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CAUCHY'S RESIDUE THEOREM FOR A CLASS OF REAL VALUED FUNCTIONS

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Abstract. Let $[a,b]$ be an interval in $\mathbb{R}$ and let F be a real valued function defined at the endpoints of $[a,b]$ and with a certain number of discontinuities within $[a,b]$. Assuming F to be differentiable on a set $[a,b]\setminus E$ to the derivative f, where E is a subset of $[a,b]$ at whose points F can take values $\pm\infty$ or not be defined at all, we adopt the convention that F and f are equal to 0 at all points of E and show that $\mathcal{KH}$-vt $\int_a^b f = F(b) - F(a)$, where $\mathcal{KH}$-vt denotes the total value of the *Kurzweil-Henstock* integral. The paper ends with a few examples that illustrate the theory.

Keywords: Kurzweil-Henstock integral, Cauchy's residue theorem

MSC 2010: 26A39, 26A24

1. Introduction

Let $[a,b]$ be a compact interval in $\mathbb{R}$. It is an old result that for an ACG_δ function $F\colon [a,b]\mapsto \mathbb{R}$ on $[a,b]$, which is differentiable almost everywhere on $[a,b]$, its derivative f is integrable (in the *Kurzweil-Henstock* sense) on $[a,b]$ and $\mathcal{KH}$-$\int_a^b f = F(b) - F(a)$, [3, Theorem 9.17]. The aim of this note is to define a new definite integral named the total *Kurzweil-Henstock* integral that can be used to extend the above mentioned result to any real valued function F defined and differentiable on $[a,b]\setminus E$, where E is a certain subset of $[a,b]$ at whose points F can take values $\pm\infty$ or not be defined at all. Unless otherwise stated, in what follows we assume that the endpoints of $[a,b]$ do not belong to E. Now, define point functions $F_{ex}\colon [a,b]\mapsto\mathbb{R}$ and $D_{ex}F\colon [a,b]\mapsto\mathbb{R}$ by extending F and its derivative f from

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$[a,b]\setminus E$ to E by $F_{ex}(x)=0$ and $D_{ex}F(x)=0$ for $x\in E$, so that

$$(1.1)\qquad F_{ex}(x)=\begin{cases}F(x) & \text{if } x\in[a,b]\setminus E,\\ 0 & \text{if } x\in E \text{ and}\end{cases}$$
$$D_{ex}F(x)=\begin{cases}f(x) & \text{if } x\in[a,b]\setminus E,\\ 0 & \text{if } x\in E.\end{cases}$$

2. Preliminaries

A partition $P[a,b]$ of $[a,b]\in\mathbb{R}$ is a finite set (collection) of interval-point pairs $\{([a_i,b_i],x_i)\colon\ i=1,\dots,\nu\}$, such that the subintervals $[a_i,b_i]$ are non-overlapping, $\bigcup\limits_{i\leqslant\nu}[a_i,b_i]=[a,b]$ and $x_i\in[a_i,b_i]$. The points $\{x_i\}_{i\leqslant\nu}$ are the tags of $P[a,b]$, [2]. It is evident that a given partition of $[a,b]$ can be tagged in infinitely many ways by choosing different points as tags. If E is a subset of $[a,b]$, then the restriction of $P[a,b]$ to E is a finite collection of $([a_i,b_i],x_i)\in P[a,b]$ such that each $x_i\in E$. In symbols, $P[a,b]\big|_E=\{([a_i,b_i],x_i)\colon\ x_i\in E, i=1,\dots,\nu\}$. Let $\mathcal{P}[a,b]$ be the family of all partitions $P[a,b]$ of $[a,b]$. Given $\delta\colon\ [a,b]\mapsto\mathbb{R}_+$, named a gauge, a partition $P[a,b]\in\mathcal{P}[a,b]$ is called *δ-fine* if $[a_i,b_i]\subseteq(x_i-\delta(x_i),x_i+\delta(x_i))$. By *Cousin's* lemma the set of *δ-fine* partitions of $[a,b]$ is nonempty, [4].

The collection $\mathcal{I}([a,b])$ is the family of compact subintervals I of $[a,b]$. The *Lebesgue* measure of the interval I is denoted by $|I|$. Any real valued function defined on $\mathcal{I}([a,b])$ is an interval function. For a function $f\colon\ [a,b]\mapsto\mathbb{R}$, the associated interval function of f is an interval function $f\colon\ \mathcal{I}([a,b])\mapsto\mathbb{R}$, again denoted by f, [5]. If $f\equiv 0$ on $[a,b]$ then its associated interval function is trivial.

