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A STRONG INVARIANCE PRINCIPLE FOR NEGATIVELY ASSOCIATED RANDOM FIELDS

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Abstract. In this paper we obtain a strong invariance principle for negatively associated random fields, under the assumptions that the field has a finite $(2+\delta)$th moment and the covariance coefficient $u(n)$ exponentially decreases to 0. The main tools are the Berkes-Morrow multi-parameter blocking technique and the Csörgő-Révész quantile transform method.

Keywords: strong invariance principle, negative association, random field, blocking technique, quantile transform

MSC 2010: 60B10, 60F15, 60F17

1. Introduction and the result

A finite family of random variables $\{X_i;\ 1 \leqslant i \leqslant n\}$ is said to be negatively associated (NA, for short) if for every pair of disjoint subsets A and B of $\{1,2,\ldots,n\}$,

$$\operatorname{Cov}(f(X_i;\ i\in A), g(X_j;\ j\in B)) \leqslant 0 \tag{1.1}$$

whenever f and g are coordinate-wise nondecreasing and the covariance exists. An infinite family is negatively associated if every finite subfamily is negatively associated. The concept of the negative association was introduced by Alam and Saxena (1981) and Joag-Dev and Proschan (1983). As pointed out and proved by Joag-Dev and Proschan (1983), a number of well-known multivariate distributions possess the NA property. NA has found important and wide applications in multivariate statistical analysis and reliability theory. In the past two decades, a lot of effort has been dedicated to prove limit theorems for random fields $\{X_{\mathbf{n}};\ \mathbf{n}\in\mathbb{Z}_+^d\}$ of NA random

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variables. In the case $d = 1$, we refer to Joag-Dev and Proschan (1983) for fundamental properties, Newman (1984) for the central limit theorem, Su et al. (1997) for the moment inequality and functional central limit theorem and Shao and Su (1999) for the law of the iterated logarithm, and we refer to Roussas (1994) for the central limit theorem, Zhang and Wen (2001) for the moment inequality and weak convergence.

Let $d \geqslant 2$ be a positive integer, let $\mathbb{Z}^d$ and $\mathbb{Z}^d_+$ denote the d-dimensional lattices of integers and positive integers, respectively. The notation $\mathbf{m} \leqslant \mathbf{n}$, where $\mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d_+$ and $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d_+$, means that $m_k \leqslant n_k$ for $k = 1, \ldots, d$. We also use $|\mathbf{n}|$ for $\prod\limits_{k=1}^{d} n_k$, and $\|\mathbf{n}\|$ for $\max(|n_1|, |n_2|, \ldots, |n_d|)$. $\mathbf{n} \to \infty$ is to be interpreted as $n_k \to \infty$ for $k = 1, \ldots, d$ and $\mathbf{n} - \mathbf{m} = (n_1 - m_1, \ldots, n_d - m_d)$. For any finite subset $V \subseteq \mathbb{Z}^d_+$ we let $|V|$ be the cardinality of V, $S(V) = \sum\limits_{\mathbf{j} \in V} X_{\mathbf{j}}$, and $\sigma^2(V) = \operatorname{Var}(S(V))$. Finally, for any $\tau \in (0, 1)$ put

$$G_\tau = \bigcap_{j=1}^{d} \Big\{ \mathbf{n} \in \mathbb{Z}^d_+ ;\ n_j \geqslant \prod_{1 \leqslant l \leqslant d \colon l \neq j} n_l^\tau \Big\}. \tag{1.2}$$

Our main goal is to prove the following theorem:

Theorem 1.1. *Let $\{X_{\mathbf{n}} ;\ \mathbf{n} \in \mathbb{Z}^d_+\}$ $(d \geqslant 2)$ be a NA field of random variables with $EX_{\mathbf{n}} = 0$. Denote $S_{\mathbf{n}} = \sum\limits_{\mathbf{k} \leqslant \mathbf{n}} X_{\mathbf{k}}$, $\sigma^2_{\mathbf{n}} = \operatorname{Var}(S_{\mathbf{n}})$ for $\mathbf{n} \in \mathbb{Z}^d_+$. Assume that*

(C1) *$\{X_{\mathbf{n}} ;\ \mathbf{n} \in \mathbb{Z}^d_+\}$ is also a weakly stationary field, i.e., there exists a function $r \colon \mathbb{Z}^d_+ \to \mathbb{R}$, such that*

$$\operatorname{Cov}(X_{\mathbf{i}}, X_{\mathbf{j}}) = r(\mathbf{i} - \mathbf{j}) = r(\mathbf{j} - \mathbf{i})\ \forall\, \mathbf{i}, \mathbf{j} \in \mathbb{Z}^d_+,$$

(C2) $u(n) := \sum\limits_{\mathbf{i} \in \mathbb{Z}^d \colon \|\mathbf{i}\| \geqslant n} \{-r(\mathbf{i})\} = O(\mathrm{e}^{-\lambda n^\varepsilon})$ *for some* $\lambda, \varepsilon > 0$,

(C3) $\sup\limits_{\mathbf{n} \in \mathbb{Z}^d_+} E|X_{\mathbf{n}}|^{2+\delta} < \infty$ *for some* $\delta > 0$,

(C4) $\sigma^2 := \sum\limits_{\mathbf{i} \in \mathbb{Z}^d} r(\mathbf{i}) > 0$.

Then without changing its distribution we can redefine the random field $\{X_{\mathbf{n}} ;\ \mathbf{n} \in \mathbb{Z}^d_+\}$ on a richer probability space together with a d-parameter Wiener process $\{W_t ;\ t \in [0, \infty)^d\}$ with variance σ^2 such that

$$S_{\mathbf{N}} - W_{\mathbf{N}} = O(|\mathbf{N}|^{1/2 - \varepsilon}) \ \text{a.s.} \tag{1.3}$$

for $\mathbf{N} \in G_\tau$. Here ε is a positive constant depending on the field $\{X_{\mathbf{n}} ;\ \mathbf{n} \in \mathbb{Z}^d_+\}$.

Remark 1.1. The τ in Theorem 1.1 must be fixed. The statement is true for each fixed $\tau \in (0, 1)$, but the constructed Wiener process and the constant in (1.3) strongly depend on it.

