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Milan Mareš

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ENTROPIES OF VAGUE INFORMATION SOURCES

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The information-theoretical entropy is an effective measure of uncertainty connected with an information source. Its transfer from the classical probabilistic information theory models to the fuzzy set theoretical environment is desirable and significant attempts were realized in the existing literature. Nevertheless, there are some open topics for analysis in the suggested models of fuzzy entropy – the main of them regard the formal aspects of the fundamental concepts. Namely their rather additive (i. e., probability-like) than monotonous (typical for fuzzy set theoretical models) structure. The main goal of this paper is to describe briefly the existing state of art, and to suggest and analyze alternative, more fuzzy set theoretical, approaches to the fuzzy entropy developed as a significant characteristic of the information sources, in the information-theoretical sense.

Keywords: information source, message, uncertainty, fuzzy set, fuzzy entropy, fuzzy information

Classification: 94A17, 94A15, 94D05, 03B52

1. INTRODUCTION

The serious attempts to measure the information (or, vice versa, uncertainty) processed in the frame of data acquisition and analysis can be found in the literature. The Fisher information measure (cf. [5]) belongs to the first concepts of the considered type. The main methodology of information analysis and processing was developed by Shannon and Weaver in [22], and by many of their successors (cf. [8, 9, 21], for example). All these works were based on the probabilistic models, representing the randomness as the best developed type of uncertainty.

After the Zadeh's introduction of the vagueness as an alternative uncertainty concept [24], formally described by the fuzzy set theoretical tools, several attempts to define a quantitative measure of the degree of vagueness were suggested (let us mention [6, 11, 19, 20] for illustration). Some of those fuzzy uncertainty measures explicitly refer the information theoretical concept of entropy (see [1, 4, 7, 13, 18, 23] and recently also [2]). Let us note that in those works, the application of the entropy-like concepts was mainly motivated by the endeavour to measure the indeterminism of fuzzy sets, and the eventual regard to the data and knowledge acquisition and transmission has more or less secondary effect. An elementary attempt to study the fuzzy set theoretical aspects of vague data sources was done, e. g., in [17] and [16].

The formal definitions of the fuzzy entropy presented in the referred literature, often repeat the structure of the Shannon's probabilistic model of the entropy or some of its specific components, in spite of the fact that there are significant differences between the essence of randomness and vagueness. Namely, the suggested definitions often prefer the additivity of uncertainty to its monotonicity, characteristic for the processing of fuzzy sets which preference implies the application of some other tools (e. g., the logarithmic function) which could be possible but redundant in the case of the models based on the monotonicity paradigm.

The aim of this paper is to discuss the methodological topics introduced above, and to suggest an alternative approach to the fuzzy entropy, more consequently based on the fuzzy set theoretical methods. Some basic properties of that alternative model are presented, too.

It is obvious that relatively many concepts introduced here can be generalized, e. g., by substitution of elementary fuzzy set theoretical tools by their more advanced counterparts used in the theory of aggregated operators, copulas, and others (cf. also Conclusive Remarks). The choice of most elementary formalism is motivated by the endeavour to simplify the interpretation and orientation in new method. Eventual generalization is investigated recently.

The following sections are organized, as follows. The next Section 2 introduces the concept of fuzzy information source and its elementary properties. After a brief recollection of the probabilistic model in Section 3, the analysis of the fuzzy information source continues in Section 4 by the discussion of the main fuzzy entropy models presented in the referred literature, and in Section 5 by the analysis of the fuzzy information connected with particular symbols or finite sequences of symbols emitted by a fuzzy information source. Finally, Section 6 is devoted to the presentation of some concepts of fuzzy entropy (known from the literature, as well as suggested here) and their analysis from the point of view of general, heuristically acceptable, conditions putted on the fuzzy uncertainty concept in the wider sense. The paper is concluded by a brief remark.

2. FUZZY INFORMATION SOURCE

In accordance with both, probabilistic and fuzzy set theoretical, methods of the information theory (see, e. g., [8, 16, 19, 21, 22]) we define the source of information as a pair, composed from an alphabet and an uncertainty distribution over that alphabet (cf. [16]). In our case, the fuzzy information source is defined as a fuzzy subset of an alphabet, identified by a membership function.

More precisely, the *alphabet* is a non-empty and finite set A of *symbols*, and the *uncertainty* is represented by the membership function $\mu : A \to [0,1]$. The pair (A, μ) is called an *elementary fuzzy source*.

Any bilaterally unlimited sequence of symbols

$$\dots a_{-2}, a_{-1}, a_0, a_1, a_2, \dots, a_{n-1}, a_n, a_{n+1}, \dots$$

is called a *message*, and our main attention will be focused on finite *segments* of messages.

The vagueness is a frequently appearing type of uncertainty even in the context of the information emission and transmission, whenever the situation does not admit the application of the statisticaly estimated probabilities. The vague reading of defected symbols, subjective interpretation of noisy measurements, or approximation of continuous data by discrete values, can be mentioned as examples of fuzzy information and knowledge. The essential difference between the probabilistic and fuzzy interpretation of the data uncertainty appears to consists in the following heuristic principle. Meanwhile the probability $p(a)$, $a \in A$, in the Shannon's classical model usually represents the uncertainty with which the symbol a is expected in the future, the membership value $\mu(a)$ rather evaluates the vagueness of the interpretation or understanding the symbol $a \in A$, already received as a result of the information acquisition.

The elementary fuzzy source, introduced above, can be extended to more complex object, namely finite segments of messages (i. e., words). In the case of the probabilistic model, such extension is formally treated by the well managed concepts of conditional and associated probabilities. Analogous procedure can be used for fuzzy information sources handled in the following way.

If (A, μ) is an elementary fuzzy source, then we denote by A^* the set of all finite segments, i. e.,

$$A^* = A \cup A^2 \cup A^3 \cup \cdots \cup A^n \cup \ldots$$

and call it the *extended alphabet*. Similarly, any mapping $\mu^* : A^* \to [0, 1]$ such that for any, n, $m = 1, 2, \ldots$, and any $\boldsymbol{a} = (a_1, \ldots, a_n) \in A^n$, $\boldsymbol{b} = (b_1, \ldots, b_m) \in A^m$

$$\mu^*(\boldsymbol{a}) = \mu(a_1) \quad \text{if } n = 1,\ \boldsymbol{a} = (a_1), \tag{1}$$

$$\mu^*(\boldsymbol{a}, \boldsymbol{b}) \leq \min\left(\mu^*(\boldsymbol{a}),\ \mu^*(\boldsymbol{b})\right) \tag{2}$$

is called the extension of μ on A^*. The pair (A^*, μ^*) is called the *fuzzy source*. If the inequality (2) turns into equality for all $\boldsymbol{a}$, $\boldsymbol{b} \in A^*$, then the fuzzy source (A^*, μ^*) is called *independent*.

