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POSITIVE SOLUTIONS FOR ELLIPTIC PROBLEMS WITH CRITICAL NONLINEARITY AND COMBINED SINGULARITY

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Abstract. Consider a class of elliptic equation of the form

$$-\Delta u - \frac{\lambda}{|x|^2}u = u^{2^*-1} + \mu u^{-q} \quad \text{in } \Omega \setminus \{0\}$$

with homogeneous Dirichlet boundary conditions, where $0 \in \Omega \subset \mathbb{R}^N\,(N \geqslant 3)$, $0 < q < 1$, $0 < \lambda < (N-2)^2/4$ and $2^* = 2N/(N-2)$. We use variational methods to prove that for suitable μ, the problem has at least two positive weak solutions.

Keywords: multiple positive solutions, singular nonlinearity, critical nonlinearity, Hardy term

MSC 2010: 35J20, 35J65

1. Introduction

In this note we study the existence of multiple positive weak solutions of the equation

$$(P_{\lambda,\mu}) \qquad \begin{cases} -\Delta u - \dfrac{\lambda}{|x|^2}u = u^{2^*-1} + \mu u^{-q} & \text{in } \Omega \setminus \{0\}, \\ u(x) > 0 \quad \text{in} \quad \Omega \setminus \{0\}, \quad u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 \in \Omega$ and $\Omega \subset \mathbb{R}^N (N \geqslant 3)$ is a bounded domain with smooth boundary, $2^* = 2N/(N-2)$ is the critical Sobolev exponent, $0 < \lambda < \Lambda = ((N-2)/2)^2$ and $0 < q < 1$. We say $u \in H_0^1(\Omega)$ is a weak solution of $(P_{\lambda,\mu})$ if for any $\varphi \in H_0^1(\Omega)$, we have

$$\int \Big(\nabla u \nabla \varphi - \frac{\lambda}{|x|^2}u\varphi - \mu u^{-q}\varphi - |u|^{2^*-2}u\varphi\Big) = 0.$$

Due to the Sobolev embedding theorem and the Hardy inequality (for any $u \in H_0^1(\Omega)$, $\int_\Omega |x|^{-2}|u|^2\,\mathrm{d}x \leqslant \Lambda^{-1}|\nabla u|^2$), $(P_{\lambda,\mu})$ is variational in nature. Finding weak solutions of $(P_{\lambda,\mu})$ is equivalent to seeking critical points of the functional

$$I(u) = \frac{1}{2}\int \Big(|\nabla u|^2 - \frac{\lambda}{|x|^2}u^2\Big) - \frac{\mu}{1-q}\int |u|^{1-q} - \frac{1}{2^*}\int |u|^{2^*}, \quad u \in H_0^1(\Omega)).$$

Problems like $(P_{\lambda,\mu})$ have attracted great interests in the last two decades. When $\lambda = 0$ and u^{2^*-1} is replaced by u^p with $1 < p < 2^* - 1$, Coclite et al. [6] proved that there is μ_1 such that the problem has at least one positive solution for $0 < \mu < \mu_1$ and has no positive solution for $\mu > \mu_1$. Sun et al. [8] proved the existence of two positive solutions if $0 < q < 1$, $\lambda = 0$, $\mu > 0$ suitably small and u^{2^*-1} replaced by u^p with $1 < p < 2^* - 1$. Hirano et al. [7] proved that there is $\mu_2 > 0$ such that the problem has at least two positive solutions in the case $0 < q < 1$, $\lambda = 0$ and $0 < \mu < \mu_2$. The purpose here is to get two positive solutions of $(P_{\lambda,\mu})$ for $\lambda \neq 0$. Our main result is

Theorem 1.1. *Let $0 < \lambda < \Lambda$ and $0 < q < 1$. Then there is $\mu_* > 0$ such that for any $\mu \in (0, \mu_*)$, $(P_{\lambda,\mu})$ possesses at least two positive solutions.*

To get the existence of multiple solutions, we use variational methods. Comparing $(P_{\lambda,\mu})$ with the previous works [6], [8], [7], we are facing three difficulties at the same time: (1) because of the critical nonlinearity u^{2^*-1}, the functional I does not satisfy a global Palais-Smale ((PS) in short) conditions; (2) since $(P_{\lambda,\mu})$ contains a Hardy term, we know that the solution does not belong to $L^\infty(\Omega)$; and (3) the functional I is not differentiable due to the singular nonlinearity u^{-q}. We need to use the methods recently developed in [4], [5] and some ideas of [1], [7] to overcome them.