Remark 1.2.

i) A sequence $\{X_i;\ 1 \leqslant i \leqslant n\}$ is called associated if for every pair of subsets A and B of $\{1, 2, \ldots, n\}$,

$$\mathrm{Cov}(f(X_i;\ i \in A), g(X_j;\ j \in B)) \geqslant 0 \tag{1.4}$$

whenever f and g are coordinatewise nondecreasing and the covariance exists. For weakly stationary associated fields Balan (2005) obtained a strong invariance principle under a finite $(2+\delta)$th moment and a certain restriction on the covariance function, i.e., $\sum_{\mathbf{i} \in \mathbb{Z}^d :\ \|\mathbf{i}\| \geqslant n} r(\mathbf{i}) = O(\mathrm{e}^{-\lambda n})$ for some $\lambda > 0$. It is easy to see that the decay rate of the covariance coefficient $u(n)$ in Theorem 1.1 is slightly weaker than the above covariance coefficient.

ii) The restriction "$\mathbf{N} \in G_\tau$" is essential here and a similar fact occurs for mixing random fields (see Berkes and Morrow 1981) and associated random fields (see Balan 2005, Bulinski and Shashkin 2005, Bulinski and Shashkin 2006).

The non-functional version of LIL obtained in Wichura (1973) for any multi-parameter process with independent increments (in particular for the Wiener process) allows us to conclude that

$$\limsup_{|\mathbf{N}| \to \infty, \mathbf{N} \in G_\tau} (2|\mathbf{N}| \log\log|\mathbf{N}|)^{-1/2} S_{\mathbf{N}} = \sigma \ \text{ a.s.} \tag{1.5}$$

The plan of the paper is as follows. In Section 2, we will introduce the main tools. In Section 3, we will give some useful lemmas which are needed for the proof of Theorem 1.1. The proof of Theorem 1.1 will be given in Section 4. From now on, without loss of generality, we assume that $0 < \delta \leqslant 1$ and C stands for a generic positive constant, independent of $\mathbf{n}$, it may take different values in each appearance.

2. Blocking technique and quantile transform method

In this section we will introduce the multi-parameter blocking technique of Berkes and Morrow (1981) and the quantile transform method of Csörgő and Révész (1975) which are the main tools that are needed for the proof of Theorem 1.1.

To use the blocking technique, for each $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}_+^d$ put

$$\mathbf{k}^\alpha = (k_1^\alpha, \ldots, k_d^\alpha), \quad N_{\mathbf{k}} = (n_{k_1}, \ldots, n_{k_d}) \quad \text{and} \quad B_{\mathbf{k}} = \{\mathbf{n} \in \mathbb{Z}_+^d;\ N_{\mathbf{k-1}} < \mathbf{n} \leqslant N_{\mathbf{k}}\},$$

where $n_l = \sum_{i=1}^{l}(i^\alpha + i^\beta) \sim (\alpha+1)^{-1} l^{1+\alpha}$ for $l \in \mathbb{Z}_+$ and some suitable chosen real numbers $\alpha > \beta > 1$. Note that for $\mathbf{k} \in \mathbb{Z}_+^d$

$$|N_{\mathbf{k}}| \sim (1+\alpha)^{-d} |\mathbf{k}|^{1+\alpha}, \qquad |B_{\mathbf{k}}| = \prod_{j=1}^{d} (k_j^\alpha + k_j^\beta) \leqslant 2^d |\mathbf{k}|^\alpha.$$

Now we define blocks H_k and I_k of consecutive positive integers, leaving no gaps between the blocks, by

$$H_{\mathbf{k}} = \{\mathbf{n} \in B_{\mathbf{k}}\,;\ N_{\mathbf{k-1}} < \mathbf{n} \leqslant N_{\mathbf{k-1}} + \mathbf{k}^\alpha\}, \qquad I_{\mathbf{k}} = B_{\mathbf{k}} \setminus H_{\mathbf{k}}.$$

Note that

$$|H_{\mathbf{k}}| = |\mathbf{k}|^\alpha \quad \text{and} \quad |\mathbf{k}|^\beta \leqslant |I_{\mathbf{k}}| \leqslant (2^d - 1)|\mathbf{k}|^\alpha.$$

Put

$$u_{\mathbf{k}} = \sum_{\mathbf{i} \in H_{\mathbf{k}}} X_{\mathbf{i}}, \qquad \lambda_{\mathbf{k}}^2 = E u_{\mathbf{k}}^2,$$

$$v_{\mathbf{k}} = \sum_{\mathbf{i} \in I_{\mathbf{k}}} X_{\mathbf{i}}, \qquad \tau_{\mathbf{k}}^2 = E v_{\mathbf{k}}^2,\ \mathbf{k} \in \mathbb{Z}_+^d,$$

where $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ are called the long blocks and the short blocks, respectively.

By (C4) and the NA property $\mathrm{Cov}(X,Y) \leqslant 0$ we get for any finite subset $V \subseteq \mathbb{Z}_+^d$

$$\frac{\sigma^2(V)}{|V|} = \frac{1}{|V|} \sum_{\mathbf{j} \in V} \mathrm{Cov}(X_{\mathbf{j}}, S(V)) \geqslant \sigma^2 > 0 \tag{2.1}$$

and thus

$$C|\mathbf{k}|^\alpha \leqslant \lambda_{\mathbf{k}}^2 \leqslant C|\mathbf{k}|^\alpha, \quad C|\mathbf{k}|^\beta \leqslant \tau_{\mathbf{k}}^2 \leqslant C|\mathbf{k}|^\alpha. \tag{2.2}$$

Let $\tau \in (0,1)$ and put $\varrho = \tau/8$. Define further

$$L = \{\mathbf{i} \in \mathbb{Z}_+^d\,;\ B_{\mathbf{i}} \subseteq G_\varrho\} \qquad \text{and} \qquad H = \bigcup_{\mathbf{i} \in L} B_{\mathbf{i}}.$$

