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UNDERLYING LIE ALGEBRAS OF QUADRATIC NOVIKOV ALGEBRAS

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Abstract. Novikov algebras were introduced in connection with the Poisson brackets of hydrodynamic type and the Hamiltonian operators in formal variational calculus. In this note we prove that the underlying Lie algebras of quadratic Novikov algebras are 2-step nilpotent. Moreover, we give the classification up to dimension 10.

Keywords: Novikov algebra, quadratic Novikov algebra, underlying Lie algebra

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1. Introduction

A Novikov algebra A is a vector space over a field $\mathbb{F}$ with a bilinear product $(x, y) \mapsto xy$ for any $x, y, z \in A$ satisfying

$$(xy)z - x(yz) = (yx)z - y(xz), \tag{1.1}$$

$$(xy)z = (xz)y. \tag{1.2}$$

Novikov algebras were introduced in connection with the Poisson brackets of hydrodynamic type [1], [4], [5] and the Hamiltonian operators in the formal variational calculus [7], [8], [9], [10], [11], [12].

The commutator

$$[x, y] = xy - yx \quad \text{for any } x, y \in A \tag{1.3}$$

makes any Novikov algebra A a Lie algebra denoted $\mathrm{Lie}(A)$ in what follows. It is called the underlying Lie algebra of A. A bilinear form $f\colon A \times A \to \mathbb{F}$ is associative if and only if

$$f(xy, z) = f(x, yz) \quad \text{for any } x, y, z \in A. \tag{1.4}$$

The pair (A, f) is called a quadratic Novikov algebra if f is a non-degenerate associative symmetric bilinear form on a Novikov algebra A.

It is proved in [13] that $(\mathrm{Lie}(A), f)$ is a quadratic Lie algebra and $\mathrm{Lie}(A)$ of dimension up to 4 is abelian. In this note we show that $\mathrm{Lie}(A)$ is 2-step nilpotent. Then we find that $\mathrm{Lie}(A)$ with an isotropic center plays an important role in the classification. Based on some lemmas, we give the classification up to dimension 10.

Throughout this paper we assume that the algebras are finite dimensional over $\mathbb{C}$. Obvious proofs are omitted.

2. Preliminary

Let (A, f) be a quadratic Novikov algebra with a basis $\{e_1, e_2, \ldots, e_n\}$. Then the bilinear form f is completely determined by the matrix $F = (f_{ij})$, where

$$f_{ij} = f(e_i, e_j). \tag{2.1}$$

Lemma 2.1 ([2]). *Quadratic Novikov algebras are associative.*

The ideal N of A is called isotropic if $f(x, y) = 0$ for any $x, y \in N$ and non-degenerate if $f|_{N\times N}$ is non-degenerate. Define $N^\perp = \{x \in A\colon f(x, y) = 0$ for any $y \in N\}$ and $Z(A) = \{x \in A\colon xy = yx = 0$ for any $y \in A\}$.

Lemma 2.2 ([13]). *In the above notation,* $Z(A) = (AA)^\perp$.

Let $\mathfrak{g}$ be a Lie algebra. Then the pair $(\mathfrak{g}, f)$ is called a quadratic Lie algebra if f is a non-degenerate symmetric bilinear form on $\mathfrak{g}$ satisfying

$$f(x, [y, z]) = f([x, y], z) \quad \text{for any } x, y, z \in \mathfrak{g}.$$

The ideal H of $\mathfrak{g}$ is called isotropic if $f(x, y) = 0$ for any $x, y \in H$ and non-degenerate if $f|_{H\times H}$ is non-degenerate. Define $H^\perp = \{x \in A\colon f(x, y) = 0$ for any $y \in H\}$ and $C(\mathfrak{g}) = \{x \in \mathfrak{g}\colon [x, y] = 0$ for any $y \in \mathfrak{g}\}$.

Lemma 2.3 ([6]). *Let $(\mathfrak{g}, f)$ be a quadratic Lie algebra.*

(1) $C(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]^\perp$.