The paper is organized as follows: in Section 2, we give some preliminaries; in Section 3, we prove Theorem 1.1.

Throughout this paper $\int_\Omega \cdot\,\mathrm{d}x$ is simply denoted by $\int \cdot$; $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is the closure of $C_0^\infty(\mathbb{R}^N)$ under the norm $\|\cdot\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = \int |\cdot|^2$; and $H_0^1(\Omega)$ is the standard Sobolev space with the usual norm.

2. Preliminaries

The following proposition was taken from [3], [9] and will play an important role in what follows.

Proposition 2.1. *For $0 < \lambda < \Lambda = (N-2)^2/4$, equation*

$$-\Delta u - \frac{\lambda}{|x|^2} u = |u|^{2^*-2} u, \qquad x \in \mathbb{R}^N \setminus \{0\},\ u(x) \to 0 \text{ as } |x| \to +\infty, \tag{2.1}$$

has a family of solutions

$$U_\varepsilon(x) = \frac{[4\varepsilon(\Lambda-\lambda)N/(N-2)]^{(N-2)/4}}{[\varepsilon|x|^{\gamma'/\sqrt{\Lambda}} + |x|^{\gamma/\sqrt{\Lambda}}]^{(N-2)/2}}, \quad \varepsilon > 0,$$

where $\Lambda = (\frac{1}{2}(N-2))^2$, $\gamma' = \sqrt{\Lambda} - \sqrt{\Lambda - \lambda}$, $\gamma = \sqrt{\Lambda} + \sqrt{\Lambda-\lambda}$. *Moreover,* $U_\varepsilon(x)$ *is the unique positive radial symmetric solution of Eq.* (2.1) *up to a dilation, and* $U_\varepsilon(x)$ *is the extremal function of the minimization problem*

$$S_\lambda = \inf\left\{ \int_{\mathbb{R}^N} \left(|\nabla u|^2 - \frac{\lambda}{|x|^2}|u|^2\right) \mathrm{d}x\colon\ u \in \mathcal{D}^{1,2}(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |u|^{2^*}\,\mathrm{d}x = 1 \right\}.$$

Clearly,

$$\int_{\mathbb{R}^N} |U_\varepsilon(x)|^{2^*}\,\mathrm{d}x = \int_{\mathbb{R}^N} \left(|\nabla U_\varepsilon|^2 - \frac{\lambda}{|x|^2} U_\varepsilon^2\right) \mathrm{d}x = S_\lambda^{N/2}.$$

According to the proof of [4, Theorem 1.1], we have the following exact local behavior of the solutions of $(P_{\lambda,\mu})$.

Proposition 2.2. *Let $0 < \lambda < \Lambda$. If $u \in H_0^1(\Omega)$ is a positive solution of $(P_{\lambda,\mu})$, then*

$$K_2|x|^{-(\sqrt{\Lambda}-\sqrt{\Lambda-\lambda})} \leqslant |u(x)| \leqslant K_1|x|^{-(\sqrt{\Lambda}-\sqrt{\Lambda-\lambda})}, \qquad x \in B(0,r) \setminus \{0\} \tag{2.2}$$

for $r > 0$ sufficiently small and some positive constants K_1, K_2.

Define a cut-off function $\zeta(x) = 1$ if $|x| \leqslant \delta$, $\zeta(x) = 0$ if $|x| \geqslant 2\delta$, $\zeta(x) \in C_0^1(\Omega)$ and $|\zeta(x)| \leqslant 1$, $|\nabla\zeta(x)| \leqslant C$. Denote $v_\varepsilon(x) = \zeta(x)U_\varepsilon(x)$. Then using an argument similar to [5, Proposition 2.4], we have the following lemma.

Lemma 2.1. *If $u \in H_0^1(\Omega)$ is a positive solution of $(P_{\lambda,\mu})$, then for $\varepsilon > 0$ sufficiently small,*

$$\int u^{2^*-1} v_\varepsilon = O(\varepsilon^{\frac{N-2}{4}}), \qquad \int u v_\varepsilon^{2^*-1} = O(\varepsilon^{\frac{N-2}{4}}).$$