Also, for each $\mathbf{N} \in H$ and $s = 1, \ldots, d$, define

$$\mathbf{N}^s = (N_1^s, \ldots, N_d^s)$$

by $N_{s'}^s = N_{s'}$, $s' \neq s$ and $N_s^s = \min_{\mathbf{n} \in H\,;\ n_{s'} = N_{s'}, s' \neq s} n_s$. Let

$$R_{\mathbf{k}} = (M_{\mathbf{k}}, N_{\mathbf{k}}] \subseteq H,$$

where $M_{\mathbf{k}} = ((N_{\mathbf{k}}^1)_1, \dots, (N_{\mathbf{k}}^d)_d)$. We note that

$$L_{\mathbf{k}} = \{\mathbf{i} \in \mathbb{Z}_+^d;\ B_{\mathbf{i}} \subseteq R_{\mathbf{k}}\} \subseteq L \cap \{\mathbf{i} \leqslant \mathbf{k}\}.$$

To use the quantile-transform method, let $\{\omega_{\mathbf{k}};\ \mathbf{k} \in \mathbb{Z}_+^d\}$ be a field of independent $N(0, \tau_{\mathbf{k}}^2)$-distributed random variables which is also independent of $\{u_{\mathbf{k}};\ \mathbf{k} \in \mathbb{Z}_+^d\}$. Put

$$\xi_{\mathbf{k}} = (u_{\mathbf{k}} + \omega_{\mathbf{k}})/\sqrt{\lambda_{\mathbf{k}}^2 + \tau_{\mathbf{k}}^2}, \quad \mathbf{k} \in \mathbb{Z}_+^d. \tag{2.3}$$

Let $F_{\mathbf{k}}$ denote the distribution function of $\xi_{\mathbf{k}}$. Note that $F_{\mathbf{k}}$ is continuous since the smooth random variable $\omega_{\mathbf{k}}$ is used.

Define

$$\eta_{\mathbf{k}} = \Phi^{-1}(F_{\mathbf{k}}(\xi_{\mathbf{k}})), \quad \mathbf{k} \in \mathbb{Z}_+^d, \tag{2.4}$$

where Φ^{-1} is the inverse of the standard Gaussian distribution function Φ. It is easy to see that $\eta_{\mathbf{k}}$ is a standard Gaussian random variable (see Yu 1996). Using the fact that $\{\xi_{\mathbf{k}};\ \mathbf{k} \in \mathbb{Z}_+^d\}$ constructed from (2.3) is NA by applying (P6) and (P7) of Joag-Dev and Proschan (1983) and the fact that $\Phi^{-1}(F_{\mathbf{k}}(\cdot))$ is a nondecreasing function, we conclude that $\{\eta_{\mathbf{k}};\ \mathbf{k} \in \mathbb{Z}_+^d\}$ is a new NA random field. Thus, by the quantile transform method, we have constructed a new NA random field with Gaussian marginals.

3. Useful lemmas

For $\mathbf{k}$ satisfying $N_{\mathbf{k}} < \mathbf{N} \leqslant N_{\mathbf{k}+\mathbf{1}}$ we can write

$$S_{\mathbf{N}} = (S_{\mathbf{N}} - S_{N_{\mathbf{k}}}) + S(R_{\mathbf{k}}) + S((0, N_{\mathbf{k}}] \setminus R_{\mathbf{k}}), \tag{3.1}$$

$$W_{\mathbf{N}} = (W_{\mathbf{N}} - W_{N_{\mathbf{k}}}) + W(R_{\mathbf{k}}) + W((0, N_{\mathbf{k}}] \setminus R_{\mathbf{k}}), \tag{3.2}$$

where W is a d-parameter Wiener process. Further, we have

$$W(R_{\mathbf{k}}) = \sum_{\mathbf{i} \in L_{\mathbf{k}}} W(B_{\mathbf{i}}), \tag{3.3}$$

$$\begin{aligned} S(R_{\mathbf{k}}) = &\sum_{\mathbf{i} \in L_{\mathbf{k}}} \sqrt{\lambda_{\mathbf{i}}^2 + \tau_{\mathbf{i}}^2}(\xi_{\mathbf{i}} - \eta_{\mathbf{i}}) + \sum_{\mathbf{i} \in L_{\mathbf{k}}} \sqrt{|B_{\mathbf{i}}|}\left(\sqrt{\frac{\lambda_{\mathbf{i}}^2 + \tau_{\mathbf{i}}^2}{|B_{\mathbf{i}}|}} - \sigma\right)\eta_{\mathbf{i}} \\ &+ \sum_{\mathbf{i} \in L_{\mathbf{k}}} \sigma\sqrt{|B_{\mathbf{i}}|}\eta_{\mathbf{i}} - \sum_{\mathbf{i} \in L_{\mathbf{k}}} \omega_{\mathbf{i}} + \sum_{\mathbf{i} \in L_{\mathbf{k}}} \nu_{\mathbf{i}}. \end{aligned} \tag{3.4}$$

In this section we will give some lemmas to show that the sums in the above decomposition, except $\sum\limits_{\mathbf{i}\in L_{\mathbf{k}}}\sigma\sqrt{|B_{\mathbf{i}}|}\eta_{\mathbf{i}}$ and $W(R_{\mathbf{k}})$, can be made sufficiently small. We first need an estimate for the difference of the characteristic function of $S_{\mathbf{n}}/\sigma_{\mathbf{n}}$ and that of the standard Gaussian distribution $N(0,1)$. This is essential for us to use the quantile transform method successfully.

Proposition 3.1. *Suppose that* (C1), (C3), (C4) *and*

$$\text{(C2}'\text{)}\qquad u(n)=O(n^{-2d\mu(1+\delta)(2+\delta)/\delta^2})\ \ \textit{for some}\ \ \mu>1$$

hold. We have

$$\begin{aligned}&|E\exp(\mathrm{i}tS_{\mathbf{n}}/\sigma_{\mathbf{n}})-\mathrm{e}^{-t^2/2}|\\&\quad\leqslant C\big[(|\mathbf{n}|^{-\frac{\mu+\mu\delta+1-\delta}{2(1+\mu)(1+\delta)}}\log^{d-1}|\mathbf{n}|t^3+|\mathbf{n}|^{-\frac{\mu\delta}{2(1+\mu)}}\log^{(d-1)(1+\delta)/2}|\mathbf{n}|t^{2+\delta})\mathrm{e}^{-t^2/4}\\&\qquad\qquad+|\mathbf{n}|^{-\frac{2\mu+2\mu\delta+2-\delta}{2(1+\mu)(1+\delta)}}\log^{\frac{d-1}{2}}|\mathbf{n}|t+|\mathbf{n}|^{-\frac{3\mu-1}{2(1+\mu)}}t\big]\end{aligned}$$

for all $0\leqslant t\leqslant C|\mathbf{n}|^{\frac{1}{2}-\frac{\delta}{(1+\mu)(1+\delta)}}\log^{1-d}|\mathbf{n}|$.