(2) *Let H be an ideal of $\mathfrak{g}$. Then $H^\perp$ is an ideal of $\mathfrak{g}$. Furthermore assume that H is non-degenerate. Then $H^\perp$ is also non-degenerate and $\mathfrak{g} = H \oplus H^\perp$.*

Lemma 2.4 ([13]). *Let (A, f) be a quadratic Novikov algebra.*

(1) *$(\mathrm{Lie}(A), f)$ is a quadratic Lie algebra.*

(2) *If $\dim A \leqslant 4$, then $\mathrm{Lie}(A)$ is abelian.*

3. Underlying Lie algebras of quadratic Novikov algebras

Theorem 3.1. *Let (A, f) be a quadratic Novikov algebra. Then $[\mathrm{Lie}(A), \mathrm{Lie}(A)] \subseteq Z(A)$. As a consequence, $\mathrm{Lie}(A)$ is 2-step nilpotent.*

Proof. For any $a, b, c, d \in A$,

$$\begin{aligned} f(a[b,c],d) = f(a,[b,c]d) = f(a,b(cd)-c(bd)) &= f((ab)c-(ab)c,d) \\ &= f((ab)c-(ac)b,d) = 0. \end{aligned}$$

It follows that $[b, c] \subseteq Z(A) \subseteq C(\mathrm{Lie}(A))$ since $f(a[b, c], d) = f([b, c]d, a)$. Furthermore, $\mathrm{Lie}(A)$ is 2-step nilpotent. □

On the other hand, for a 2-step nilpotent quadratic Lie algebra (A, f), define a bilinear product on A by $xy = \frac{1}{2}[x, y]$ for any $x, y \in A$, then A is a Novikov algebra by Proposition 2.5 of [3]. Furthermore, (A, f) is a quadratic Novikov algebra.

Thus, to get the classification of $\mathrm{Lie}(A)$, it is enough to get the classification of 2-step nilpotent quadratic Lie algebras. Moreover, we have the following well-known fact:

Proposition 3.2. *Let (A, f) be a quadratic Novikov algebra. Then its underlying Lie algebra $\mathrm{Lie}(A)$ is a direct sum $\mathrm{Lie}(A) = \mathfrak{g}_a \oplus \mathfrak{g}_i$, where $\mathfrak{g}_a$ is an abelian ideal with a non-degenerate restriction f on it and $\mathfrak{g}_i$ is an ideal with an isotropic center.*

By Proposition 3.2, it is enough to discuss the classification with the additional condition that the center of the underlying Lie algebra is isotropic. First, we establish some lemmas.

Lemma 3.3. *In the above notation, $\dim[\mathrm{Lie}(A), \mathrm{Lie}(A)] \leqslant \frac{1}{2} \dim \mathrm{Lie}(A)$. If $C(\mathrm{Lie}(A))$ is isotropic, then $[\mathrm{Lie}(A), \mathrm{Lie}(A)] = C(\mathrm{Lie}(A))$. As a consequence, $\dim \mathrm{Lie}(A)$ is even.*

Proof. Since $[\mathrm{Lie}(A), \mathrm{Lie}(A)] \subseteq C(\mathrm{Lie}(A))$ and $(\mathrm{Lie}(A), f)$ is a quadratic Lie algebra, we have $\dim[\mathrm{Lie}(A), \mathrm{Lie}(A)] \leqslant \frac{1}{2} \dim \mathrm{Lie}(A)$. If $C(\mathrm{Lie}(A))$ is isotropic, by Lemma 2.3 we have

$$C(\mathrm{Lie}(A)) \subseteq C(\mathrm{Lie}(A))^{\perp} = [\mathrm{Lie}(A), \mathrm{Lie}(A)].$$

It follows that $C(\mathrm{Lie}(A)) = [\mathrm{Lie}(A), \mathrm{Lie}(A)]$ and $\dim \mathrm{Lie}(A)$ is even. □

Lemma 3.4. $[\mathrm{Lie}(A), \mathrm{Lie}(A)] \subseteq AA \subseteq C(\mathrm{Lie}(A))$ *and* $[\mathrm{Lie}(A), \mathrm{Lie}(A)] \subseteq Z(A) \subseteq C(\mathrm{Lie}(A))$*. In particular,* $Z(A) = AA$ *if* $C(\mathrm{Lie}(A))$ *is isotropic.*

Proof. It is enough to show that $AA \subseteq C(\mathrm{Lie}(A))$. In fact, for any $a, b, c, d \in A$,

$$f(a, [b, cd]) = f([a, b], cd) = 0.$$

It follows that $[b, cd] = 0$ by the non-degeneracy of f. Namely $AA \subseteq C(\mathrm{Lie}(A))$. Furthermore if $C(\mathrm{Lie}(A))$ is isotropic, then $Z(A) = AA$, by Lemma 3.3. □