Remark 1. If (A^*, μ^*) is a fuzzy source, then obviously $\mu^*(\boldsymbol{a}) \leq \min(\mu(a_1), \ldots$ $\ldots, \mu(a_n))$ for any $\boldsymbol{a} = (a_1, \ldots, a_n) \in A^*$, as follows from (2) and (1).

The previous definition (1), (2), and Remark 1 immediately mean that any independent fuzzy source (A^*, μ^*) is fully defined by an elementary fuzzy source (A, μ).

3. BRIEF RECOLLECTION OF PROBABILISTIC MODEL

The Shannon's probabilistic pattern [22] of the basic structure of an information source and the information/uncertainty measure is universal and applicable in various specific models (cf. [16], e. g.), and it is frequently referred in the following sections, too. Hence, it deserves a brief summary of its main concepts.

The *random information source* is defined as a pair (A, p), where A is an alphabet and p is a probability distribution over A. For any $a \in A$, $p(a)$ is the probability with which the emission of a by the source is expected. The probability distribution p can be extended on the extended alphabet A^* by means of well known probabilistic tools (see, e. g., [8, 9, 21, 22] and many others). The source is said to be independent if for any pair of symbols $(a_1, a_2) \in A^2$, the extended probability $p(a_1, a_2) = p(a_1) \cdot p(a_2)$.

The probabilistic *information* transmitted by a symbol $a \in A$ is defined as a value $I_p(a)$, where

$$I_p(a) = -\log_2 p(a) = \log_2(1/p(a)). \tag{3}$$

The uncertainty characterizing the entire source (A, p) is defined as the mean value of individual probabilistic information measures. We denote it by $H_p(A, p)$, call it the *probabilistic entropy*, and define by

$$H_p(A, p) = \sum\nolimits_{a\in A} p(a) \cdot I_p(a) = -\sum\nolimits_{a\in A} p(a) \cdot \log_2 p(a). \tag{4}$$

There exist several attempts to transfer the above concepts into the fuzzy set theoretical environment, and many others can be suggested. Those which appear to be most significant are shown and discussed in the following sections.

4. FUZZY ANALOGIES OF ENTROPY – DISCUSSION

Soon after the publication of the seminal work [22], interesting generalizations or modifications of its main concepts were suggested (see, e. g., [8, 9, 19, 20, 21]). Nevertheless, the main stream of such works has appeared after the introduction of the concept of vagueness, and its representative, fuzzy set, by Zadeh's paper [24]. Let us mention [1, 4, 11], followed by [7, 13, 18, 23] and other papers, till the generalizing work [2].

In this section, we focus our attention on the works dealing with the, in principle information theoretical, measures of vagueness formally described by a fuzzy set. Such endeavour usually resulted into the introduction of concepts analogous to Shannon's entropy (4). Quite often, the fuzzified entropy included the logarithmic function in a position similar to the classical probabilistic entropy.

Namely, if we use the notations and terms of the previous section, then the fuzzified entropy of a fuzzy source (A^*, μ^*) can be defined (due to De Luca and Termini [4]) as the value

$$H_{\mathrm{LT}}(A^*, \mu^*) = -K \cdot \sum\nolimits_{\boldsymbol{a}\in A^*} \mu^*(\boldsymbol{a}) \cdot \log_2 \mu^*(\boldsymbol{a}), \tag{5}$$

where K is a positive constant normalizing the final value of the entropy. We call that type of entropy *De Luca–Termini fuzzy entropy.*

Rather more complex definition of logarithmic fuzzy entropy was suggested by Kolesárová and Vivona in [13], where the following modification of the fuzzy entropy concept is suggested. For a fuzzy source (A^*, μ^*), the value

$$H_{\mathrm{KV}}(A^*, \mu^*) = -K \sum\nolimits_{\boldsymbol{a}\in A^*} \left(\mu^*(\boldsymbol{a}) \cdot \log_2 \mu^*(\boldsymbol{a}) + (1 - \mu^*(\boldsymbol{a})) \cdot \log(1 - \mu^*(\boldsymbol{a}))\right), \tag{6}$$

where K is, analogously to the previous case, a normalizing positive constant. In the rest of this paper, we call the value $H_{\mathrm{KV}}(A^*, \mu^*)$ *Kolesárová–Vivona fuzzy entropy.* In both definitoric equations, (5) and (6), the convention "$0 \cdot \log_2 0$"$= 0$ is respected.

The above "logarithmic" approaches to fuzzy entropy are correct and they have significant advantages, including their nearness to the probabilistic pattern. Nevertheless, there are some aspects of their structure which deserve to be discussed. Most of them are related to the fact that each entropy, including the fuzzy ones, represents an aggregation operator (see [3], for example) over the values of fuzzy information transmitted by individual symbols. We turn our attention to the following features of the above fuzzy entropies, especially.

4.1. The Additivity of Information

The Shannon's concepts of entropy (4) and information (3) are defined for the random uncertainty characterized by a probability distribution. Those probabilities are naturally processed by algebraic tools, like the operations of sum and product, and this algebraic approach is reflected also in the formal properties of $H_p(A, p)$ and I_p.

On the other hand, the vagueness assumed and dealt by fuzzy information processing, is usually characterized by its monotonicity, and usual operations with fuzzy concepts are rather monotonous and essentially set theoretical (union, intersection, complement) represented by monotonous operators like minimum and maximum. In this sense, the "logarithmic" structures of information and entropy are not typical for fuzzy set theoretical paradigm.

4.2. The Logarithmic Scale of Information

The probabilistic information measure (3) is demanded to be additive – the associated probabilistic information is to be a sum

$$I_p(a, b) = I_p(a) + I_p(b), \quad a,\, b \in A,$$

if the symbols a, b are independent. On the other hand, the associated probability $p(a, b)$ of independent symbols is the product $p(a) \cdot p(b)$. Hence, the use of the logarithm in (3) is not only natural but also unavoidable.

In the contrary, the fuzzy information and consecutive concepts are to be rather monotonous than additive (cf. subsection 4.1), and also the processing of fuzzy sets and related notions is based on the monotonicity of the used operations. It means that the use of logarithmic function is possible and admissible but it is not necessary. For some marginal values of the membership functions in the fuzzy sources (A, μ) and (A^*, μ^*), namely for vanishing values $\mu(a)$ or $\mu^*(\boldsymbol{a})$, the logarithms would be treated with certain care.