Next, we define some Nehari type sets, which are relevant in getting multiple positive solutions. Denote $\|u\|_\lambda^2 = \int(|\nabla u|^2 - \lambda|x|^{-2}u^2)$ and set

$$\begin{aligned}
\mathcal{M} &:= \Big\{u \in H_0^1(\Omega)\colon\ \|u\|_\lambda^2 = \mu\int|u|^{1-q} + \int|u|^{2^*}\Big\},\\
\mathcal{M}^+ &:= \Big\{u \in \mathcal{M}\colon\ (1+q)\|u\|_\lambda^2 > (2^*-1+q)\int|u|^{2^*}\Big\},\\
\mathcal{M}^0 &:= \Big\{u \in \mathcal{M}\colon\ (1+q)\|u\|_\lambda^2 = (2^*-1+q)\int|u|^{2^*}\Big\} \quad \text{and}\\
\mathcal{M}^- &:= \Big\{u \in \mathcal{M}\colon\ (1+q)\|u\|_\lambda^2 < (2^*-1+q)\int|u|^{2^*}\Big\}.
\end{aligned}$$

Define also the minimization problems

$$d_+ = \inf_{u\in\mathcal{M}^+} I(u). \tag{2.3}$$

It is easy to see that $d_+ < 0$ for $\mu > 0$ and $d_+ \to 0$ as $\mu \to 0$. Take $\mu_3 > 0$ such that $d_+ + N^{-1}S_\lambda^{N/2} > 0$ for any $\mu \in (0,\mu_3)$. Denote

$$\mu_4 = \frac{2^*-2}{2^*-1+q}\Big(\frac{1+q}{2^*-1-q}\Big)^{\frac{N-2}{4}(1+q)} S_\lambda^{\frac{N}{4}(1+q)+\frac{1-q}{2}}|\Omega|^{\frac{1-q-2^*}{2^*}}.$$

Set

$$\mu_* = \min\{\mu_3,\ \mu_4\}.$$

Lemma 2.2. *If $\mu \in (0,\mu_*)$, then $\mathcal{M}^0 = \{0\}$. Moreover, for any $u \neq 0$ there exists a unique $t^+ = t^+(u) > 0$ such that $t^+(u)u \in \mathcal{M}^-$ and*

$$t^+ > T_m := \Big(\frac{\|u\|_\lambda^2}{(2^*-1)\int|u|^{2^*}}\Big)^{\frac{1}{2^*-2}}$$

and

$$I(t^+u) = \max_{t\geqslant T_m} I(tu),$$

and there exists a unique $t^- = t^-(u) > 0$ such that $t^-(u)u \in \mathcal{M}^+$, $t^- < T_{\max}$ and

$$I(t^-u) = \inf_{0\leqslant t\leqslant T_m} I(tu).$$

Proof. The proof is similar to [5, Lemma 3.2]. We omit the details. □

3. Proof of theorem 1.1

In this section we will prove Theorem 1.1. The proof of Theorem 1.1 is based on solving the minimization problem (2.3) and the minimization problem

$$d_- = \inf_{u \in \mathcal{M}^-} I(u). \tag{3.1}$$

We divide the proof into two steps. In the first step, we prove that if there is $w \in \mathcal{M}^+$ such that $d_+ = I(w)$ and there is $v \in \mathcal{M}^-$ such that $d_- = I(v)$, then w and v are two positive weak solutions of $(P_{\lambda,\mu})$. In the second step, we prove that the minima d_+ in (2.3) and d_- in (3.1) are achieved, respectively.

Step 1. Let $w \in \mathcal{M}^+$ be such that $d_+ = I(w)$ and $v \in \mathcal{M}^-$ such that $d_- = I(v)$.

Lemma 3.1. *For each $\varphi \in H_0^1(\Omega)$ and $\varphi \geqslant 0$, we have*

(i) *there is $\varrho_0 > 0$ such that $I(w + \varrho_0\varphi) \geqslant I(w)$ for each $0 \leqslant \varrho < \varrho_0$;*

(ii) *$t_\varrho^- \to 1$ as $\varrho \to 0+$, where t_ϱ^- is the unique positive number satisfying $t_\varrho^- \times (v + \varrho\varphi) \in \mathcal{M}^-$.*

Proof. The proof follows exactly the scheme in the proof of Lemma 3 in [7]. □

Lemma 3.2. *For each $\varphi \in H_0^1(\Omega)$ and $\varphi \geqslant 0$ we have that $w^{-q}\varphi,\ v^{-q}\varphi \in L^1(\Omega)$. Moreover,*

$$\int \Big(\nabla w \nabla \varphi - \frac{\lambda}{|x|^2} w\varphi - \mu w^{-q}\varphi - w^{2^*-1}\varphi\Big) \geqslant 0 \tag{3.2}$$

and

$$\int \Big(\nabla v \nabla \varphi - \frac{\lambda}{|x|^2} v\varphi - \mu v^{-q}\varphi - v^{2^*-1}\varphi\Big) \geqslant 0. \tag{3.3}$$

In particular, $w,\ v > 0$ a.e. in $\Omega \setminus \{0\}$.