Theorem 3.5. *Let* (A, f) *be a quadratic Novikov algebra of dimension* 6*. If* $C(\mathrm{Lie}(A))$ *is isotropic, then there exists a basis* $\{e_i, f_i,\ 1 \leqslant i \leqslant 3\}$ *of* $\mathrm{Lie}(A)$ *such that the non-zero products are given by* $[f_1, f_2] = e_3,\ [f_1, f_3] = -e_2,\ [f_2, f_3] = e_1$ *and* $f(e_i, f_j) = \delta_{ij}$*, where* $1 \leqslant i, j \leqslant 3$.

Proof. Since $C(\mathrm{Lie}(A))$ is isotropic, we have $\dim C(\mathrm{Lie}(A)) = 3$, and then there exists a basis $\{e_i, f_i,\ 1 \leqslant i \leqslant 3\}$ of $\mathrm{Lie}(A)$ such that $\{e_1, e_2, e_3\}$ is a basis of $C(\mathrm{Lie}(A)) = [\mathrm{Lie}(A), \mathrm{Lie}(A)]$ and $f(e_i, f_j) = \delta_{ij}$. Since $f(f_i, [f_i, f_j]) = f([f_i, f_i], f_j) = 0$, we have

$$[f_1, f_2] = ae_3,\ [f_2, f_3] = be_1,\ [f_3, f_1] = ce_2.$$

Since $f(f_3, [f_1, f_2]) = f(f_1, [f_2, f_3]) = f(f_2, [f_3, f_1])$ and $\dim[\mathrm{Lie}(A), \mathrm{Lie}(A)] = 3$, we know that $a = b = c \neq 0$. Replacing e_i by $\sqrt[3]{a} e_i$ and f_i by $(1/\sqrt[3]{a}) f_i$, we have $a = 1$. □

Theorem 3.6. *Let* (A, f) *be a quadratic Novikov algebra of dimension* 8*. Then* $C(\mathrm{Lie}(A))$ *is nonisotropic.*

Proof. Assume that $C(\mathrm{Lie}(A))$ is isotropic. By Lemma 3.3, $\dim C(\mathrm{Lie}(A)) = 4$, and then there exists a basis $\{e_i, f_i,\ 1 \leqslant i \leqslant 4\}$ of A such that $\{e_1, e_2, e_3, e_4\}$ is a basis of $C(\mathrm{Lie}(A)) = [\mathrm{Lie}(A), \mathrm{Lie}(A)]$ and $f(e_i, f_j) = \delta_{ij}$. Since $f([f_i, f_j], f_i) = f([f_i, f_j], f_j) = 0$, we have

$$\begin{aligned}
[f_1, f_2] &= a_{12}^3 e_3 + a_{12}^4 e_4, & [f_1, f_3] &= a_{13}^2 e_2 + a_{13}^4 e_4,\\
[f_1, f_4] &= a_{14}^2 e_2 + a_{14}^3 e_3, & [f_2, f_3] &= a_{23}^1 e_1 + a_{23}^4 e_4,\\
[f_2, f_4] &= a_{24}^1 e_1 + a_{24}^3 e_3, & [f_3, f_4] &= a_{34}^1 e_1 + a_{34}^2 e_2.
\end{aligned}$$

Since $f([f_i, f_j], f_k) = f([f_j, f_k], f_i) = f([f_k, f_i], f_j)$, we have

$$\begin{aligned}
a_{12}^3 &= a_{23}^1 = -a_{13}^2, & a_{12}^4 &= a_{24}^1 = -a_{14}^2,\\
a_{13}^4 &= a_{34}^1 = -a_{14}^3, & a_{23}^4 &= a_{34}^2 = -a_{24}^3.
\end{aligned}$$

Let

$$B = \begin{pmatrix} 0 & 0 & 0 & a_{12}^3 & a_{12}^4 & a_{13}^4 \\ 0 & -a_{12}^3 & a_{12}^4 & 0 & 0 & a_{23}^4 \\ a_{12}^3 & 0 & -a_{13}^4 & 0 & -a_{23}^4 & 0 \\ a_{12}^4 & a_{13}^4 & 0 & a_{23}^4 & 0 & 0 \end{pmatrix}.$$