4.3. Limited Regard to the Information of Individual Symbols

The analysis of the uncertainty existing in a random information source starts from the uncertainty of individual symbols – represented by the information measure I_p. The uncertainty of the entire information source (A, p) is defined as an aggregation operator (in that case the mean value) over the set of those individual uncertainties.

The models of fuzzy entropy suggested in the referred works, including [3] and [13], are not aimed to the characterization of uncertainty of emitted or transmitted data, but to the measurement of vagueness characterized by a fuzzy set. This procedure is correct and rational but it means some weakening of the link between the suggested definitions of entropy and the reality of vague data source.

In the following sections, alternative approaches to the fuzzy entropy are analyzed and discussed, with special regard to the three aspects mentioned in this section.

5. FUZZY INFORMATION MEASURE

The elementary treatment of fuzzy information defined for particular symbols from A or finite "words" from A^* was introduced in [16] and [17], already. Here, we analyze its basic properties with regard to the aggregation of such individual information values into a global characteristics of the entire information source.

The existing or potential definitions of the individual information measure on A or A^* would respect some general demands. Namely, if (A, μ) is an elementary fuzzy information source then the *fuzzy information* measure I_μ is a mapping of A into real numbers depending on the membership values $\mu(A)$, $a \in A$, and such that for $a,\, b \in A$

$$I_\mu(a) \geq 0, \quad I_\mu(a) = 0 \quad \text{iff} \quad \mu(a) = 1, \tag{7}$$

$$\text{if } \mu(a) \geq \mu(b) \quad \text{then} \quad I_\mu(a) \leq I_\mu(b). \tag{8}$$

The definition of fuzzy information can be easily extended on the "alphabet" A^*, where the membership values $\mu^*(\boldsymbol{a})$ for $\boldsymbol{a} \in A^*$ define the information values $I_\mu(\boldsymbol{a}^*)$. The following conditions are to be fulfilled for any $\boldsymbol{a},\, \boldsymbol{b} \in A^*$,

$$I_\mu(\boldsymbol{a}) \geq 0, \tag{9}$$

$$\begin{gathered} I_\mu(\boldsymbol{a}, \boldsymbol{b}) \geq \max\left(I_\mu(\boldsymbol{a}),\, I_\mu(\boldsymbol{b})\right), \\ \text{if } \mu^*(\boldsymbol{a}, \boldsymbol{b}) = \min(\mu^*(\boldsymbol{a}), \mu^*(\boldsymbol{b})) \text{ then } I_\mu(\boldsymbol{a}, \boldsymbol{b}) = \max(I_\mu(\boldsymbol{a}), I_\mu(\boldsymbol{b})). \end{gathered} \tag{10}$$

Remark 2. If the conditions (7), (8), (9), (10) are fulfilled then obviously

$$I_\mu(\boldsymbol{a}) \geq \max\left(I_\mu(a_1), \dots, I_\mu(a_n)\right)$$

for any $\boldsymbol{a} = (a_1, \dots, a_n) \in A^*$.

Remark 3. If the assumptions of the previous Remark 2 are fulfilled and if the fuzzy source (A^*, μ^*) is independent then

$$I_\mu(\boldsymbol{a}) = \max\left(I_\mu(a_1), \dots, I_\mu(a_n)\right).$$

Let us note, with regard to the Shannon's probabilistic theory, that the mutual relation between probabilistic independence and its eventual fuzzy set theoretical counterparts is discussed on a heuristic level since the seventies. The very special concept of independence used here is not in contradiction with the main ideas formulated during this discussion, however the structures of fuzziness and randomness are too different to search for direct analogies (see also [12]).

Lemma 1. If the fuzzy source (A^*, μ^*) is independent then for any $\boldsymbol{a} = (a_1, \dots, a_n) \in A^*$, $I_\mu(\boldsymbol{a}) = 0$ if and only if $\mu(a_i) = 1$ for all $i = 1, \dots, n$.

Proof. If for independent source (A^*, μ^*), $\mu(a_i) = 1$ for all $i = 1, \dots, n$, then Remark 3 and condition (7) imply that $I_\mu(\boldsymbol{a}) = 0$. Let, on the other hand, $\mu(a_j) < 1$ for some $j \in \{1, \dots, n\}$. Then, due to (7), $I_\mu(a_j) \neq 0$ and, by Remark 3, also $I_\mu(\boldsymbol{a}) \neq 0$. □

Let us verify the validity of the previous conditions (7), (8), (9), (10) for some significant specifications of the fuzzy information. In the following paragraphs, we consider an elementary fuzzy source (A, μ) and its extension, fuzzy source (A^*, μ^*).

5.1. Logarithmic Fuzzy Information

The first type of information measure characterizing particular symbols and segments of messages is the concealed background of the fuzzy entropies: $H_{\mathrm{LT}}(A^*, \mu^*)$ and $H_{\mathrm{KV}}(A^*, \mu^*)$, namely the mapping $I_\mu^L : A \to R^+$ defined by

$$I_\mu^L(a) = -\log_2 \mu_a \quad \text{for } a \in A, \tag{11}$$

called the *logarithmic fuzzy information*. Let us note that the binary type of the logarithmic function in (11) is an arbitrary choice respecting rather the information theoretical habits. The essential properties of $I_\mu^L(a)$ keep preserved for any other base of logarithm.

Lemma 2. The fuzzy information I_μ fulfils (7) and (8).

Proof. The statement is an elementary consequence of the properties of logarithms. □

It is easy to extend I_μ from the fuzzy source (A, μ) on its extension (A^*, μ^*) by means of

$$I_\mu(\boldsymbol{a}) = -\log_2 \mu^*(\boldsymbol{a}) \quad \text{for } \boldsymbol{a} \in A^*. \tag{12}$$

Lemma 3. The extension (12) of the logarithmic fuzzy information I_μ^L over (A^*, μ^*) fulfils for any $\boldsymbol{a} = (a_1, \dots, a_n) \in A^*$ the inequality

$$I_\mu^L(\boldsymbol{a}) \geq \max\left(I_\mu^L(a_1), \dots, I_\mu^L(a_n)\right).$$

If the fuzzy source is independent then

$$I_\mu^L(\boldsymbol{a}) = \max\left(I_\mu^L(a_1), \dots, I_\mu^L(a_n)\right).$$

Proof. The statement follows from Remark 2, Remark 3 and (12). □

Lemma 4. The extended logarithmic fuzzy information over (A^*, μ^*) defined by (12) fulfils conditions (9) and (10).

