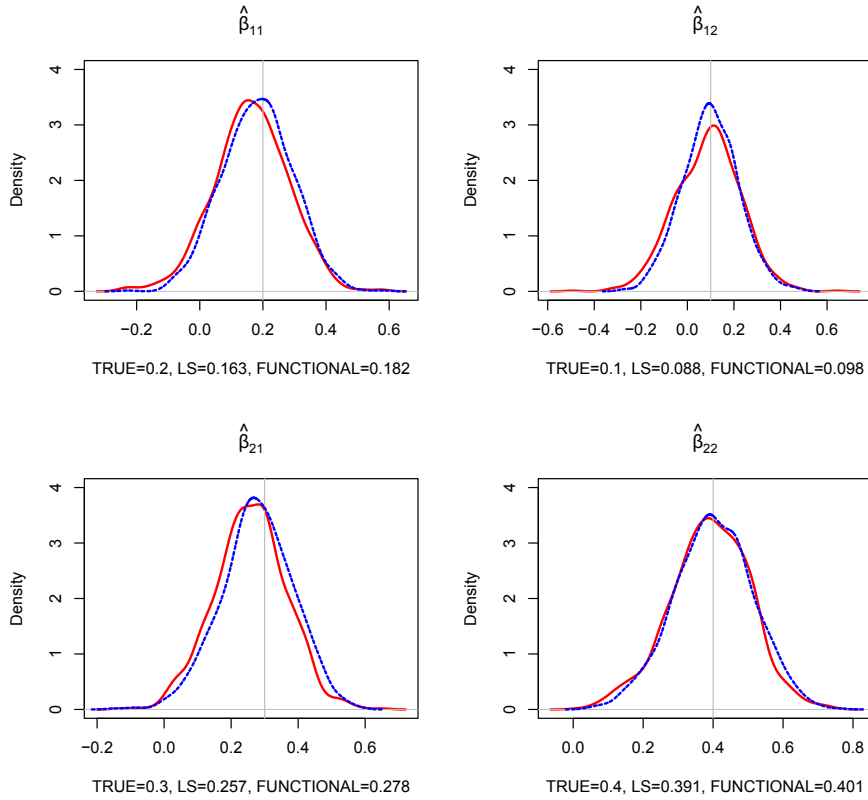
parameter	true value	LS est.	$\phi(\mathbf{z}) = \frac{\mathbf{z}}{1+\mathbf{z}'\mathbf{z}}$
$\beta_{11}$	0.2	0.163	0.182
$\beta_{12}$	0.1	0.088	0.098
$\beta_{21}$	0.3	0.257	0.278
$\beta_{22}$	0.4	0.391	0.401

**Tab. 1.** Average values of estimated parameters in simulated model (23).

We also simulated 500 observations of the one-dimensional RCA process of second order given by

$$X_t = (0.4 + B_{t1})X_{t-1} + (0.2 + B_{t2})X_{t-2} + Y_t. \quad (24)$$

The random coefficients  $B_t = (B_{t1}, B_{t2})'$  were independent identically normally distributed with zero mean and diagonal variance matrix  $\text{Diag}[\sigma_1^2, \sigma_2^2]$ , where  $\sigma_1^2, \sigma_2^2$  will be specified below, and  $\{Y_t\}$  was a sequence of *iid* random errors with standard normal distribution, independent of  $\{B_t\}$ .