A function $f\colon\ [a,b]\mapsto\mathbb{R}$ is said to be *Kurzweil-Henstock* integrable on $[a,b]$ to a real number A if for every $\varepsilon>0$ there exists a gauge $\delta_\varepsilon\colon\ [a,b]\mapsto\mathbb{R}_+$ such that $\Big|\sum\limits_{i\leqslant\nu}\big[f(x_i)\big|[a_i,b_i]\big|\big]-A\Big|<\varepsilon$, whenever $P[a,b]$ is a *δ_ε-fine* partition of $[a,b]$. In symbols, $A=\mathcal{KH}\text{-}\int_a^b f$.

3. Main results

In what follows we will use the notation

$$(3.1)\qquad \Xi_f(P[a,b])=\sum_{i\leqslant\nu}[f(x_i)|b_i-a_i|]\quad\text{and}\quad \Sigma_\Phi(P[a,b])=\sum_{i\leqslant\nu}[\Phi(b_i)-\Phi(a_i)].$$

Now, we are in a position to introduce the total *Kurzweil-Henstock* integral.

Definition 3.1. For any compact interval $[a,b] \in \mathbb{R}$ let E be a non-empty subset of $[a,b]$. A function $f\colon [a,b] \mapsto \mathbb{R}$ is said to be totally *Kurzweil-Henstock* integrable to a real number $\Im$ on $[a,b]$ if there exists a nontrivial interval function $\Phi\colon \mathcal{I}([a,b]) \mapsto \mathbb{R}$ with the following property: for every $\varepsilon > 0$ there exists a gauge δ_ε on $[a,b]$ such that $|\Xi_f(P[a,b]) - \Sigma_\Phi(P[a,b]|_{[a,b]\setminus E})| < \varepsilon$ and $\Sigma_\Phi(P[a,b]) = \Im$, whenever $P[a,b] \in \mathcal{P}[a,b]$ is a δ_ε-*fine* partition and $P[a,b]|_{[a,b]\setminus E}$ is its restriction to $[a,b] \setminus E$. In symbols, $\mathcal{KH}\text{-vt}\int_a^b f = \Im$.

Definition 3.2. Let E be a non-empty subset of $[a,b]$. Then an interval function $\Phi\colon \mathcal{I}([a,b]) \mapsto \mathbb{R}$ is said to be basically summable ($\mathrm{BS}_{\delta_\varepsilon}$) to the sum $\Re$ on E, if there exists a real number $\Re$ with the following property: given $\varepsilon > 0$ there exists a gauge δ_ε on $[a,b]$ such that $|\Sigma_\Phi(P[a,b]|_E) - \Re| < \varepsilon$, whenever $P[a,b] \in \mathcal{P}[a,b]$ is a δ_ε-*fine* partition and$P[a,b]|_E$ is its restriction to E. If E can be written as a countable union of sets on each of which the interval function Φ is $\mathrm{BS}_{\delta_\varepsilon}$, then Φ is said to be $\mathrm{BSG}_{\delta_\varepsilon}$ on E.

Our main result reads as follows.

Theorem 3.1. *For any compact interval $[a,b] \in \mathbb{R}$ let E be a non-empty subset of $[a,b]$ at whose points a real valued function F can take values $\pm\infty$ or not be defined at all. If F is defined and differentiable on the set $[a,b] \setminus E$, then $D_{ex}F$ is totally Kurzweil-Henstock integrable on $[a,b]$ and*

$$\mathcal{KH}\text{-vt}\int_a^b D_{ex}F = F(b) - F(a). \tag{3.2}$$

If the associated interval function of F_{ex} defined by (1.1) *is in addition basically summable (BS_{δ_ε}) to the sum $\Re$ on E, then*

$$F(b) - F(a) = \mathcal{KH}\text{-}\int_a^b D_{ex}F + \Re. \tag{3.3}$$

Before starting with the proof we give the following lemma.