Proof. We only prove (3.2) since the proof of (3.3) is similar. Let $\varphi \geqslant 0$ and $\varepsilon > 0$. By (i) of Lemma 3.1 and simple computations we have that

$$\begin{aligned}\frac{\mu}{1-q} \int \frac{(w+\varepsilon\varphi)^{1-q} - w^{1-q}}{\varepsilon} \leqslant{}& \frac{1}{2\varepsilon}\left(\|w+\varepsilon\varphi\|_\lambda^2 - \|w\|_\lambda^2\right) \\ &- \frac{1}{2^*\varepsilon}\left(|w+\varepsilon\varphi|^{2*} - |w|^{2*}\right).\end{aligned}$$

Since the right hand side of the inequality has a finite limit value as $\varepsilon \downarrow 0$ for each $x \in \Omega \setminus \{0\}$, we conclude $\varepsilon^{-1}((w+\varepsilon\varphi)^{1-q} - w^{1-q})$ increases monotonically as $\varepsilon \downarrow 0$

and

$$\lim_{\varepsilon\downarrow 0}\frac{(w+\varepsilon\varphi)^{1-q}-w^{1-q}}{\varepsilon}=\begin{cases}0 & \text{if } \varphi(x)=0,\\ (1-q)w^{-q}\varphi & \text{if } \varphi(x)>0 \text{ and } w(x)>0,\\ \infty & \text{if } \varphi(x)>0 \text{ and } w(x)=0.\end{cases}$$

The monotone convergence theorem yields $w^{-q}\varphi\in L^1(\Omega)$ and we get (3.2). $\square$

Proposition 3.1. *We have that w and v are positive weak solutions of $(P_{\lambda,\mu})$.*

P r o o f. We borrow some ideas from [6], [8]. For any $\varphi\in H_0^1(\Omega)$ and $\varrho>0$, we define $\psi=(w+\varrho\varphi)$ and $\psi^+=\max\{\psi,0\}$. Then $\psi^+\in H_0^1(\Omega)$. Since $w\in\mathcal{M}$, we obtain from (3.2) that

$$\begin{aligned}0&\leqslant\int\Big(\nabla w\nabla\psi^+-\frac{\lambda}{|x|^2}w\psi^+-\mu w^{-q}\psi^+-w^{2^*-1}\psi^+\Big)\\&=\int_{[w+\varrho\varphi>0]}\Big(\nabla w\nabla\psi^+-\frac{\lambda}{|x|^2}w\psi^+-\mu w^{-q}\psi^+-w^{2^*-1}\psi^+\Big)\\&=\int\Big(\nabla w\nabla\psi-\frac{\lambda}{|x|^2}w\psi-\mu w^{-q}\psi-w^{2^*-1}\psi\Big)\\&\quad-\int_{[w+\varrho\varphi\leqslant 0]}\Big(\nabla w\nabla\psi^+-\frac{\lambda}{|x|^2}w\psi^+-\mu w^{-q}\psi^+-w^{2^*-1}\psi^+\Big)\\&\leqslant\varrho\int\Big(\nabla w\nabla\varphi-\frac{\lambda}{|x|^2}w\varphi-\mu w^{-q}\varphi-w^{2^*-1}\varphi\Big)-\varrho\int_{[w+\varrho\varphi\leqslant 0]}\nabla w\nabla\varphi.\end{aligned}$$

Dividing by ϱ and letting $\varrho\to 0$, since the measure of $[w+\varrho\varphi\leqslant 0]$ tends to 0 as $\varrho\to 0$, we get that $\int_{[w+\varrho\varphi\leqslant 0]}\nabla w\nabla\varphi\to 0$. Therefore

$$\int\Big(\nabla w\nabla\varphi-\frac{\lambda}{|x|^2}w\varphi-\mu w^{-q}\varphi-w^{2^*-1}\varphi\Big)\geqslant 0.$$

Since φ is arbitrary, we get that w is a solution of $(P_{\lambda,\mu})$. Similarly, we can prove that v is also a solution of $(P_{\lambda,\mu})$. $\square$

S t e p 2. *The minima d_+ and d_- are achieved.* We only prove that d_- is achieved by some $v\in\mathcal{M}^-$ since proving that d_+ is achieved is similar but quite simpler. Since we are faced with critical nonlinearity and the Hardy term, the functional I does not satisfy (PS) conditions. We need some technique developed in [4], [5] and some ideas from [1], [7] to overcome them. We point out that v_ε and the exact local behavior of w (see Proposition 2.2) play essential roles. From Proposition 2.2, we also know that there is $m>0$ such that $w(x)\geqslant m$ for $x\in\operatorname{supp} w\setminus\{0\}$.