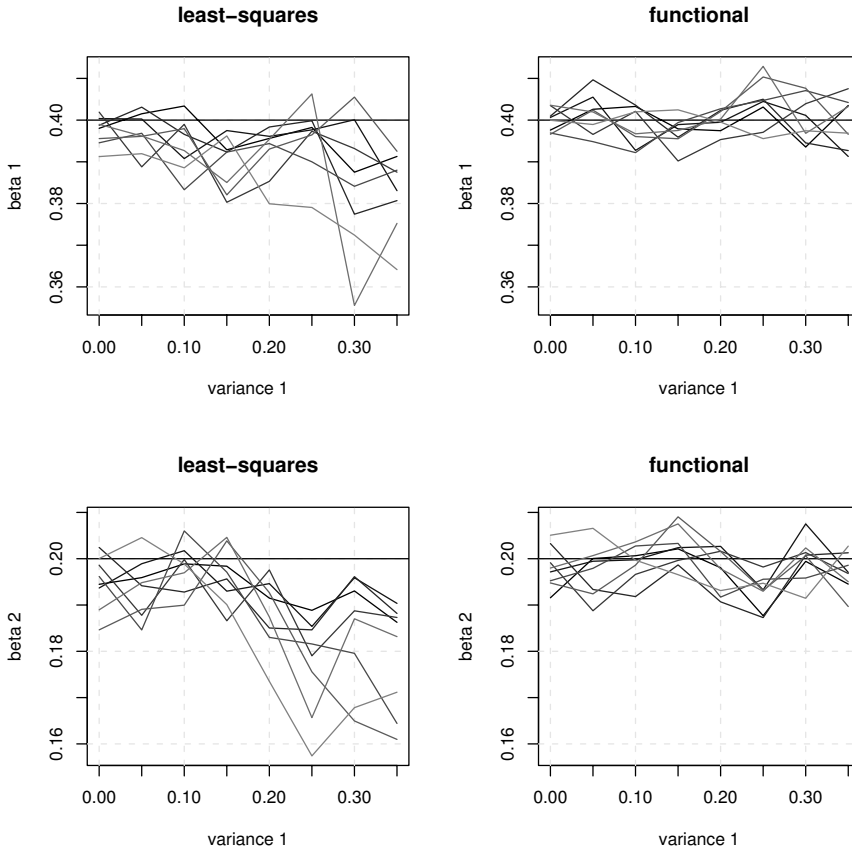
**Fig. 1.** Density of estimated values of the parameters (solid line for the least-squares estimator, dotted line for Schick-type estimator) in simulated 2-dimensional RCA(1) process (23).

In our case,

$$\boldsymbol{\beta} = \begin{pmatrix} 0.4 & 0.2 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{B}_t = \begin{pmatrix} B_{t1} & B_{t2} \\ 0 & 0 \end{pmatrix}$$

and for  $\sigma_1^2, \sigma_2^2 \in (0; 0.05; 0.10; \dots; 0.35)$ , all the eigenvalues of  $E(\mathbf{B}_0 \otimes \mathbf{B}_0) + \boldsymbol{\beta} \otimes \boldsymbol{\beta}$  are in the unit circle and thus the assumption A2 is satisfied. We estimated the parameter  $\boldsymbol{\beta} = (\beta_1, \beta_2)'$  using  $\hat{\beta}_{LS}$  and  $\hat{\beta}_\phi$  with  $\phi(\mathbf{z}) = \frac{\mathbf{z}}{1+\mathbf{z}'\mathbf{z}}$ , respectively, and compared the mean square errors. The results are displayed in Figures 2 and 3 as functions of  $\sigma_1^2$  for a fixed level of  $\sigma_2^2$ . The estimators  $\hat{\beta}_{LS}$  and  $\hat{\beta}_\phi$  with  $\phi$  as above behave similarly when the random coefficients have low variances, however,  $\hat{\beta}_{LS}$  underestimates the true value of the parameter when the variances increase. On the other hand,  $\hat{\beta}_\phi$  is stable no matter how large the variances of the random coefficients are and the estimated values do not vary so much even if only 100 repetitions are

used.