Lemma 3.1. *Let E be a non-empty subset of $[a,b]$. If a function $f\colon [a,b] \mapsto \mathbb{R}$ is totally Kurzweil-Henstock integrable on $[a,b]$ and Φ is basically summable (BS_{δ_ε}) to the sum $\Re$ on E, then f is Kurzweil-Henstock integrable on $[a,b]$ and*

$$\mathcal{KH}\text{-vt}\int_a^b f = \mathcal{KH}\text{-}\int_a^b f + \Re. \tag{3.4}$$

P r o o f. Given $\varepsilon > 0$ we will construct a gauge for f as follows. Since f is totally *Kurzweil-Henstock* integrable on $[a, b]$ it follows from Definition 3.1 that there exist a real number $\Im$ and an interval function Φ with the following property: for every $\varepsilon > 0$ there exists a gauge δ_ε^* on $[a, b]$ such that $|\Xi_f(P[a, b]) - \Sigma_\Phi(P[a, b]|_{[a,b]\setminus E})| < \varepsilon$ and $\Sigma_\Phi(P[a, b]) = \Im$, whenever $P[a, b] \in \mathcal{P}[a, b]$ is a δ_ε^*-*fine* partition and $P[a, b]|_{[a,b]\setminus E}$ is its restriction to $[a, b] \setminus E$. Choose a gauge $\delta_\varepsilon^\star(x)$ as required in Definition 3.2 above. The function $\delta_\varepsilon = \min(\delta_\varepsilon^*, \delta_\varepsilon^\star)$ is a gauge on $[a, b]$. We now let $P[a, b] = \{([a_i, b_i], x_i)\colon\ i = 1, \ldots, \nu\}$ be a δ_ε-*fine* partition of $[a, b]$. It is readily seen that

$$\begin{aligned} |\Xi_f(P[a, b]) - \Im + \Re| &= |\Xi_f(P[a, b]) - \Im + \Sigma_\Phi(P[a, b]|_E) - [\Sigma_\Phi(P[a, b]|_E) - \Re]| \\ &\leqslant |\Xi_f(P[a, b]) - \Sigma_\Phi(P[a, b]|_{[a,b]\setminus E})| + |\Sigma_\Phi(P[a, b]|_E) - \Re| < 2\varepsilon. \end{aligned}$$

Therefore, f is *Kurzweil-Henstock* integrable on $[a, b]$ and $\mathcal{KH}\text{-}\int_a^b f = \Im - \Re$, that is

$$\mathcal{KH}\text{-vt}\int_a^b f = \mathcal{KH}\text{-}\int_a^b f + \Re.$$

□

We now turn to the proof of Theorem 3.1.

P r o o f. Given $\varepsilon > 0$, by definition of f at the point $x \in [a, b] \setminus E$ there exists $\delta_\varepsilon(x) > 0$ such that if $x \in [u, v] \subseteq [x - \delta_\varepsilon(x), x + \delta_\varepsilon(x)]$ and $x \in [a, b] \setminus E$, then

$$|F(v) - F(u) - f(x)(v - u)| < \varepsilon(v - u).$$

For F_{ex} defined by (1.1) let $F_{ex}\colon \mathcal{I}([a, b]) \mapsto \mathbb{R}$ be its associated interval function. We now let $P[a, b] = \{([a_i, b_i], x_i)\colon\ i = 1, \ldots, \nu\}$ be a δ_ε-*fine* partition of $[a, b]$. Since $F(b) - F(a) = \sum\limits_{i=1}^{\nu} [F_{ex}(b_i) - F_{ex}(a_i)]$ and (remember if $x \in E$, then $D_{ex}F = 0$)

$$\begin{aligned} &|\Xi_{D_{ex}F}(P[a, b]) - \Sigma_{F_{ex}}(P[a, b]|_{[a,b]\setminus E})| \\ &\qquad = |\Xi_f(P[a, b]|_{[a,b]\setminus E}) - \Sigma_F(P[a, b]|_{[a,b]\setminus E})| < \varepsilon(b - a), \end{aligned}$$

it follows from Definition 3.1 that $D_{ex}F$ is totally *Kurzweil-Henstock* integrable on $[a, b]$ and

$$\mathcal{KH}\text{-vt}\int_a^b D_{ex}F = F(b) - F(a).$$

Finally, by virtue of *Lemma* 3.1

$$F(b) - F(a) = \mathcal{KH}\text{-}\int_a^b D_{ex}F + \Re.$$

□

By *Definition* 3.2 one can easily see that if $\Re = 0$ then F has negligible variation on E, [1, Definition 5.11]. So, we are now in position to define a residual function of F.