Lemma 3.3. *Under the assumptions of Theorem* 1.1,

$$d_- < I(w) + \frac{1}{N} S_\lambda^{N/2}.$$

P r o o f. First, using an argument similar to the proofs in [7, Lemma 8], we have $t_* > 0$ such that $w + t_* v_\varepsilon \in \mathcal{M}^-$. It remains to prove that

$$\sup\{I(w + tv_\varepsilon)\colon\ t > 0\} < I(w) + \frac{1}{N} S_\lambda^{N/2}. \tag{3.4}$$

Since w is a solution, we obtain by direct computation that

$$\begin{aligned} I(w + tv_\varepsilon) - I(w) &= \frac{t^2}{2}\|v_\varepsilon\|_\lambda^2 + t\int \Big(\nabla w \nabla v_\varepsilon - \frac{\lambda}{|x|^2} w v_\varepsilon\Big) \\ &\quad - \mu \int \Big(\frac{(w + tv_\varepsilon)^{1-q}}{1-q} - \frac{w^{1-q}}{1-q}\Big) - \int \Big(\frac{(w + tv_\varepsilon)^{2^*}}{2^*} - \frac{w^{2^*}}{2^*}\Big) \\ &= \frac{t^2}{2}\|v_\varepsilon\|_\lambda^2 - \mu \int \Big(\frac{(w + tv_\varepsilon)^{1-q}}{1-q} - \frac{w^{1-q}}{1-q} - w^{-q} t v_\varepsilon\Big) \\ &\quad - \int \Big(\frac{(w + tv_\varepsilon)^{2^*}}{2^*} - \frac{w^{2^*}}{2^*} - w^{2^*-1} t v_\varepsilon\Big). \end{aligned}$$

Note that the following inequality (see [7]) holds: there is $\alpha > 0$ and $0 < \delta < N/(N-2)$ such that

$$\mu\Big(\frac{(r+s)^{1-q}}{1-q} - \frac{r^{1-q}}{1-q} - r^{-q}s\Big) \geqslant -\alpha s^\delta \quad \text{for each } r \geqslant m \text{ and } s \geqslant 0.$$

Another useful inequality is: for $r,\ s > 0$ we have

$$\frac{(r+s)^{2^*}}{2^*} - \frac{r^{2^*}}{2^*} - \frac{s^{2^*}}{2^*} - r^{2^*-1}s \geqslant r s^{2^*-1}.$$

Thus we get that

$$I(w + tv_\varepsilon) - I(w) \leqslant \frac{t^2}{2}\|v_\varepsilon\|_\lambda^2 - \frac{t^{2^*}}{2^*}\int |v_\varepsilon|^{2^*} - t^{2^*-1}\int w v_\varepsilon^{2^*-1} + \alpha t^\delta \int v_\varepsilon^\delta.$$

So when $t \to 0$ and $t \to \infty$, then $I(w + tv_\varepsilon) \to 0$. Hence we only consider the right hand side of the above inequality in the case of $t \in [t_0, t_1]$ for some $0 < t_0 < t_1 < \infty$.

Hence, we obtain from Lemma 2.1 that

$$\begin{aligned}\sup_{t>0} I(w+tv_\varepsilon)-I(w) &\leqslant \frac{1}{N}\Big(\int(|\nabla v_\varepsilon|^2-\frac{\lambda}{|x|^2}|v_\varepsilon|^2)\Big)^{\frac{2^*}{2^*-2}} \\ &\quad -\Big(\int|v_\varepsilon|^{2^*}\Big)^{-\frac{2}{2^*-2}} - O(\varepsilon^{\frac{N-2}{4}})+O(\varepsilon^{\frac{N-2}{4}\delta}) \\ &= \frac{1}{N}S_\lambda^{\frac{N}{2}}+O(\varepsilon^{\frac{N-2}{2}})-O(\varepsilon^{\frac{N-2}{4}})+O(\varepsilon^{\frac{N-2}{4}\delta}) \\ &< \frac{1}{N}S_\lambda^{\frac{N}{2}} \quad \text{for} \quad \varepsilon>0 \quad \text{sufficiently small.}\end{aligned}$$

The proof is complete. □

Lemma 3.4. *The minimum* d_- *in* (3.1) *is achieved by* $v\in\mathcal{M}^-$ *with* $I(v)=d_-$.