P r o o f. Definition (12) immediately implies the positivity of $I^L_\mu(\boldsymbol{a})$ for any $\boldsymbol{a} \in A^*$. Let us consider $\boldsymbol{a}$, $\boldsymbol{b} \in A^*$. Then, due to (2),

$$\mu^*(\boldsymbol{a}, \boldsymbol{b}) \leq \min\left(\mu^*(\boldsymbol{a}),\, \mu^*(\boldsymbol{b})\right),$$

and, by (12), condition (10) is fulfilled. □

5.2. Monotonous Fuzzy Information

The information measure suggested in this subsection more reflects the paradigm of monotonicity of the operations based on fuzzy set theoretical concepts which we intend to respect in the following parts of this paper.

If (A, μ) is an elementary fuzzy source, we define a mapping $I^M_\mu : A \to R$ such that for any $a \in A$

$$I^M_\mu(a) = 1 - \mu(a), \tag{13}$$

which is called *monotonous fuzzy information*. The following statement follows from (13), immediately.

Lemma 5. The monotonous fuzzy information I^M_μ fulfils conditions (7) and (8).

It is also easy to extend I^M_μ on the information source (A^*, μ^*), and by means of

$$I^M_\mu(\boldsymbol{a}) = 1 - \mu^*(\boldsymbol{a}), \quad \text{for } \boldsymbol{a}^* \in A^*. \tag{14}$$

The following statement means that the extension is formally correct.

Lemma 6. The extension (14) of I^M_μ on (A^*, μ^*) fulfils conditions (9) and (10).

P r o o f. The statement is an immediate consequence of (14), (2) and of the properties of membership functions. □

Lemma 7. The extension (14) of I^M_μ fulfils the following inequality for any $\boldsymbol{a} = (a_1, \ldots, a_n) \in A^*$

$$I^M_\mu(\boldsymbol{a}) \geq \max\left(I^M_\mu(a_1), \ldots, I^M_\mu(a_n)\right).$$

If, moreover, the fuzzy source (A^*, μ^*) is independent then the above inequality turns into equality.

P r o o f. Analogously to Lemma 3, also this statement follows from Remark 2 and Remark 3 applied on (14). □

The previous statements show that the monotonous approach to the concept of fuzzy information offers a formal structure which is comparable with the one following from the traditional approach and that its formal processing can be easier.

In the next chapter, we analyze the influence of both conceptions on the notion of fuzzy informational entropy.

6. AGGREGATED FUZZY INFORMATION MEASURES

In this section, we compare both approaches, the logarithmic and the monotonous ones, from the point of view of the construction of some aggregated measure of information regarding the entire fuzzy information source. As the term "entropy" is traditionally closely connected with the logarithmic approach to the uncertainty of information source (probabilistic, as well as fuzzy), we prefer here the terms "aggregated information" and "aggregated fuzzy information" generalizing various models of the entropy.

The notion of aggregated fuzzy information is quite wide, which implies the need to specify its general boundaries. We do it in the first subsection, meanwhile the following two subsections of this section are devoted to the analysis and differences between the known logarithmic model, newly suggested monotonous models, and the general demands. Let us note, on the extremally heuristic level, that each aggregated fuzzy information would be a function of the individual information values for particular symbols, and that it would be the larger the less variable (or structured) these individual information values are.

6.1. Postulates of Aggregated Fuzzy Information

The rough heuristic idea formulated in the previous paragraphs is to be specified in several formal postulates. Some attempts to formulate them were undertaken in a few papers dealing with that topic (let us mention, e. g., [2, 4, 6, 13, 16, 19, 20] and others). However these attempts cover the desired properties of the aggregation fuzzy information measures, they often differ from each other in significant details. Let us summarize, here, the expected properties of the aggregated fuzzy information (in other words, of the fuzzy entropy) in a system of five postulates. The author takes for his honour to express here his thanks to prof. Radko Mesiar from Bratislava for informal inspirational discussion regarding this topic. Very probably, some of the models of fuzzy entropy (already suggested or potentially expectable) will not fulfil all the postulates below. Nevertheless, even a limited set of properties characterizes the actual entropy concepts.

Let us consider an elementary information source (A, μ) and its extension (A^*, μ^*) (see Section 2). Let I_μ be the fuzzy information on (A, μ) fulfilling (7), (8), which can be extended on (A^*, μ^*) where (1), (2), as well as (9), (10) are fulfilled.

Let us consider a mapping H connecting the elementary fuzzy source (A, μ) with a real number $H(A, \mu)$, where the following postulates are fulfilled

(H1) $H(A, \mu) = 0$ if for any $a \in A$ $\mu(a) \in \{0, 1\}$.

(H2) Let A, B be finite alphabets and (A, μ), (B, ν) be fuzzy sources such that B is a permutation of A and the set $\{\nu(b) : b \in B\}$ is a permutation of $\{\mu(a) : a \in A\}$. Then $H(A, \mu) = H(B, \nu)$.

(H3) If (A, μ), (A, μ') are elementary fuzzy sources and for all $a \in A$

$$\mu(a) \le \mu'(a) \quad \text{if} \quad \mu'(a) \le 1/2,$$
$$\mu(a) \ge \mu'(a) \quad \text{if} \quad \mu'(a) \ge 1/2,$$

then $H(A,\mu) \le H(A,\mu')$.

(H4) If (A,μ), (A,μ') are such that for all $a \in A$, $\mu(a) = 1 - \mu'(a)$ then $H(A,\mu) = H(A,\mu')$.

(H5) $H(A,\mu) \ge H(A,\mu')$ for all (A,μ') if $\mu(a) = 1/2$ for all $a \in A$.

Moreover, in some of the following subsections we consider the following postulate, too.

(H6) The mapping H can be extended on the fuzzy source (A^*,μ^*) in a way consistent with (H1),...,(H5), where the consistency means that all postulates are fulfilled for the segments $\boldsymbol{a} \in A^*$ and values $\mu^*(\boldsymbol{a})$ for which $\boldsymbol{a} = (a)$ is a one-symbol segment.

The previous axioms deserve a few comments. As mentioned in Section 2, the information source (A,μ) is another term from a fuzzy subset of A and, consequently, any measure of uncertainty connected with (A,μ), measures the degree of its vagueness, i. e., fuzziness. This interpretation of the entropy concept, is especially well cognizable in axioms (H3) and (H5). Namely, the doubt if a symbol $x \in A$ belongs to the fuzzy set (A,μ) is the more intensive the nearer the membership value $\mu(x)$ is to 1/2. Similarly, the more do the values of $\mu(x)$ being near to 1 or to 0 prevail, the "sharper" the "contures" of (A,μ) are and the more degligible the rate of vagueness in its shape is. The interpretation of the fuzzy entropy as a measure of vagueness is easily cognizible in other axioms, as well.

The mapping H introduced above is called the *aggregated fuzzy information* of fuzzy sources.

Theorem 1. If the property (H3) is fulfilled then also (H5) is true.