Definition 3.3. Let $F\colon [a,b] \mapsto \mathbb{R}$. A function $\mathcal{R}\colon [a,b] \mapsto \mathbb{R}$ is said to be a residual function of F on $[a,b]$ if given $\varepsilon > 0$ there exists a gauge δ_ε on $[a,b]$ such that $|F(b_i) - F(a_i) - \mathcal{R}(x_i)| < \varepsilon$, whenever $P[a,b] \in \mathcal{P}[a,b]$ is a δ_ε-*fine* partition.

Definition 3.4. Let E be a non-empty subset of $[a,b]$ and let $F\colon [a,b] \mapsto \mathbb{R}$ be a function whose associated interval function $F\colon \mathcal{I}([a,b]) \mapsto \mathbb{R}$ is $\mathrm{BS}_{\delta_\varepsilon}$ ($\mathrm{BSG}_{\delta_\varepsilon}$) to the sum $\Re$ on E. Then, a residual function $\mathcal{R}\colon [a,b] \mapsto \mathbb{R}$ of F is said to be also $\mathrm{BS}_{\delta_\varepsilon}$ ($\mathrm{BSG}_{\delta_\varepsilon}$) to the same sum $\Re$ on E. In symbols, $\sum\limits_{x\in E} \mathcal{R}(x) = \Re$.

Clearly, *Definition* 3.4 establishes a causal connection between *Definitions* 3.2 and 3.3. If E is a countable set, the causality is so obvious. However, if E is an infinite set, then this connection is not necessarily a causal connection. Namely, if $F\colon [a,b] \mapsto \mathbb{R}$ has negligible variation on a subset E of $[a,b]$, which is a countably infinite set, then its residual function $\mathcal{R}$ vanishes identically on E, so that the sum $\sum\limits_{x\in E} \mathcal{R}(x)$ is reduced to the so-called indeterminate expression $\infty \cdot 0$ that has, in this case, the null value. On the contrary, if F has no negligible variation on E, and its residual function $\mathcal{R}$ also vanishes identically on E, as in the case of the *Cantor* function, then the sum $\sum\limits_{x\in E} \mathcal{R}(x)$ is reduced to the indeterminate expression $\infty \cdot 0$ that actually has, in Cantor's case, the numerical value of 1. By *Definition* 3.4, we may rewrite (3.3) as

$$F(b) - F(a) = \mathcal{KH}\text{-}\int_a^b D_{ex}F + \sum_{x\in E} \mathcal{R}(x). \tag{3.5}$$

If f in addition vanishes identically on $[a,b] \setminus E$, then

$$F(b) - F(a) = \sum_{x\in E} \mathcal{R}(x). \tag{3.6}$$

This result is an extension of Cauchy's residue theorem in $\mathbb{R}$.

4. Examples

For an illustration of (3.5) and (3.6) we consider the Heaviside unit function defined by

$$F(x) = \begin{cases} 0 & \text{if } a \leqslant x \leqslant 0, \\ 1 & \text{if } 0 < x \leqslant b. \end{cases} \tag{4.1}$$

In this case, if $a < 0$, then $\mathcal{KH}$-vt $\int_a^b D_{ex}F = 1$, in spite of the fact that $D_{ex}F \equiv 0$ on $[a, b]$. Accordingly, it follows from (3.5) and (3.6) that $\mathcal{R}(0) = 1$, since

$$f(x) = \begin{cases} +\infty & \text{if } x = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{4.2}$$

where f is the derivative of F, and $\mathcal{KH}$-$\int_a^b D_{ex}F = 0$.

Let $[a, b] \subset \mathbb{R}$ be an arbitrary compact interval within which is the point $x = 0$. For an illustration of the result (3.2) of *Theorem* 3.1 we consider the real valued function $F(x) = 1/x$ that is differentiable to $f(x) = -1/x^2$ at all but the exceptional set $\{0\}$ of $[a, b]$. In spite of the fact that f is not *Kurzweil-Henstock* integrable on $[a, b]$ it follows from (3.2) that $\mathcal{KH}$-vt $\int_a^b D_{ex}F = (a - b)/(ab)$. In this case, $\mathcal{R}(x)$ is not defined at the point $x = 0$, that is

$$\mathcal{R}(x) = \begin{cases} +\infty & \text{if } x = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{4.3}$$

and $\mathcal{KH}$-vt $\int_a^b D_{ex}F$ is reduced to the so-called indeterminate expression $\infty - \infty$ (in the sense of the difference of limits) that actually has, in this situation, the real numerical value of $(a - b)/(ab)$.

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