P r o o f. Let $\{v_n\}_{n\in\mathbb{N}}\subset\mathcal{M}^-$ be such that $I(v_n)\to d_-$. It is easy to see that $\{v_n\}$ is bounded in $H_0^1(\Omega)$. We may assume that $v_n \rightharpoonup v$ weakly in $H_0^1(\Omega)$. Set $z_n=v_n-v$ and assume that

$$\|z_n\|_\lambda^2\to a^2 \quad \text{and} \quad \int|z_n|^{2^*}\to b^{2^*}.$$

Since $v_n\in\mathcal{M}$, by using the Brezis-Lieb lemma and the Sobolev embedding theorem we get that

$$a^2+\|v\|_\lambda^2=\mu\int|v|^{1-q}+b^{2^*}+\int|v|^{2^*}.$$

We claim that $v\geqslant 0$ and $v\neq 0$. Indeed, if $v=0$, then $a\neq 0$ (since for any $u\in\mathcal{M}^-$, $\|u\|_\lambda$ is bounded away from zero) and this means that

$$d_-=\lim_{n\to\infty}I(v_n)=I(0)+\frac{1}{2}a^2-\frac{b^{2^*}}{2^*}\geqslant\frac{1}{N}S_\lambda^{N/2},$$

which contradicts the previous lemma.

From the assumption on $\mu\in(0,\mu_*)$ we have $0<t^+<T_m<t^-$ such that $t^+v\in\mathcal{M}^+$ and $t^-v\in\mathcal{M}^-$. For $t>0$, we define

$$\eta(t)=\frac{a^2}{2}t^2-\frac{b^{2^*}}{2^*}t^{2^*} \quad \text{and} \quad g(t)=I(tv)+\eta(t).$$

Now, we consider the cases

(i) $t^-<1$;

(ii) $t^-\geqslant 1$ and $b>0$, and

(iii) $t^-\geqslant 1$ and $b=0$.

C a s e (i). From $t^- < 1$, $g'(1) = 0$ and $g'(t^-) > 0$ we can see that g is increasing on $[t^-, 1]$. Then we have

$$d_- = g(1) > g(t^-) \geqslant I(t^- v) + \frac{(t^-)^2}{2}(a^2 - b^{2^*}) > I(t^- v) \geqslant d_-,$$

which is a contradiction.

C a s e (ii). We set $T_0 = (a^2/b^{2^*})^{(N-2)/4}$. We know that η attains the unique maximum at T_0 and $\eta(T_0) \geqslant N^{-1} S_\lambda^{N/2}$. Moreover, $\eta'(t) > 0$ for $0 < t < T_0$ and $\eta'(t) < 0$ for $t > T_0$.

By the assumption $\mu \in (0, \mu_*)$, we also know $g(1) \geqslant g(T_0)$. If $T_0 \leqslant 1$, we have

$$d_- = g(1) \geqslant g(T_0) = I(T_0 v) + \eta(T_0) \geqslant I(T_0 v) + \frac{1}{N} S_\lambda^{N/2},$$

which contradicts the previous lemma. Thus we have $T_0 > 1$. By virtue of $g'(t) \leqslant 0$ for $t \geqslant 1$, we obtain $\frac{\partial}{\partial t} I(tv) \leqslant -\eta'(t) \leqslant 0$ for $1 \leqslant t \leqslant T_0$ and

$$d_- = g(1) = I(v) + \frac{1}{2}a^2 - \frac{b^{2^*}}{2^*} \geqslant I(v) + \frac{1}{N} S_\lambda^{N/2},$$

which also contradicts the previous lemma.

C a s e (iii). If $a \neq 0$, then we obtain from the fact that $v_n \in \mathcal{M}^-$ by some computations that $(\partial/\partial t) I(tv)|_{t=1} < 0$ and $(\partial^2/\partial t^2) I(tv)|_{t=1} < 0$, which contradicts $t^- \geqslant 1$. Thus $a = 0$ and $v_n \to v$ strongly in $H_0^1(\Omega)$. Hence, we have $v \in \mathcal{M}^-$ and $I(v) = d_-$.

The proof of Lemma 3.4 is complete. □

P r o o f of Theorem 1.1. The proof follows directly from Lemma 3.4 and Proposition 3.1. □

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