Proof. Let us consider two elementary fuzzy sources (A,μ) and (A,μ'), and let us suppose that

$$\mu(a) = 1/2 \quad \text{for all } a \in A,$$

meanwhile $\mu' : A \to [0,1]$ is a general membership function. Then, $\mu'(a) \le 1/2$ implies $\mu(a) \ge \mu'(a)$, and $\mu'(a) \ge 1/2$ implies $\mu(a) \le \mu'(a)$. Hence, the assumptions of (H3) are fulfilled and $H(A,\mu) \ge H(A,\mu')$. □

Remark 4. If the elementary fuzzy source (A,μ) is independent and such that $\mu(a) \in \{0,1\}$ for all $a \in A$ then also its extension (A^*,μ^*) fulfils $\mu^*(\boldsymbol{a}) \in \{0,1\}$ for all $\boldsymbol{a} \in A^*$.

Remark 5. Let (A,μ) and (A,ν) be elementary independent fuzzy sources and (A^*,μ^*), (A^*,ν^*) their extensions fulfilling (1), (2). Then

$$\mu^*(\boldsymbol{a}) \le \nu^*(\boldsymbol{a}) \quad \text{if} \quad \nu^*(\boldsymbol{a}) \le 1/2,$$
$$\mu^*(\boldsymbol{a}) \ge \nu^*(\boldsymbol{a}) \quad \text{if} \quad \nu^*(\boldsymbol{a}) \ge 1/2.$$

6.2. Logarithmic Model

The definitions of aggregated fuzzy information treated in this subsection were already presented in the literature (namely in [4] and [13]), and they were briefly recollected in Section 4, formulas (5) and (6). We call them in this paper De Luca–Termini and Kolesárová–Vivona fuzzy entropies. Both of them are based on the aggregation of logarithmic fuzzy information measure I^L_μ defined by (11). Its general properties are described in subsection 5.1. Among others, it is shown there that I^L_μ fulfils (7) and (8), and that it can be extended on the fuzzy source (A^*, μ^*).

The extension of the fuzzy information from the elementary fuzzy source (A, μ) on (A^*, μ^*) starts at the extension of μ on μ^* fulfilling (1), (2), as well as the extensions of I_μ from (A, μ) on (A^*, μ^*) is limited by (9), (10), only. In this subsection, dealing with specific logarithmic approach to the information measuring, we define μ^* and I^L_μ in more specialized form. Namely, we put

$$\mu^*(\boldsymbol{a}) = \mu(a_1) \cdot \ldots \cdot \mu(a_n), \quad \boldsymbol{a} = (a_1, \ldots, a_n) \in A^*. \tag{15}$$

Remark 6. Obviously, the membership function $\mu^* : A^* \to [0, 1]$ defined by (15) fulfils conditions (1) and (2).

Remark 7. If the fuzzy information source (A^*, μ^*) with μ^* is defined by (15), and if (A, μ) is crisp, i. e. $\mu : A \to \{0, 1\}$, then (A^*, μ^*) is independent.

The approach to the logarithmic processing of information represented by (15) and by consequent concepts, appears acceptable if we take into account the fact that the fuzzy entropy model suggested in [4] and, in a more developed form, in [13] emulates the probabilistic processing of uncertainty and information formulated in [22] quite consequently (cf. Section 4). Definition (15) respects this close analogy of the mentioned approaches.

The logarithmic fuzzy information measure I^L_μ is defined by (11) and (12), and it is easy to prove that even if μ^* is defined by (15), its basic properties (7), (8), (9), (10) are fulfilled (cf., Lemmas 3, 4, 5).

Remark 8. If (A^*, μ^*) is defined by (15) and I^L_μ by (12) then for any $\boldsymbol{a} = (a_1, \ldots, a_n) \in A^*$

$$I^L_\mu(\boldsymbol{a}) = I^L_\mu(a_1) + \cdots + I^L_\mu(a_n).$$

6.2.a De Luca–Termini Fuzzy Entropy

The model presented in [4] deals with fuzzy entropy (5), where we simplify the formalism by putting $K = 1$,

$$H_{\mathrm{LT}}(A, \mu) = -\sum\nolimits_{a \in A} \mu(a) \log_2 \mu(a) = \sum\nolimits_{a \in A} \mu(a) \cdot I^L_\mu(a),$$

eventually, with its extension,

$$H_{\mathrm{LT}}(A^*, \mu^*) = -\sum\nolimits_{\boldsymbol{a} \in A^*} \mu^*(\boldsymbol{a}) \cdot \log_2 \mu^*(\boldsymbol{a}) = \sum\nolimits_{\boldsymbol{a} \in A^*} \mu^*(\boldsymbol{a}) \cdot I^L_\mu(\boldsymbol{a}). \tag{16}$$

Lemma 8. Let us denote by $H^n_{\mathrm{LT}}(A^n,\mu^*)$ the entropy over the alphabet A^n and relevant uncertainty values of $\mu^*(\boldsymbol{a})$ for $\boldsymbol{a}\in A^n$. Let $\beta\in[0,1]$ be such that for all $a\in A$, $\mu(a)\le\beta$. Finally, let h be the number of symbols in the alphabet A. Then

$$H^n_{\mathrm{LT}}(A^n,\mu^*)\le\beta^n\cdot h\cdot n\cdot H_{\mathrm{LT}}(A,\mu).$$

Proof. Due to (15), Remark 7 and (16), denoting $\boldsymbol{a}=(a_1,\dots,a_n)$ for any $\boldsymbol{a}\in A^n$,

$$\begin{aligned}
&H^n_{\mathrm{LT}}(A^n,\mu^*)=\sum\nolimits_{\boldsymbol{a}\in A^n}\mu^*(\boldsymbol{a})\cdot\left(I^L_\mu(a_1)+\dots+I^L_\mu(a_n)\right)\\
=\;&\sum\nolimits_{\boldsymbol{a}\in A^n}(\mu(a_1)\cdot\ldots\cdot\mu(a_n))\cdot I^L_\mu(a_1)+\dots+\sum_{\boldsymbol{a}\in A^n}(\mu(a_1)\cdot\ldots\cdot\mu(a_n))\cdot I^L_\mu(a_n)\\
\le\;&\sum\nolimits_{\boldsymbol{a}\in A^n}\left(\mu(a_1)\cdot\beta^{n-1}\cdot I^L_\mu(a_1)+\dots+\mu(a_n)\,\beta^{n-1}\cdot I^L_\mu(a_n)\right)\\
=\;&\beta^{n-1}\left(\sum\nolimits_{a\in A}\mu(a)\cdot H_{\mathrm{LT}}(A,\mu)\right)\cdot n\le\beta^n\cdot n\cdot h\cdot H_{\mathrm{LT}}(A,\mu).
\end{aligned}$$

□

Corollary. If for all $a\in A$, $\mu(a)=\alpha\in[0,1]$ then $H^n_{\mathrm{LT}}(A^n,\mu^*)\le\alpha^n\cdot n\cdot h\cdot H_{\mathrm{LT}}(A,\mu)$.