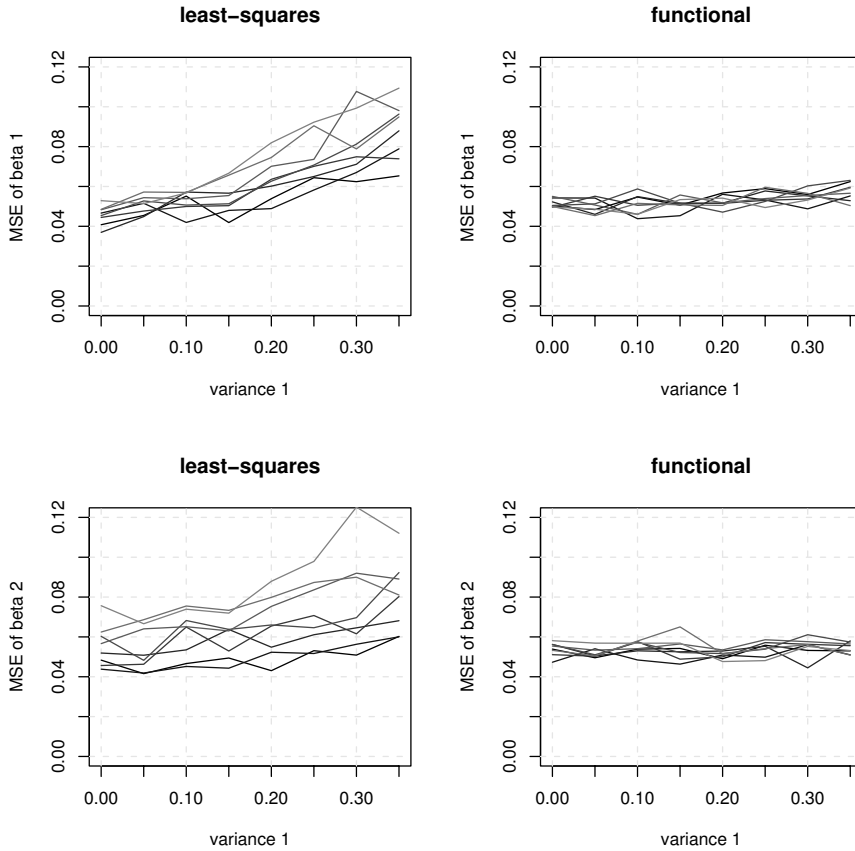


**Fig. 2.** Estimates of true parameter  $\beta = (0.4, 0.2)'$  in simulated model (24). Upper panels stand for estimation of  $\beta_1$ , lower for  $\beta_2$ , left for the least-squares estimator, right for the Schick-type estimator. Horizontal axis stands for variance  $\sigma_1^2$ , curves correspond to various choices of variance  $\sigma_2^2$ .

## A. APPENDIX

**Definition A.1.** We introduce the following notions.

1. Let  $\mathbf{A}$  be an  $m \times n$  matrix. Then the  $mn$ -component vector  $\text{vec}(\mathbf{A})$  is defined by stacking the columns of  $\mathbf{A}$ , one on top of the other in order from left to right.



**Fig. 3.** MSE of estimates of parameters  $\beta$  in simulated model (24). Upper panels for  $\beta_1$ , lower for  $\beta_2$ , left panels for least-squares estimator, right for the Schick-type estimator. Horizontal axis stands for variance  $\sigma_1^2$ , curves correspond to various choices of variance  $\sigma_2^2$ .

2. Let  $\mathbf{A}$  be an  $n \times n$  symmetric matrix. Then the  $n(n + 1)/2$ -component vector  $vech(\mathbf{A})$  is defined as stacking those parts of the columns of  $\mathbf{A}$  on and below the main diagonal, one on top of the other in order from left to right.
  
3. Let  $\mathbf{A}$  be an  $m \times n$  matrix and  $\mathbf{B}$  be a  $p \times q$  matrix. Then the *Kronecker product*  $\mathbf{A} \otimes \mathbf{B}$  of  $\mathbf{A}$  and  $\mathbf{B}$  is the  $mp \times nq$  matrix whose  $(i, j)$ th block is the  $p \times q$  matrix  $a_{i,j} \cdot \mathbf{B}$ , where  $a_{i,j}$  is the  $(i, j)$ th element of  $\mathbf{A}$ .



**Lemma A.2.** Let  $\mathbf{A}_i$  and  $\mathbf{B}_i$ ,  $i = 1, 2, \dots, n$ , be any matrices. Then

$$\left( \prod_{i=1}^n \mathbf{A}_i \right) \otimes \left( \prod_{j=1}^n \mathbf{B}_j \right) = \prod_{i=1}^n (\mathbf{A}_i \otimes \mathbf{B}_i).$$

*Proof.* See [13], Paragraph 3 in Chapter 7.  $\square$

**Lemma A.3.** Let  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{C}$  be any matrices and  $\mathbf{u}$ ,  $\mathbf{v}$  be any vectors such that the expressions below are well defined. Then

- (a)  $\text{vec}(\mathbf{ABC}) = (\mathbf{C}' \otimes \mathbf{A}) \cdot \text{vec}(\mathbf{B})$
- (b)  $\mathbf{Au} = (\mathbf{u}' \otimes \mathbf{I}) \cdot \text{vec}(\mathbf{A})$
- (c)  $\text{vec}(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B}' \otimes \mathbf{I}) \cdot \text{vec}(\mathbf{A})$
- (d)  $\text{vec}(\mathbf{uv}') = (\mathbf{v} \otimes \mathbf{I}) \cdot \mathbf{u}$ .

*Proof.* Properties (a) and (b) are proved in [13], Paragraph 5 in Chapter 7. Properties (c) and (d) are direct applications of property (a).  $\square$

**Lemma A.4.** Let  $\mathbf{A}$  be an  $n \times n$  symmetric matrix. Then there exist constant  $(n(n+1)/2) \times n^2$  matrices  $\mathbf{K}_n$  and  $\mathbf{H}_n$  for which

$$\begin{aligned} \text{vech}(\mathbf{A}) &= \mathbf{H}_n \text{vec}(\mathbf{A}) \\ \text{vec}(\mathbf{A}) &= \mathbf{K}'_n \text{vech}(\mathbf{A}) \end{aligned}$$

such that  $\mathbf{H}_n \mathbf{K}'_n = \mathbf{I}_{n(n+1)/2}$ . Matrices  $\mathbf{H}_n$  and  $\mathbf{K}_n$  are sometimes called the elimination and duplication matrices, respectively.