The proof of this proposition can be found in Cai and Wang (2009).

Lemma 3.1. *Under the assumptions of Proposition* 3.1, *if* $2(1+\mu)/\mu(2+\delta)<\alpha/\beta<2(1+\mu)(1+\delta)/(1+\mu+\mu\delta+3\delta)$ *and* $\mu>(3-\delta)/(1+\delta)$, *then*

$$\sup_x|F_{\mathbf{k}}(x)-\Phi(x)|\leqslant C|\mathbf{k}|^{-\frac{\delta\beta}{2+\delta}}\qquad\textit{and}\qquad\sup_x|f_{\mathbf{k}}(x)-\varphi(x)|\leqslant C,$$

where $f_{\mathbf{k}}(x)$ *is the density function of* $\xi_{\mathbf{k}}$ *and* $\varphi(x)$ *is the density function of* $\Phi(x)$.

P r o o f. By the smoothing lemma of Berry (see Feller 1971) and the independence between $\{u_{\mathbf{k}}\}$ and $\{\omega_{\mathbf{k}}\}$, for any $T>0$ we have

$$\begin{aligned}\sup_x|F_{\mathbf{k}}(x)-\Phi(x)|&\leqslant\frac{1}{\pi}\int_{-T}^{T}\Big|\frac{E\exp(\mathrm{i}t\xi_{\mathbf{k}})-\exp(-t^2/2)}{t}\Big|\,\mathrm{d}t+\frac{24}{\pi T}\\&\leqslant C\int_{-T}^{T}\Big|\frac{E\exp(\mathrm{i}tu_{\mathbf{k}}/\lambda_{\mathbf{k}})-\exp(-t^2/2)}{t}\Big|\exp\Big(-\frac{\tau_{\mathbf{k}}^2t^2}{2\lambda_{\mathbf{k}}^2}\Big)\,\mathrm{d}t+\frac{C}{T}.\end{aligned}$$

Then replacing $S_{\mathbf{n}}/\sigma_{\mathbf{n}}$ by $u_{\mathbf{k}}/\lambda_{\mathbf{k}}$ in Proposition 3.1, for $|\mathbf{n}|=|\mathbf{k}|^{\alpha}$ and $T=C|\mathbf{n}|^{1/2-\delta/(1+\mu)(1+\delta)}\log^{1-d}|\mathbf{n}|$ we have

$$\sup_x|F_{\mathbf{k}}(x)-\Phi(x)|\leqslant C|\mathbf{k}|^{-\mu\delta\alpha/2(1+\mu)}\log^{(1+\delta)(d-1)/2}|\mathbf{k}|\leqslant C|\mathbf{k}|^{-\delta\beta/(2+\delta)}.$$

For the second inequality we use a technique similar to that used to prove relationship (3.3) of Yu (1996). For any $T > 0$,

$$\sup_x |f_{\mathbf{k}}(x) - \varphi(x)|$$
$$\leqslant \frac{C}{2\pi}\int_{-T}^{T} |E\exp(\mathrm{i}t\xi_{\mathbf{k}}) - \exp(-t^2/2)|\,\mathrm{d}t + \frac{C(\lambda_{\mathbf{k}}^2+\tau_{\mathbf{k}}^2)}{T\tau_{\mathbf{k}}^2}\exp\Big(-\frac{T^2\tau_{\mathbf{k}}^2}{2(\lambda_{\mathbf{k}}^2+\tau_{\mathbf{k}}^2)}\Big).$$

Similarly, by Proposition 3.1 and (2.2) the conclusion follows by choosing $T = |\mathbf{k}|^{\alpha-\beta}$ (such a choice is possible since $\alpha/\beta < 2(1+\mu)(1+\delta)/(1+\mu+\mu\delta+3\delta)$ and $\mu > (3-\delta)/(1+\delta)$).

The next result follows exactly as Theorem 2.1 of Yu (1996), using Lemma 3.1.

Lemma 3.2. *Under the assumptions of Lemma* 3.1, *for any* $0 < \theta < 1/2$ *we have*

$$-E\eta_{\mathbf{i}}\eta_{\mathbf{j}} \leqslant C[(|\mathbf{i}||\mathbf{j}|)^{-\alpha/2}(-Eu_{\mathbf{i}}u_{\mathbf{j}})]^{\theta/(1+\theta)}, \quad \forall\, \mathbf{i} \neq \mathbf{j}.$$

Lemma 3.3. *Under the assumptions of Lemma* 3.1, *we have*

$$|\eta_{\mathbf{k}} - \xi_{\mathbf{k}}| \leqslant C|\mathbf{k}|^{-(\delta\beta/(2+\delta)-K^2/2)}$$

provided that $|\xi_{\mathbf{k}}| \leqslant K(\log|\mathbf{k}|)^{1/2}$ *and* $0 < K < (2\delta\beta/(2+\delta))^{1/2}$.

Proof. By Lemma 3.1, the proof is basically the same as that of Lemma 2.5.1 of Csörgő and Révész (1981).

Using (2.2), Lemma 3.1, Lemma 3.3 and Lemma A.1 of Zhang and Wen (2001), and employing the same techniques which were used in the proofs of Lemma 3.10 of Yu (1996) and Lemma 3.6 of Balan (2005), we get the following result.