The properties of the De Luca–Termini fuzzy entropy were investigated in [4]. Nevertheless, let us verify its correspondence with conditions (H1), (H2), (H3), (H4) and (H5).

Lemma 9. If for any $a\in A$, $\mu(a)\in\{0,1\}$ then $H_{\mathrm{LT}}(A,\mu)$ and $H_{\mathrm{LT}}(A^*,\mu^*)$ vanish.

Proof. If for any $a\in A$, $\mu(a)=0$ or $\mu(a)=1$, then the product $\mu(a)\cdot\log_2\mu(a)=0$, and $H_{\mathrm{LT}}(A,\mu)=0$. Moreover, for any $\boldsymbol{a}=(a_1,\dots,a_n)\in A^*$, either $\mu^*(\boldsymbol{a})=0$, if at least one $\mu(a_i)=0$, $i\in\{1,\dots,n\}$, or $\mu^*(\boldsymbol{a})=1$ if $\mu(a_i)=1$ for all $i=1,\dots,n$. Hence, $\mu^*(\boldsymbol{a})\cdot\log_2\mu^*(\boldsymbol{a})=0$ for any $\boldsymbol{a}\in A^*$ and $H_{\mathrm{LT}}(A^*,\mu^*)=0$, (see (15)). □

Remark 9. The algebraic structure of (5) and (16) immediately implies that the value $H_{\mathrm{LT}}(A^*,\mu^*)$, as well as $H_{\mathrm{LT}}(A,\mu)$ is invariant regarding permutation of A.

Remark 10. Due to [4], the fuzzy entropy $H_{\mathrm{LT}}(A,\mu)$ fulfils conditions (H3) and (H5). The proof of these statements can be used to prove the validity of (H3) and (H5) for $H_{\mathrm{LT}}(A^*,\mu^*)$, as well.

Example 1. Let us consider a binary alphabet $A=\{a,b\}$ and elementary fuzzy sources (A,μ_1), (A,μ_2), (A,μ_3) such that

$$\mu_1(a)=\mu_1(b)=1/2,\quad \mu_2(a)=\mu_2(b)=1/4,\quad \mu_3(a)=\mu_3(b)=3/4.$$

It is easy to compute the logarithmic fuzzy entropies

$$H_{\mathrm{LT}}(A,\mu_1)=1,\quad H_{\mathrm{LT}}(A,\mu_2)=1,\quad H_{\mathrm{LT}}(A,\mu_3)\doteq 0,6226.$$

Remark 11. The foregoing example shows that (H4) is not fulfilled for the elementary fuzzy source. As one-element segments of messages are special case of elements of A^* and $\mu^*(a) = \mu(a)$ for any $a \in A$, (H4) is not generally fulfilled for $H_{\mathrm{LT}}(A^*, \mu^*)$.

Remark 12. Example 1 illustrates also the fact that condition (H5) if fulfilled as an implication but not as logical equivalence.

Theorem 2. The De Luca–Termini fuzzy entropy fulfils conditions (H1), (H2), (H3), (H5) but it does not generally fulfil (H4).

P r o o f. The statement is a summary of the above remarks. □

6.2.b Kolesárová–Vivona Fuzzy Entropy

The fuzzy entropy model suggested in [13] preserves the advantages of the previous De Luca–Termini aggregated fuzzy information (fuzzy entropy) and overcomes its discrepancy consisting in the invalidity of (H4).

It is defined by (6), and when putting $K = 1$ for the simplification of the formalism, we get

$$H_{\mathrm{KV}}(A, \mu) = \sum\nolimits_{a \in A} \left((\mu(a) \cdot \log_2 \mu(a)) + (1 - \mu(a)) \cdot \log(1 - \mu(a)) \right).$$

For the extended fuzzy source, we put

$$H_{\mathrm{KV}}(A^*, \mu^*) = \sum_{\boldsymbol{a} \in A^*} \left((\mu^*(\boldsymbol{a}) \cdot \log_2 \mu^*(\boldsymbol{a})) + (1 - \mu^*(\boldsymbol{a})) \cdot \log_2(1 - \mu^*(\boldsymbol{a})) \right). \quad (17)$$

The properties of this type of logarithmic fuzzy entropy are investigated in [13]. In the context of this paper, we briefly recollect the validity of (H1),. . . ,(H5).

Theorem 3. The Kolesárová–Vivona fuzzy entropies $H_{\mathrm{KV}}(A\mu)$ and $H_{\mathrm{KV}}(A^*, \mu^*)$ defined by (6) and (17) fulfil all conditions (H1), (H2), (H3), (H4), (H5).

P r o o f. Formulas (6) and (17) imply that

$$\begin{aligned} H_{\mathrm{KV}}(A, \mu) &= H_{\mathrm{LT}}(A, \mu) + H_{\mathrm{LT}}(A, 1 - \mu), && (18) \\ H_{\mathrm{KV}}(A^*, \mu^*) &= H_{\mathrm{LT}}(A^*, \mu^*) + H_{\mathrm{LT}}(A^*, 1 - \mu^*) && (19) \end{aligned}$$

where for any $a \in A$, $\boldsymbol{a} \in A^*$, $(1 - \mu)(a) = 1 - \mu(a)$ and

$$(1 - \mu^*)(\boldsymbol{a}) = 1 - \mu^*(\boldsymbol{a}).$$

The fuzzy entropies H_{LT} referred in (18) and (19) fulfil the statement of Theorem 2. Hence, (H1), (H2), (H3) are fulfilled obviously. The validity of (H4), which is not fulfilled for H_{LT}, follows from the symmetry of (17) regarding the positions of $\mu^*(\boldsymbol{a})$ and $1 - \mu^*(\boldsymbol{a})$. Finally, (H5) follows from Theorem 2, too as for $\mu^*(\boldsymbol{a}) = 1/2$ also $1 - \mu^*(\boldsymbol{a}) = 1/2$ and $H_{\mathrm{KV}}(A^*, \mu^*)$ is the sum of two maximal values. □

6.3. Monotonous Model

Following the heuristic discussion presented in Section 4, we suggest, here, the concept of aggregated fuzzy information measure respecting the methodological principles more adequate to the environment of the fuzzy set theory. Namely, we pay special attention to the application of monotonous, set theoretical operations like complement, union and intersection, and we accept the presumption that such monotonous operations do not demand the simplifying properties of logarithms.