It is easy to check that

$$\det B_{ijkl} = 0, \quad 1 \leqslant i < j < k < l \leqslant 6.$$

Here $B_{ijkl} = (c_i(B), c_j(B), c_k(B), c_l(B))$, where $c_i(B)$ denotes the i^{th}-column of B. It follows that $\dim[\mathrm{Lie}(A), \mathrm{Lie}(A)] < 4$. This is a contradiction. □

Theorem 3.7. *Let (A, f) be a quadratic Novikov algebra of dimension 10. If $C(\mathrm{Lie}(A))$ is isotropic, then there exists a basis $\{e_i, f_i,\ 1 \leqslant i \leqslant 5\}$ of $\mathrm{Lie}(A)$ such that the non-zero products are given by $[f_1, f_2] = e_3$, $[f_1, f_3] = -e_2$, $[f_2, f_3] = e_1$, $[f_1, f_4] = e_5$, $[f_1, f_5] = -e_4$, $[f_4, f_5] = e_1$ and $f(e_i, f_j) = \delta_{ij}$, where $1 \leqslant i, j \leqslant 5$.*

P r o o f. Since $C(\mathrm{Lie}(A))$ is isotropic, we have $\dim C(\mathrm{Lie}(A)) = 5$, and then there exists a basis $\{e_i, f_i,\ 1 \leqslant i \leqslant 5\}$ of A such that $\{e_1, e_2, e_3, e_4, e_5\}$ is a basis of $C(\mathrm{Lie}(A)) = [\mathrm{Lie}(A), \mathrm{Lie}(A)]$ and $f(e_i, f_j) = \delta_{ij}$.

Without loss of generality, assume that $[f_1, f_2] \neq 0$. Suppose that $k_1e_1 + k_2e_2 + k_3[f_1, f_2] = 0$. Then $k_1 = f(k_1e_1, f_1) = f(k_1e_1 + k_2e_2 + k_3[f_1, f_2], f_1) = 0$. Similarly, $k_2 = 0$, and then $k_3 = 0$. That is, $e_1, e_2, [f_1, f_2]$ are linearly independent. We can replace e_3 by $[f_1, f_2]$ since $f([f_1, f_2], f_i) = 0$ for $i = 1, 2$. By the identity (1.4), we have

$$[f_1, f_3] = -e_2 + ae_4 + be_5, \quad [f_2, f_3] = e_1 + ce_4 + de_5.$$

Replace e_1 by $[f_2, f_3]$ and e_2 by $-[f_1, f_3]$. Clearly f is non-degenerate restricted on the linear subspace spanned by $e_1, e_2, e_3, f_1, f_2, f_3$. Consider the orthogonal complement to this linear subspace. We can assume that

$$f(e_i, f_j) = \delta_{ij}, \quad [f_1, f_2] = e_3, \quad [f_1, f_3] = -e_2, \quad [f_2, f_3] = e_1. \tag{3.1}$$

Since $\dim[\mathrm{Lie}(A), \mathrm{Lie}(A)] = 5$, we can assume that $[f_1, f_4] \neq 0$. That is,

$$[f_1, f_4] = a_1e_2 + a_2e_3 + a_3e_5 \neq 0.$$

By the identities (1.4) and (3.1), $a_1 = a_2 = 0$. Moreover, we can assume that $[f_1, f_4] = e_5$ by replacing e_5 by ae_5 and f_5 by $a^{-1}f_5$. Furthermore, we have

$$[f_1, f_5] = -e_4, \quad [f_4, f_5] = e_1.$$

Also we have $[f_i, f_j] = 0$ for $i = 2, 3$ and $j = 4, 5$. In fact, assume that $[f_2, f_4] = ae_1 + be_3 + ce_5$. Then $a = f(ae_1, f_1) = f(ae_1 + be_3 + ce_5, f_1) = f([f_2, f_4], f_1) = f(f_2, [f_4, f_1]) = 0$. Similarly, $b = c = 0$. Then the theorem follows. □

Remark 3.8. It is easy to check that the above quadratic Lie algebras are indecomposable, i.e., they can't decompose into the orthogonal direct sum of two non-degenerate Lie ideals. Thus any quadratic Novikov algebra on these quadratic Lie algebras is indecomposable since any ideal of a Novikov algebra is a Lie ideal of the underlying Lie algebra.

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