Lemma 3.4. *Under the assumptions of Lemma* 3.1 *and* $\beta > (1+2/\delta)(3+4/\delta)$, *there exist* $\varepsilon_0 = 2\beta\delta^2/(2+\delta)(4+3\delta)$ *and* $\varepsilon_1 > 0$ *such that for every* $\mathbf{k} \in \mathbb{Z}_+^d$

$$E(\lambda_{\mathbf{k}}^2+\tau_{\mathbf{k}}^2)(\xi_{\mathbf{k}}-\eta_{\mathbf{k}})^2 \leqslant C|\mathbf{k}|^{\alpha-\varepsilon_0}$$

and

$$\sum_{\mathbf{i}\in L_{\mathbf{k}}}\Big|\sqrt{\lambda_{\mathbf{i}}^2+\tau_{\mathbf{i}}^2}(\xi_{\mathbf{i}}-\eta_{\mathbf{i}})\Big| \leqslant C|N_{\mathbf{k}}|^{1/2-\varepsilon_1} \quad \text{a.s.}$$

Lemma 3.5. *If* (C1) *and*

$$\text{(C2}''\text{)} \qquad u(n) = O(n^{-\tau_1}), \quad \textit{for some } \tau_1 > 1$$

hold, then

$$0 \leqslant \frac{\sigma^2(V)}{|V|} - \sigma^2 \leqslant C|V|^{-1/d}. \tag{3.5}$$

P r o o f. The left inequality of (3.5) follows from (2.1). Let $V = (\mathbf{m}, \mathbf{n}]$ be a square, i.e., $n_s - m_s = k$ for $s = 1, \ldots, d$. Note that $|V| = k^d$. By stationarity and the NA property we have $\mathrm{Cov}(X, Y) \leqslant 0$,

$$\begin{aligned}\sigma^2(V) &= |V|r(\mathbf{0}) + \sum_{\|\mathbf{i}\| \leqslant k-1, \mathbf{i} \neq \mathbf{0}} \prod_{s=1}^{d} (k - |i_s|) r(\mathbf{i}) \\ &\leqslant |V|(r(\mathbf{0}) + \sum_{\|\mathbf{i}\| \leqslant k-1, \mathbf{i} \neq \mathbf{0}} r(\mathbf{i})) - Ck^{d-1} \sum_{\|\mathbf{i}\| \leqslant k-1, \mathbf{i} \neq \mathbf{0}} \|\mathbf{i}\| r(\mathbf{i}).\end{aligned}$$

Thus, by (C2$''$) we have

$$\begin{aligned}\frac{\sigma^2(V)}{|V|} - \sigma^2 &\leqslant \sum_{\|\mathbf{i}\| \geqslant k} \{-r(\mathbf{i})\} + Ck^{-1} \sum_{\|\mathbf{i}\| \leqslant k-1, \mathbf{i} \neq \mathbf{0}} \|\mathbf{i}\| \{-r(\mathbf{i})\} \\ &\leqslant u(k) + Ck^{-1} \sum_{j=1}^{\infty} u(j) \\ &\leqslant Ck^{-1} = C|V|^{-1/d}. \qquad (3.6)\end{aligned}$$

Let $V = (\mathbf{m}, \mathbf{n}]$ be not a square. It is well known that each rectangle V can be written as a finite union of disjoint squares: $V = \bigcup_{j=1}^{N} V_j$. By the NA property, we have

$$\sigma^2(V) \leqslant \sum_{j=1}^{N} \sigma^2(V_j).$$

Thus

$$\frac{\sigma^2(V)}{|V|} - \sigma^2 \leqslant \frac{1}{|V|} \sum_{j=1}^{N} |V_j| \Big(\frac{\sigma^2(V_j)}{|V_j|} - \sigma^2 \Big) \leqslant \frac{C}{|V|} \sum_{j=1}^{N} |V_j|^{1-1/d} = C|V|^{-1/d}.$$

□

Using Lemma 3.5 and employing the same method that was used in the proof of Lemma 3.8 of Balan (2005), we get the following result.

Lemma 3.6. *Suppose that* (C1) *and* (C2$''$) *hold and* $\beta > 3d$. *Then for every* $\mathbf{k} \in \mathbb{Z}_+^d$,

$$\sum_{\mathbf{i}\in L_\mathbf{k}} \sqrt{|B_\mathbf{i}|}\left(\sqrt{\frac{\lambda_\mathbf{i}^2+\tau_\mathbf{i}^2}{|B_\mathbf{i}|}}-\sigma\right)|\eta_\mathbf{i}| \leqslant C|N_\mathbf{k}|^{1/2-\varepsilon_2} \quad \text{a.s.},$$

where $\varepsilon_2 = \frac{1}{2}(1+\alpha)^{-1}$.

Using the condition "$\mathbf{i} \in G_\varrho$", and employing the same method that was used in the proof of Lemma 3.9 of Balan (2005), we get

Lemma 3.7. *If* $\alpha - \beta > 2 + 4/\varrho$, *then for every* $\mathbf{k} \in \mathbb{Z}_+^d$ *we have*

$$\sum_{\mathbf{i}\in L_\mathbf{k}} |\nu_\mathbf{i}| \leqslant C|N_\mathbf{k}|^{1/2-\varepsilon_2} \ \text{a.s.} \quad \text{and} \quad \sum_{\mathbf{i}\in L_\mathbf{k}} |\omega_\mathbf{i}| \leqslant C|N_\mathbf{k}|^{1/2-\varepsilon_2} \ \text{a.s.}$$

Lemma 3.8. *If* (C2) *holds, then*

$$-Eu_\mathbf{i}u_\mathbf{j} \leqslant C\mathrm{e}^{-\lambda M_{\mathbf{i},\mathbf{j}}^{\varepsilon\beta}}|\mathbf{j}|^\alpha,$$

where $M_{\mathbf{i},\mathbf{j}} = \max_{s\colon i_s\neq j_s}(M_s(\mathbf{i},\mathbf{j})-1)$ *and* $M_s(\mathbf{i},\mathbf{j}) = \max(i_s, j_s)$, $s = 1,\ldots,d$.