We complete the rather general assumptions (1) and (2) putted on the extension from elementary fuzzy source to (A^*, μ^*) by a more special one, namely we suppose that (A^*, μ^*) is to be independent. It means

$$\mu^*(\boldsymbol{a}) = \min\left(\mu(a_1), \ldots, \mu(a_n)\right) \quad \text{for any } \boldsymbol{a} = (a_1, \ldots, a_n) \in A^*. \tag{20}$$

We use the monotonous fuzzy information $I_\mu^M : A \to R$ defined by (13)

$$I_\mu^M(a) = 1 - \mu(a)$$

and its extension on (A^*, μ^*)

$$I_\mu^M(\boldsymbol{a}) = 1 - \mu^*(\boldsymbol{a}), \quad \text{for } \boldsymbol{a} \in A^*. \tag{21}$$

Remark 13. Using Lemma 7, Remark 4, (21) and (20) it is easy to verify that for $\boldsymbol{a} = (a_1, \ldots, a_n) \in A^*$, $\boldsymbol{b}$, $\boldsymbol{c} \in A^*$,

$$\begin{aligned}
&I_\mu^M(\boldsymbol{a}) = \max\left(I_\mu^M(a_1), \ldots, I_\mu^M(a_n)\right),\\
&I_\mu^M(\boldsymbol{a}) \in [0, 1],\\
&I_\mu^M(\boldsymbol{a}) = 0 \text{ iff } \mu^*(\boldsymbol{a}) = 1 \text{ iff } \mu(a_i) = 1 \text{ for all } i = 1, 2, \ldots, n,\\
&I_\mu^M(\boldsymbol{a}) = 1 \text{ iff } \mu^*(\boldsymbol{a}) = 0 \text{ iff } \mu(a_i) = 0 \text{ for at least one } i \in \{1, \ldots, n\},\\
&\text{If } \boldsymbol{c} = (\boldsymbol{a}, \boldsymbol{b}) \text{ then } I_\mu^M(\boldsymbol{c}) = \max\left(I_\mu^M(\boldsymbol{a}), I_\mu^M(\boldsymbol{b})\right).
\end{aligned}$$

In the rest of this subsection we suggest three definitions of monotonous aggregated information measures and verify their compatibility with the general demands of (H1),...,(H5).

6.3.a Strictly Monotonous Fuzzy Entropy

If (A, μ) is an elementary fuzzy source and (A^*, μ^*) its extension, and if the above assumptions on μ^* and I_μ^M are fulfilled, then we define the mapping $H_{\text{SM}}(A, \mu)$ and its extension on (A^*, μ^*) by

$$H_{\text{SM}}(A, \mu) = 2 \cdot \max\left(\min(\mu(a),\, 1 - \mu(a)) : a \in A\right), \tag{22}$$

$$H_{\text{SM}}(A^*, \mu^*) = 2 \cdot \max\left(\min(\mu^*(\boldsymbol{a}),\, 1 - \mu^*(\boldsymbol{a})) : \boldsymbol{a} \in A^*\right), \tag{23}$$

and call them *strictly monotonous fuzzy entropy.*

Remark 14. Equalities (21), (22) and (23) immediately imply that

$$H_{\mathrm{SM}}(A^*, \mu^*) = 2 \cdot \max\left(\min(I^M_\mu(\boldsymbol{a}),\, I^M_{1-\mu}(\boldsymbol{a})) : \boldsymbol{a} \in A^*\right)$$

where $I^M_{1-\mu}(\boldsymbol{a}) = 1 - I^M_\mu(\boldsymbol{a}) = \mu^*(\boldsymbol{a})$.

The validity of the fundamental properties of aggregated fuzzy information measure H_{SM} is formulated in the following statement.

Theorem 4. The strictly monotonous fuzzy entropy fulfils conditions (H1), (H2), (H3), (H4) and (H5).

Proof. Due to Remark 13 and Remark 14, if $\mu(a) = 0$ then $I^M_\mu(a) = 1$ and $I^M_{1-\mu}(a) = 0$. If $\mu(a) = 1$ then $I^M_\mu(a) = 0$. In any case, $H_{\mathrm{SM}}(A, \mu) = 0$ and also $H_{\mathrm{SM}}(A^*, \mu^*) = 0$. Hence (H1) is fulfilled. The validity of (H2) follows from the commutativity of the monotonous operations in (22) and (23).

If the assumptions of (H3) are fulfilled then formula (22) and Remark 14 immediately imply its validity. The symmetry of (22) and (23), implies (H4) and also (H5) follows from (23), where the maximal value of $H_{\mathrm{SM}}(A^*, \mu^*)$ achieved for (A^*, μ^*) such that $\mu(a) = 1/2$ for any $a \in A$, is equal to 1. □

6.3.b Weakly Monotonous Fuzzy Entropy

The aggregated fuzzy information measure suggested in this part represents certain compromise between the logarithmic and monotonous approach. Namely, the monotonous fuzzy information I^M_μ is appointed into the basic scheme of the De Luca–Termini fuzzy entropy. More precisely, if (A, μ) is an elementary fuzzy information source and (A^*, μ^*) is its extension then the value

$$H_{\mathrm{WM}}(A, \mu) = \sum\nolimits_{a \in A} \mu(a) \cdot I^M_\mu(a) = \sum\nolimits_{a \in A} \mu(a) \cdot (1 - \mu(a)) \qquad (24)$$

is called *weakly monotonous fuzzy entropy* and its extension on (A^*, μ^*) is obviously defined by

$$H_{\mathrm{WM}}(A^*, \mu^*) = \sum\nolimits_{\boldsymbol{a} \in A^*} \mu^*(\boldsymbol{a}) \cdot I^M_\mu(\boldsymbol{a}) = \sum\nolimits_{\boldsymbol{a} \in A^*} \mu^*(\boldsymbol{a}) \cdot (1 - \mu^*(\boldsymbol{a})). \qquad (25)$$

The adequacy of this aggregated fuzzy information measure to the general scheme of fuzzy entropy can be formulated as the following statement.

Theorem 5. If $H_{\mathrm{WM}}(A, \mu)$ and $H_{\mathrm{WM}}(A^*, \mu^*)$ represent the weakly monotonous fuzzy entropy then the conditions (H1), (H2), (H3), (H4), (H5) are fulfilled.

