**Czechoslovak Mathematical Journal, 61 (136) (2011), 3–6**

# A SHORT DIRECT CHARACTERIZATION OF GS-QUASIGROUPS

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(Received April 6, 2005)

*Abstract.* The theorem about the characterization of a GS-quasigroup by means of a commutative group in which there is an automorphism which satisfies certain conditions, is proved directly.

*Keywords*: GS-quasigroup, commutative group

*MSC 2010*: 20N05

## 1. Introduction

The concept of a GS-quasigroup is defined in [1].

A quasigroup $(Q, \cdot)$ is called a GS-quasigroup if it satisfies the (mutually equivalent) identities

$$a(ab \cdot c) \cdot c = b, \tag{1.1}$$

$$a \cdot (a \cdot bc)c = b, \tag{1.2}$$

and the identity of idempotency

$$aa = a. \tag{1.3}$$

It can be proved that GS-quasigroups are medial quasigroups, i.e. the identity

$$ab \cdot cd = ac \cdot bd \tag{1.4}$$

is valid. Namely, we have successively

$$ac \cdot (ab \cdot cd)d \overset{(1.2)}{=} a[ab \cdot (ab \cdot cd)d] \cdot (ab \cdot cd)d \overset{(1.1)}{=} b \overset{(1.2)}{=} ac \cdot (ac \cdot bd)d,$$

where from (1.4) follows.

As a consequence of the identity of mediality the considered GS-quasigroup $(Q, \cdot)$ satisfies the identities of elasticity and the left and right distributivity, i.e. we have the identities

$$a \cdot ba = ab \cdot a, \tag{1.5}$$

$$a \cdot bc = ab \cdot ac, \tag{1.6}$$

$$ab \cdot c = ac \cdot bc. \tag{1.7}$$

Further, the identities

$$a(ab \cdot c) = b \cdot bc, \tag{1.8}$$

$$(a \cdot bc)c = ab \cdot b \tag{1.9}$$

are also valid in any GS-quasigroup. Namely, we have successively

$$a(ab \cdot c) \cdot c \overset{(1.1)}{=} b \overset{(1.1)}{=} b(bb \cdot c) \cdot c \overset{(1.3)}{=} (b \cdot bc)c$$

wherefrom the identity (1.8) follows. Analogously, by virtue of

$$a \cdot (a \cdot bc)c \overset{(1.2)}{=} b \overset{(1.2)}{=} a \cdot (a \cdot bb)b \overset{(1.3)}{=} a(ab \cdot b)$$

we get the identity (1.9).

**Example 1.1.** Let $(G, +)$ be a commutative group in which there is an automorphism $\varphi$ which satisfies the identity

$$(\varphi \circ \varphi)(a) - \varphi(a) - a = 0. \tag{1.10}$$

If the binary operation on the set $G$ is defined by the identity

$$ab = a + \varphi(b - a), \tag{1.11}$$

then it can be proved that $(G, \cdot)$ is a GS-quasigroup ([1]).

## 2. A direct characterization of GS-quasigroups

We will prove that Example 1.1 is a characteristic example of a GS-quasigroup, namely, any GS-quasigroup can be obtained from a commutative group in the way given in Example 1.1.

**Theorem 2.1.** *Let $(Q,\cdot)$ be a* GS-*quasigroup, then there is a commutative group $(Q,+)$ and its automorphism $\varphi$ which satisfies the identities* (1.10) *and* (1.11).

P r o o f. Let 0 be a given point. If we define the addition of points in $Q$ by

$$a+b=0(0a\cdot b0)\cdot 0 \tag{2.1}$$

then $(Q,+)$ is a commutative group with the neutral element 0. Let us prove the above in the following way:

$$\begin{aligned} a+b &\overset{(2.1)}{=} 0(0a\cdot b0)\cdot 0 \overset{(1.4)}{=} 0(0b\cdot a0)\cdot 0 \overset{(2.1)}{=} b+a,\\ a+0 &\overset{(2.1)}{=} 0(0a\cdot 00)\cdot 0 \overset{(1.3)}{=} 0(0a\cdot 0)\cdot 0 \overset{(1.1)}{=} a. \end{aligned}$$