*Proof.* See Theorem A.1.3 in [11].  $\square$

**Lemma A.5.** Let  $\{\mathbf{A}_n, n \in \mathbb{N}\}$  and  $\{\mathbf{B}_n, n \in \mathbb{N}\}$  be random  $r \times r$ -dimensional matrix processes. Assume that  $\{\mathbf{B}_n\}$  is ergodic with finite second moments and that  $\mathbf{A}_n \xrightarrow{\text{a.s.}} \mathbf{A}$  as  $n \rightarrow +\infty$  where  $\mathbf{A} \in \mathbb{R}^{r \times r}$  is a finite constant matrix. Then

$$\frac{1}{n} \sum_{t=1}^n \mathbf{B}_t \mathbf{A}_n \mathbf{B}'_t \xrightarrow{\text{a.s.}} \mathbb{E}(\mathbf{B}_1 \mathbf{A} \mathbf{B}'_1) \text{ as } n \rightarrow +\infty.$$

*Proof.* We prove the convergence of the correspondent components using ergodicity and  $\text{vec}$  operator:

$$\begin{aligned} \text{vec} \left( \frac{1}{n} \sum_{t=1}^n \mathbf{B}_t \mathbf{A}_n \mathbf{B}'_t \right) &= \frac{1}{n} \sum_{t=1}^n \text{vec} \left( \mathbf{B}_t \mathbf{A}_n \mathbf{B}'_t \right) = \frac{1}{n} \sum_{t=1}^n \mathbf{B}_t \otimes \mathbf{B}_t \cdot \text{vec}(\mathbf{A}_n) \\ &\xrightarrow{\text{a.s.}} \mathbb{E} \left( \mathbf{B}_1 \otimes \mathbf{B}_1 \right) \cdot \text{vec}(\mathbf{A}) = \mathbb{E} \left( \text{vec} \left( \mathbf{B}_1 \mathbf{A} \mathbf{B}'_1 \right) \right) = \text{vec} \left( \mathbb{E} \left( \mathbf{B}_1 \mathbf{A} \mathbf{B}'_1 \right) \right). \end{aligned}$$

$\square$

**Lemma A.6.** Consider block matrix

$$M = \begin{pmatrix} A & B \\ B' & C \end{pmatrix},$$

where  $A$ ,  $B$ ,  $C$  are  $p \times p$ -dimensional matrices. Let  $M \geq \mathbf{0}$ ,  $B$  be an invertible matrix, and  $C$  be a symmetric invertible matrix. Then

$$B^{-1}AB'^{-1} - C^{-1} \geq \mathbf{0}.$$

*Proof.* We will prove that  $A - BC^{-1}B' \geq \mathbf{0}$  from which the result could be obtained by multiplication of matrices  $B^{-1}$  and  $B'^{-1}$ , respectively. Choose arbitrary  $\mathbf{x} \in \mathbb{R}^p$  and  $\mathbf{y} \in \mathbb{R}^p$ . Matrix  $M$  is positively semi-definite which means that

$$(\mathbf{x}', \mathbf{y}')M \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \mathbf{x}'A\mathbf{x} + \mathbf{y}'B'\mathbf{x} + \mathbf{x}'B\mathbf{y} + \mathbf{y}'C\mathbf{y} = \mathbf{x}'A\mathbf{x} + \mathbf{y}'C\mathbf{y} + 2\mathbf{x}'B\mathbf{y} \geq 0.$$

We want to prove that for any  $\mathbf{x} \in \mathbb{R}^p$ ,

$$\mathbf{x}'A\mathbf{x} - \mathbf{x}'BC^{-1}B'\mathbf{x} \geq 0.$$

Comparing the previous two inequalities we conclude that it suffices to find  $\mathbf{y} \in \mathbb{R}^p$  such that

$$\mathbf{y}'C\mathbf{y} + 2\mathbf{x}'B\mathbf{y} = -\mathbf{x}'BC^{-1}B'\mathbf{x}.$$

Denote  $\mathbf{z} = B'\mathbf{x}$ . Then  $\mathbf{y}$  has to satisfy the elliptical equation

$$\mathbf{y}'C\mathbf{y} + 2\mathbf{z}'\mathbf{y} + \mathbf{z}'C^{-1}\mathbf{z} = 0$$

and  $\mathbf{y} = -C^{-1}\mathbf{z}$  solves the latter equation which completes the proof. □

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