Proof. If $\mu^*(\boldsymbol{a}) \in \{0, 1\}$ then the product $\mu^*(\boldsymbol{a}) \cdot (1 - \mu^*(\boldsymbol{a}))$ in (25) vanishes. It is true for all $\boldsymbol{a} \in A^*$ (or all $a \in A$), condition (H1) is fulfilled. (H2) follows from (25), immediately. The course of the function $x(1 - x)$ for $x \in [0, 1]$ immediately implies (H3), as well as (H4) and (H5). □

6.3.c Extremaly Monotonous Fuzzy Entropy

The last example of aggregated fuzzy information measure, presented here, deals with the extreme values of individual information measure I_μ^M. In the contrary to the strongly monotonous fuzzy entropy, this one pays more attention to both extremes of the information measure. In the more exact formulation, we consider an elementary fuzzy information source (A, μ) and its extension (A^*, μ^*) and define the number

$$H_{\mathrm{EM}}(A, \mu) = 1 - \left(\max(I_\mu^M(a) : a \in A) - \min(I_\mu^M(a) : a \in A)\right), \tag{26}$$

eventually

$$H_{\mathrm{EM}}(A^*, \mu^*) = 1 - \left(\max(I_\mu^M(\boldsymbol{a}) : \boldsymbol{a} \in A^*) - \min(I_\mu^M(\boldsymbol{a}) : \boldsymbol{a} \in A^*)\right), \tag{27}$$

which we call *extremaly monotonous fuzzy entropy.* Its conformity with the general demands on fuzzy entropies formulated in Section 6.1 is analyzed in the following statements.

Theorem 6. The extremaly monotonous fuzzy entropy $H_{\mathrm{EM}}(A^*, \mu^*)$ eventually $H_{\mathrm{EM}}(A, \mu)$ fulfils the condition (H2), (H4), (H5).

Proof. The condition (H2) follows from the commutativity of operations in (27). If we substitute $\mu^*(\boldsymbol{a})$ by $1 - \mu^*(\boldsymbol{a})$ then we only replace the $\max(I_\mu^M(\boldsymbol{a}) : \boldsymbol{a} \in A^*)$ by $1 - \min(I_\mu^M(\boldsymbol{a}) : \boldsymbol{a} \in A^*)$ and vice versa but the difference in (27) keeps unchanged. That proves the validity of (H4). Evidently, the membership function $\mu(\boldsymbol{a}) = 1/2$ minimizes (vanishes) the difference in brackets in (27) and, consequently, it maximizes the value $H_{\mathrm{EM}}(A^*, \mu^*) = 1$. Hence, (H5) is fulfilled. □

Remark 15. The proof of the previous theorem obviously implies that condition (H5) is fulfilled not only as an implication but also as a logical equivalence. Moreover, condition $\mu^*(\boldsymbol{a}) = 1/2$ for all $\boldsymbol{a} \in A^*$ is fulfilled if and only if $\mu(a) = 1/2$ for all $\boldsymbol{a} \in A$.

The remaining two conditions (H1) and (H3) are fulfilled in rather modified, weakened, form.

Theorem 7. The extremaly monotonous fuzzy entropy $H_{\mathrm{EM}}(A^*, \mu^*) = 0$ if there exists at least one $c \in A$ such that $\mu(c) = 0$ and at least one $b \in A$ such that $\mu(b) = 1$.

Proof. Under the above assumptions, there exist one-symbol segments $\boldsymbol{b} = (b) \in A^*$, and $\boldsymbol{c} = (c) \in A^*$ such that $\mu^*(\boldsymbol{b}) = 1$, $\mu^*(\boldsymbol{c}) = 0$ and, consequently, the difference between both extremes is equal to 1. Hence $H_{\mathrm{EM}}(A^*, \mu^*) = 0$. □

Regarding (H3), the following weakened form is true.

Theorem 8. If (A^*, μ^*), (A^*, ν^*) are two fuzzy sources such that

$$\begin{array}{rcl} \max\left(\mu^*(\boldsymbol{a}) : \boldsymbol{a} \in A^*\right) & \geq & \max\left(\nu^*(\boldsymbol{a}) : \boldsymbol{a} \in A\right) \\ \min\left(\mu^*(\boldsymbol{a}) : \boldsymbol{a} \in A^*\right) & \leq & \min\left(\nu^*(\boldsymbol{a}) : \boldsymbol{a} \in A\right) \end{array}$$

then

$$H_{\mathrm{EM}}(A^*, \mu^*) \leq H_{\mathrm{EM}}(A^*, \nu^*).$$

Proof. If we denote by I_μ^M and I_ν^M the respective monotonous fuzzy information measures for particular symbols and segments, then the first of the assumed inequalities means

$$\begin{array}{rl} & \max\left(I_\mu^M(\boldsymbol{a}) : \boldsymbol{a} \in A^*\right) = 1 - \min\left(\mu^*(\boldsymbol{a}) : \boldsymbol{a} \in A^*\right) \\ \geq & 1 - \min\left(\nu^*(\boldsymbol{a}) : \boldsymbol{a} \in A^*\right) = \max\left(I_\nu^M(\boldsymbol{a}) : \boldsymbol{a} \in A^*\right), \end{array}$$

and similarly

$$\min\left(I_\mu^M(\boldsymbol{a}) : \boldsymbol{a} \in A^*\right) \leq \min\left(I_\nu^M(\boldsymbol{a}) : \boldsymbol{a} \in A^*\right).$$

Hence, due to (27), the desired inequality is true. □

7. CONCLUSIVE REMARKS

The main goal of this paper was to suggest some alternative approaches to the fuzzy entropy concept, which could be more adequate to the fuzzy set theoretical character of the vague information produced by the fuzzy information sources.

Three such alternative aggregated measures of uncertainty regarding the entire information source were suggested. They were compared with former definitions of similar information measures known in the literature and inspired by the probabilistic pattern presented in [22]. The first simple comparison of the classical and new concepts of fuzzy entropy (more generally of aggregated information measure) appears to characterize the suggested fuzzy entropies as adequate to the structure of fuzzy information and fuzzy knowledge. Moreover, their formal processing by means of the fuzzy set theoretical tools appears to be easier than in the former logarithmic model.

The fuzzification of the information theoretical concepts presented here is not the single possible one. Nevertheless, the methods based on the application of fuzzy quantities theory (see, e. g., [14, 15, 25]) or its generalizations (e. g., [3,10]), represent deeper changes of the basic model, and they have to be postponed to further research. That regards also the application of modern advanced measure theory (see [12] as the top representative) and its analysis of the difference between additive and monotonous measures.

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Milan Mareš, Institute of Information Theory and Automation – Academy of Sciences of the Czech Republic, Pod Vodárenskou věží 4, 182 08 Praha 8, and Faculty of Sciences, Institute of Applied Information Sciences, South Bohemian University, Branišovská 31, 370 05 České Budějovice. Czech Republic.

e-mail: mares@utia.cas.cz