For

$$-a=0a\cdot 0$$

we get

$$a+(-a)\overset{(2.1)}{=} 0[0a\cdot(0a\cdot 0)0]\cdot 0 \overset{(1.5)}{=} 0\cdot[0a\cdot(0a\cdot 0)0]0 \overset{(1.1)}{=} 0\cdot 0 \overset{(1.3)}{=} 0.$$

Now, we shall prove the associativity. If we introduce the abbreviation $a+b=d$ we get

$$\begin{aligned} (a+b)+c &= d+c \overset{(2.1)}{=} 0(0d\cdot c0)\cdot 0 \overset{(1.6),(1.7)}{=} (0\cdot 0d)0\cdot(0\cdot c0)0\\ &\overset{(1.5)}{=} 0(0d\cdot 0)\cdot(0\cdot c0)0 \overset{(1.8),(1.9)}{=} (d\cdot d0)(0c\cdot c). \end{aligned}$$

Because of

$$(d\cdot d0)0 \overset{(1.9)}{=} dd\cdot d \overset{(1.3)}{=} d=a+b \overset{(2.1)}{=} 0(0a\cdot b0)\cdot 0$$

we get

$$d\cdot d0=0(0a\cdot b0)\overset{(1.6)}{=}(0\cdot 0a)(0\cdot b0).$$

On the other hand, the following identities

$$\begin{aligned} (a+b)+c &= (d\cdot d0)(0c\cdot c)=(0\cdot 0a)(0\cdot b0)\cdot(0c\cdot c)\overset{(1.4)}{=}(0\cdot 0a)(0c)\cdot(0\cdot b0)c\\ &\overset{(1.6)}{=} 0(0a\cdot c)\cdot(0\cdot b0)c \overset{(1.4)}{=} 0(0\cdot b0)\cdot(0a\cdot c)c \end{aligned}$$

are valid. Similarly we have the identity

$$(c+b)+a=0(0\cdot b0)\cdot(0c\cdot a)a.$$

However, we have

$$(0c \cdot a)a \overset{(1.7)}{=} (0a \cdot ca)a \overset{(1.9)}{=} (0a \cdot c)c.$$

So, the previous equality yields

$$a + (b + c) = (c + b) + a = 0(0 \cdot b0) \cdot (0c \cdot a)a = 0(0 \cdot b0) \cdot (0a \cdot c)c = (a + b) + c.$$

The mapping $\varphi\colon\ Q \to Q$ defined by $\varphi(a) = 0a$ is an automorphism of the group $(Q, +)$ so that the identities (1.10) and (1.11) hold. Let us prove it like this:

$$\begin{aligned}\varphi(a) + \varphi(b) = 0a + 0b &\overset{(2.1)}{=} [0 \cdot (0 \cdot 0a)(0b \cdot 0)]0 \overset{(1.5)}{=} [0 \cdot (0 \cdot 0a)(0 \cdot b0)]0\\ &\overset{(1.6)}{=} [0 \cdot 0(0a \cdot b0)]0 \overset{(1.5)}{=} 0[0(0a \cdot b0) \cdot 0] \overset{(2.1)}{=} 0(a + b) = \varphi(a + b).\end{aligned}$$

Analogously, it can be proved that the mapping $\psi\colon\ Q \to Q$ defined by $\psi(a) = a0$ is also an automorphism of the group $(Q, +)$.

For any points $a$, $b$ the following identities hold:

$$\psi(a)+\varphi(b) = a0+0b \overset{(2.1)}{=} [0\cdot(0\cdot a0)(0b\cdot 0)]0 \overset{(1.6),(1.7)}{=} [0(0\cdot a0)\cdot 0][0(0b\cdot 0)\cdot 0] \overset{(1.5),(1.1)}{=} ab.$$

This equality and (1.3) immediately imply

$$\psi(a) = a - \varphi(a).$$

By virtue of

$$-a = 0a \cdot 0 = \psi(\varphi(a)) = \varphi(a) - \varphi(\varphi(a))$$

the identity (1.10) follows.

Finally, it remains to prove the identity (1.11) which can actually be achieved from the following

$$ab = \psi(a) + \varphi(b) = a - \varphi(a) + \varphi(b) = a + \varphi(b - a).$$

□

**Acknowledgement.** The author would like to thank the Editorial Board on the information about the paper from 1993 in which Havel and Sedlářová also considered the characterization of GS-quasigroups. This paper is included in the references under [2].

*References*

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