P r o o f. We follow the lines of the proof of Lemma 4.2 of Balan (2005). Let $D = \min_{\mathbf{k}\in H_\mathbf{j}} d(\mathbf{k}, H_\mathbf{i})$ be the distance between $H_\mathbf{i}$ and $H_\mathbf{j}$, where $d(\mathbf{k}, H_\mathbf{i}) = \min_{\mathbf{k}'\in H_\mathbf{i}} \|\mathbf{k}-\mathbf{k}'\|$. Then by (C2)

$$-Eu_\mathbf{i}u_\mathbf{j} = \sum_{\mathbf{k}\in H_\mathbf{j}}\left\{-E\left(X_\mathbf{k}\cdot\sum_{\mathbf{k}'\in H_\mathbf{i}} X_\mathbf{k}\right)\right\} \leqslant \sum_{\mathbf{k}\in H_\mathbf{j}} u(d(\mathbf{k},H_\mathbf{i})) \leqslant C\mathrm{e}^{-\lambda D^\varepsilon}|\mathbf{j}|^\alpha$$

and $D = \max_{s=1,\ldots,d}\min_{\mathbf{k}\in H_\mathbf{j},\mathbf{k}'\in H_\mathbf{i}} |k_s - k_s'| \geqslant M_{\mathbf{i},\mathbf{j}}^\beta$. □

Lemma 3.9. *Suppose that the conditions of Lemma* 3.1 *hold and put*

$$S(\mathbf{m},\mathbf{n}) = \sum_{\mathbf{j}\leqslant\mathbf{n}} X_{\mathbf{m}+\mathbf{j}}, \quad M(\mathbf{m},\mathbf{n}) = \max_{\mathbf{j}\leqslant\mathbf{n}}|S(\mathbf{m},\mathbf{j})|$$

for $\mathbf{m},\mathbf{n}\in\mathbb{Z}_+^d$. *Then we have*

$$P(M(\mathbf{m},\mathbf{n}) \geqslant x|\mathbf{n}|^{1/2}) \leqslant Cx^{-(2+\delta)}, \quad x > 0.$$

Further, there exists $\gamma > 0$ *such that*

$$P(M(\mathbf{m},\mathbf{n}) \geqslant |\mathbf{n}|^{1/2}\log^{d+1}|\mathbf{n}|) \leqslant C|\mathbf{n}|^{-\gamma}.$$

P r o o f. Using the Markov inequality and Lemma 3.4 of Zhang (2000) we get

$$P(M(\mathbf{m},\mathbf{n}) \geqslant x|\mathbf{n}|^{1/2}) \leqslant x^{-(2+\delta)}|\mathbf{n}|^{-(2+\delta)/2}EM^{2+\delta}(\mathbf{m},\mathbf{n})$$
$$\leqslant Cx^{-(2+\delta)}.$$

The second inequality follows exactly as the second inequality of Lemma 7 of Berkes and Morrow (1981), using the moment inequality given by Lemma A.1 of Zhang and Wen (2001) and the rate of convergence in the CLT given by Proposition 3.1 and Berry's lemma. □

Define

$$D_s(\mathbf{N}) = \max_{\mathbf{n}\leqslant \mathbf{N}^s} |S_{\mathbf{n}}|, \quad D'_s(\mathbf{N}) = \max_{\mathbf{n}\leqslant \mathbf{N}^s} |W_{\mathbf{n}}|$$

for each $s = 1,\dots,d$ and $\mathbf{N} \in H$. We have

$$S((0,N_{\mathbf{k}}] \setminus R_{\mathbf{k}}) \leqslant \sum_{s=1}^{d} 2^{d-s} D_s(N_{\mathbf{k}}), \quad W((0,N_{\mathbf{k}}] \setminus R_{\mathbf{k}}) \leqslant \sum_{s=1}^{d} 2^{d-s} D'_s(N_{\mathbf{k}}).$$

On the other hand, for any nonempty subset J of $\{1,2,\dots,d\}$ and any $\mathbf{k} \in \mathbb{Z}_+^d$, define

$$M_{\mathbf{k}}^J = \max_{n_{k_s} < N_s \leqslant n_{k_s+1}, s\in J} \Bigg| \sum_{1\leqslant v_s \leqslant n_{k_s}, s\in J^c; n_{k_s} < v_s \leqslant N_s, s\in J} X_{\mathbf{v}} \Bigg|. \tag{3.7}$$

Define by $M_{\mathbf{k}}'^{J}$ the analogous quantity for the Wiener process, i.e., the quantity we get if we replace the sum in (3.7) by the increment of W over the given rectangle. Then we have

$$\max_{N_{\mathbf{k}} < \mathbf{N} \leqslant N_{\mathbf{k+1}}} |S_{\mathbf{N}} - S_{N_{\mathbf{k}}}| \leqslant \sum_J M_{\mathbf{k}}^J, \quad \max_{N_{\mathbf{k}} < \mathbf{N} \leqslant N_{\mathbf{k+1}}} |W_{\mathbf{N}} - W_{N_{\mathbf{k}}}| \leqslant \sum_J M_{\mathbf{k}}'^{J}.$$

The next two lemmas follow exactly as Lemma 6 and Lemma 9 of Berkes and Morrow (1981), using Lemma 3.9.

Lemma 3.10. *Under the assumptions of Lemma* 3.1, *if* $\alpha > \frac{16}{3}\tau^{-1} - 1$, *then for every* $N_{\mathbf{k}} \in G_\tau$ *and* $0 < \varepsilon_3 < \tau/32$, *we have*

$$\max_{s=1,\dots,d} \max(D_s(N_{\mathbf{k}}), D'_s(N_{\mathbf{k}})) \leqslant C|N_{\mathbf{k}}|^{1/2-\varepsilon_3} \quad a.s.$$

Lemma 3.11. *Under the assumptions of Lemma 3.1, if $\alpha > 2/\gamma$ (γ is the constant given by Lemma 3.9), then for every $N_{\mathbf{k}} \in G_\varrho$ and $0 < \varepsilon_4 < \varrho/(8\alpha)$ we have*

$$\max_J \max(M_{\mathbf{k}}^J, {M'_{\mathbf{k}}}^J) \leqslant C|N_{\mathbf{k}}|^{1/2-\varepsilon_4} \quad \text{a.s.}$$

Lemma 3.12 (Balan 2005). *There exists a bijection $\psi\colon\ \mathbb{Z}_+ \to L$ such that*

$$l < m \Rightarrow \exists s^* = s^*(l,m) \quad \text{such that} \quad \psi(l)_{s^*} \leqslant \psi(m)_{s^*} \tag{3.8}$$

$$\exists m_0 \in \mathbb{Z}_+ \quad \text{such that} \quad m \leqslant C|\psi(m)|^{\gamma_0} \quad \forall\, m \geqslant m_0 \tag{3.9}$$

for any $\gamma_0 > (1+1/\varrho)(1-1/d)$.

4. Proof of Theorem 1.1

By virtue of (3.1) $\sim$ (3.4), Lemma 3.4, Lemma 3.6 $\sim$ 3.7 and Lemma 3.10 $\sim$ 3.11, in order to prove Theorem 1.1 it suffices to show that for every $\mathbf{k} \in \mathbb{Z}_+^d$

$$\sum_{\mathbf{i}\in L_{\mathbf{k}}} \sigma\sqrt{|B_{\mathbf{i}}|}|\eta_{\mathbf{i}} - \frac{W(B_{\mathbf{i}})}{\sigma\sqrt{|B_{\mathbf{i}}|}}| \leqslant C|N_{\mathbf{k}}|^{1/2-\varepsilon_2} \quad \text{a.s.} \tag{4.1}$$

We follow the lines of the proof of Theorem 4.4 in Balan (2005). Let $\psi\colon\ \mathbb{Z}_+ \to L$ be the bijection given by Lemma 3.12. Define

$$Y_m = \eta_{\psi(m)}, \quad \text{for } m \in \mathbb{Z}_+.$$

Observing that $Y_1, \ldots, Y_m$ are NA random variables, we have by Lemma 2.1 of Zhang (2001)

$$\begin{aligned} &\left|E\exp\left(\mathrm{i}\sum_{j=1}^{m} t_jY_j\right) - E\exp\left(\mathrm{i}\sum_{j=1}^{m-1} t_jY_j\right)E\exp(\mathrm{i}t_mY_m)\right| \\ &\leqslant \left|\mathrm{Cov}\left(\cos\sum_{j=1}^{m-1} t_jY_j, \cos(t_mY_m)\right)\right| + \left|\mathrm{Cov}\left(\cos\sum_{j=1}^{m-1} t_jY_j, \sin(t_mY_m)\right)\right| \\ &\quad + \left|\mathrm{Cov}\left(\sin\sum_{j=1}^{m-1} t_jY_j, \cos(t_mY_m)\right)\right| + \left|\mathrm{Cov}\left(\sin\sum_{j=1}^{m-1} t_jY_j, \sin(t_mY_m)\right)\right| \\ &\leqslant 4\sum_{j=1}^{m-1} |t_jt_m|\{-EY_jY_m\} \end{aligned} \tag{4.2}$$

for all $t_1, \ldots, t_m \in \mathbb{R}$ with $\sum_{j=1}^{m} t_j^2 \leqslant U_m^2$, where $U_m > 10^{10}$.

By Lemma 3.2 and Lemma 3.8, for $0 < \theta < 1/2$ and suitable constants α, β satisfying the Lemmas in Section 3 we have

$$(4.3) \quad -EY_jY_m \leqslant C(|\psi(j)||\psi(m)|)^{-\frac{\alpha\theta}{2+2\theta}} \exp\Big(-\frac{\lambda\theta}{1+\theta} M^{\beta\varepsilon}_{\psi(j),\psi(m)}\Big)|\psi(m)|^{\frac{\alpha\theta}{1+\theta}}$$
$$\leqslant C|\psi(j)|^{-\frac{\alpha\theta}{2+2\theta}}|\psi(m)|^{\frac{\alpha\theta}{2+2\theta}} \exp\Big(-\frac{\lambda\theta}{1+\theta}|\psi(m)|^{\beta\varepsilon\varrho/2}\Big),$$

where in the second inequality above we used (3.8) to obtain an s^* for which $M_{s^*}(\psi(j),\psi(m)) = \psi(m)_{s^*}$; since $\psi(m) \in L$, we have $M_{\psi(j),\psi(m)} \geqslant \psi(m)_{s^*} - 1 \geqslant C|\psi(m)|^{\varrho/2}$.

Thus, by (4.2) and (4.3), by the Cauchy inequality and (3.9) we get

$$(4.4) \quad \Big|E\exp\Big(\mathrm{i}\sum_{j=1}^{m} t_jY_j\Big) - E\exp\Big(\mathrm{i}\sum_{j=1}^{m-1} t_jY_j\Big)E\exp(\mathrm{i}t_mY_m)\Big|$$
$$\leqslant C\exp\Big(-\frac{\lambda\theta}{1+\theta}|\psi(m)|^{\beta\varepsilon\varrho/2}\Big)\sum_{j=1}^{m-1}|t_jt_m||\psi(j)|^{-\frac{\alpha\theta}{2+2\theta}}|\psi(m)|^{\frac{\alpha\theta}{2+2\theta}}$$
$$\leqslant C\exp\Big(-\frac{\lambda\theta}{1+\theta}|\psi(m)|^{\beta\varepsilon\varrho/2}\Big)\Big[\sum_{j=1}^{m-1} t_j^2|\psi(j)|^{-\frac{\alpha\theta}{1+\theta}} + (m-1)t_m^2|\psi(m)|^{\frac{\alpha\theta}{1+\theta}}\Big]$$
$$\leqslant C\exp\Big(-\frac{\lambda^*\theta}{1+\theta}|\psi(m)|^{\beta\varepsilon\varrho/2}\Big)U_m^2 := \varrho_m$$

for m large enough and $0 < \lambda^* < \lambda$.

Hence by Theorem 5 of Berkes and Philipp (1979), without changing its distribution we can redefine the sequence $\{Y_m\,;\ m \in \mathbb{Z}_+\}$ on a rich enough probability space together with a sequence $\{Z_m\,;\ m \in \mathbb{Z}_+\}$ of independent $N(0,1)$-distributed random variables such that

$$P(|Y_m - Z_m| \geqslant a_m) \leqslant a_m, \quad \forall\, m \in \mathbb{Z}_+,$$

where $a_m \leqslant C[U_m^{-1/4}\log U_m + \exp(-3U_m^{1/2}/16)m^{1/2}U_m^{1/4} + \varrho_m^{1/2}U_m^{m+1/4}]$. Then we select $U_m = m^q$ with $q > 8$. Clearly

$$U_m^{-1/4}\log U_m \leqslant Cm^{-2},$$

and

$$\exp(-3U_m^{1/2}/16)m^{1/2}U_m^{1/4} \leqslant \exp(-U_m^{1/2}/8) \leqslant Cm^2$$

for m large enough. Let $\beta > 4\gamma_0/\varrho\varepsilon$, then by (3.9) we have

$$\varrho_m^{1/2}U_m^{m+1/4} = \exp\Big(-\frac{\lambda^*\theta}{2+2\theta}|\psi(m)|^{\beta\varepsilon\varrho/2}\Big)m^{q(m+5/4)} \leqslant Cm^{-2}$$

since $[2+q(m+5/4)]\log m \leqslant Cm^{1+\varepsilon} \leqslant C|\psi(m)|^{(1+\varepsilon)\gamma_0} \leqslant C|\psi(m)|^{\beta\varepsilon\varrho/2}$ for m large enough and any small $0<\varepsilon<1$. Thus

$$a_m \leqslant Cm^{-2} \quad \text{for } m \text{ large enough.} \tag{4.5}$$

Then, by the Borel-Cantelli lemma, we have

$$|Y_m - Z_m| \leqslant a_m \quad \text{a.s.} \tag{4.6}$$

Since $\{Z_m;\ m\in\mathbb{Z}_+\}$ is an independent Gaussian sequence, we assume without loss of generality that there exists a Wiener process with variance σ^2 satisfying

$$Z_m = W(B_{\psi(m)})/(\sigma\sqrt{|B_{\psi(m)}|}), \quad \forall\, m\in\mathbb{Z}_+.$$

Hence

$$\Bigl|\eta_{\mathbf{i}} - \frac{W(B_{\mathbf{i}})}{\sigma\sqrt{|B_{\mathbf{i}}|}}\Bigr| \leqslant Ca_{\psi^{-1}(\mathbf{i})} \ \text{ a.s.} \tag{4.7}$$

for $\forall\,\mathbf{i}\in L$. because $\sum\limits_{l\in\mathbb{Z}_+} a_l < \infty$ and $|B_{\mathbf{i}}| \leqslant |B_{\mathbf{k}}| \leqslant C|\mathbf{k}|^{\alpha}$ for $\forall\,\mathbf{i}\in L_{\mathbf{k}}$. Thus we have

$$\begin{aligned}\sum_{\mathbf{i}\in L_{\mathbf{k}}} \sigma\sqrt{|B_{\mathbf{i}}|}\Bigl|\eta_{\mathbf{i}} - \frac{W(B_{\mathbf{i}})}{\sigma\sqrt{|B_{\mathbf{i}}|}}\Bigr| &\leqslant C|\mathbf{k}|^{\alpha/2}\sum_{\mathbf{i}\in L_{\mathbf{k}}} a_{\psi^{-1}(\mathbf{i})}\\ &\leqslant C|\mathbf{k}|^{\alpha/2}\sum_{l\in Z_+} a_l \leqslant C|\mathbf{k}|^{\alpha/2} \leqslant C|N_{\mathbf{k}}|^{1/2-\varepsilon_2}.\end{aligned}$$

□

References

[1] *K. Alam and K. M. L. Saxena*: Positive dependence in multivariate distributions. Comm. Statist. *A 10* (1981), 1183–1196.

[2] *R. M. Balan*: A strong invariance principle for associated random fields. Ann. Probab. *33* (2005), 823–840.

[3] *I. Berkes and W. Philipp*: Approximation theorems for independent and weakly dependent random vectors. Ann. Probab. *7* (1979), 19–54.

[4] *I. Berkes and G. J. Morrow*: Strong invariance principles for mixing random fields. Z. Wahrsch. view. Gebiete *57* (1981), 15–37.

[5] *Bulinski and Shashkin*: The strong invariance principles for dependent multi-indexed random variables. Dokl. Acad. Nauk *403* (2005), 155–158.

[6] *Bulinski and Shashkin*: Strong invariance principles for dependent random fields. ISM Lect. Notes-Monograph Series Dynamics and Stochastics *48* (2006), 128–143.

[7] *G. H. Cai and J. F. Wang*: Uniform bounds in normal approximation under negatively associated random fields. Statistics and Probability Letters *79* (2009), 215–222.

[8] *M. Csörgő and P. Révész*: I. A new method to prove Strassen type laws of invariance principle. Z. Wahrsch. view. Gebiete *31* (1975), 255–260.

[9] *M. Csörgő and P. Révész*: Strong Approximations in Probability and Statistics. Academic Press, New York, 1981.

[10] *W. Feller*: An Introduction to Probability Theory and its Applications 2. 2nd ed John Wiley. New York, 1971.

[11] *K. Joag-Dev and F. Proschan*: Negative association of random variables with applications. Ann. Statist. *11* (1983), 286–295.

[12] *C. M. Newman*: Asympotic independence and limit theorems for positively and negatively dependent random variables. Inequalities in Statistics and Probability (Tong, Y. L., ed., Institute of Mathematical Statistics, Hayward, CA), 1984, pp. 127–140.

[13] *G. G. Roussas*: Asymptotic normality of random fields of positively or negatively associated processes. J. Multivariate Anal. *50* (1994), 152–173.

[14] *Q. M. Shao and C. Su*: The law of the iterated logarithm for negatively associated random variables. Stochastic Process. Appl. *83* (1999), 139–148.

[15] *C. Su, L. Zhao and Y. Wang*: Moment inequalities and weak convergence for negatively associated sequences. Sci. China Ser. *A 40* (1997), 172–182.

[16] *M. J. Wichura*: Some Strassen-type laws of the iterated logarithm for multiparameter stochastic processes with independent increments. Ann. Probab. *1* (1973), 272–296.

[17] *H. Yu*: A strong invariance principle for associated sequences. Ann. Probab. *24* (1996), 2079–2097.

[18] *L. X. Zhang*: A functional central limit theorem for asymptotically negatively dependent random fields. Acta Math. Hungar. *86* (2000), 237–259.

[19] *L. X. Zhang*: The weak convergence for functions of negatively associated random variables. J. Multivariate Anal. *78* (2001), 272–298.

[20] *L. X. Zhang and J. W. Wen*: A weak convergence for negatively associated fields. Statist. Probab. Lett. *53* (2001), 259–267.

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