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# INTERSECTION GRAPHS OF SUBGROUPS OF FINITE GROUPS

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*Abstract.* In this paper we classify finite groups with disconnected intersection graphs of subgroups. This solves a problem posed by Csákány and Pollák.

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## 1. Introduction

In [5] Csákány and Pollák defined intersection graphs of nontrivial proper subgroups of groups. This study was inspired by the definition of intersection graphs of nontrivial proper subsemigroups of semigroups due to Bosák (see [2]). Zelinka continued the work on intersection graphs of subgroups of finite abelian groups [9]. Recently, the intersection graph of ideals of rings was studied by Chakrabarty, Ghosh, Mukherjee and Sen [4].

Let $G$ be a finite group different from a cyclic group of prime order. The intersection graph $\Gamma(G)$ of $G$ is the undirected graph (without loops and multiple edges) whose vertices are the nontrivial proper subgroups of $G$ and two vertices are joined by an edge if and only if they have a non-unit intersection, i.e., an intersection containing a non-unit element. If $\Gamma(G)$ has one vertex, then $G$ is a cyclic group of order $p^2$ where $p$ is a prime. Denote the components of the intersection graph $\Gamma(G)$ by $\Gamma_1(G), \Gamma_2(G), \ldots, \Gamma_k(G)$. For every component $\Gamma_i(G)$ for $i = 1, 2, \ldots, k$ we define the block $B_i$ to be the union of all vertices of $\Gamma_i(G)$. Obviously, the block $B_i$ is a union of some maximal subgroups of $G$.

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If the intersection graph of the group $G$ is connected (or disconnected), we say briefly that $G$ is connected (or disconnected). We will speak about the distance of nontrivial proper subgroups $H$ and $K$ of the group $G$, and also about the diameter of a connected group $G$. Denote by $\varrho(H,K)$ the distance of $H$ and $K$, and by $\delta(G)$ the diameter of the connected group $G$. Let again $H$ and $K$ be nontrivial proper subgroups in $G$.

**Definition.** If there exist nontrivial subgroups $L_1,\ldots,L_n$ of $G$ such that $H\sim L_1$, $L_1\sim L_2$, …, $L_{n-1}\sim L_n$, $L_n\sim K$, then we say that $H$ and $K$ are connected by the chain $H\sim L_1\sim L_2\sim\ldots\sim L_n\sim K$. Clearly, in this case $\varrho(H,K)\leqslant n+1$.

In the end of the paper [5], Csákány and Pollák put forward the problem to classify disconnected groups. In this paper we solve it and prove the following theorem. In the sequel all groups are finite.

**Theorem.** *A finite group with a disconnected intersection graph is $\mathbb{Z}_p\times\mathbb{Z}_q$, where both $p$, $q$ are primes, or a Frobenius group whose complement is a prime order group and the kernel is a minimal normal subgroup.*

## 2. Some lemmas

First, we cite some lemmas due to Csákány and Pollák. Recall that a subset $S$ of the group $G$ is called a normal subset if $S^g=S$ for every element $g$ of $G$.

**Lemma 1.** *If $G$ is connected, then the diameter $\delta(G)$ is equal to $\max\{\varrho(P,Q)\colon$ both $P$, $Q$ are subgroups of prime order of $G\}$.*

**Lemma 2.** *Let $B$ be a block of $G$ and $M$ a proper subgroup of the group $G$. If $B\cap M\neq 1$, then $M\subseteq B$.*

**Lemma 3.** *Let $B$ be a block of $G$. Then $B$ is a subgroup of $G$ or a normal subset of $G$.*

**Lemma 4.** *Let $G$ be disconnected and let $\mathcal{B}=\{B_1,B_2,\ldots,B_l\}$ be the set of all the subgroup blocks of $G$. Then any conjugate of $B_i$ is also contained in $\mathcal{B}$ for $i=1,2,\ldots,l$.*

Next we investigate non-simple groups, i.e., those containing at least one nontrivial normal subgroup.

**Lemma 5.** *If $G$ is not a simple group, then one of the following cases occurs:*

(1) *the diameter $\delta(G) \leqslant 4$.*

(2) *$G$ is $\mathbb{Z}_p \times \mathbb{Z}_q$, where $p$, $q$ are primes.*

(3) *$G$ is a Frobenius group whose complement is a group of prime order and the kernel is a minimal normal subgroup.*

P r o o f. Suppose that $N$ is a nontrivial proper normal subgroup of $G$. By Lemma 1, the required result $\delta(G) \leqslant 4$ is equivalent to $\varrho(P,Q) \leqslant 4$ for any prime order subgroups $P$, $Q$ with $P \neq Q$. Let $|P| = |\langle a\rangle| = p$ and $|Q| = |\langle b\rangle| = q$.

*Case 1.* $PN = G$.

(a) If $Q \cap N = \langle b\rangle \cap N = 1$, then $b \in G \setminus N$. Since $G/N \cong P$, the order of every element of $G \setminus N$ is a multiple of $p$. So the order of $b$ is $p$, that is $o(b) = p = q$.

If $C_G(a) = G$, then $G = P \times N$. Since $o(a) = o(b) = p$, we can assume that $Q = \langle (a,x)\rangle$, where $x \in N$ and $o(x) = p$. Now we set $H = \{(y,z)\colon\ y \in \langle a\rangle,\ z \in \langle x\rangle\}$. If $|N| \neq p$, then $H$ is a proper subgroup of $G$, so that we have a chain $P \sim H \sim Q$. Thus $\varrho(P,Q) \leqslant 2$. Certainly, when $G$ is $\mathbb{Z}_p \times \mathbb{Z}_p$, there are $p+1$ nontrivial subgroups such that the intersection of any two of them is trivial, i.e., the intersection graph $\Gamma(G)$ is the $p+1$ isolated vertices graph.

If $C_G(b) = G$, then $\langle b\rangle \lhd G$. Since $b \notin N$, we have $G = \langle b\rangle \times N$ by virtue of $|G| = p|N|$. So we can assume that $\langle a\rangle = \langle b, x\rangle$, where $x \in N$ and $o(x) = p$. Similarly, we choose a group $H = \{(y,z)\colon\ y \in \langle b\rangle,\ z \in \langle x\rangle\}$. When $|N| \neq p$, then $H$ is a proper subgroup of $G$, so $P$ and $Q$ are connected by a chain $P \sim H \sim Q$. Thus we have also $\varrho(P,Q) \leqslant 2$.

Now we suppose that $C_G(a) \neq G$ and $C_G(b) \neq G$.

If $C_G(a) \cap N \neq 1$ and $C_G(b) \cap N \neq 1$, then $\langle a\rangle \sim C_G(a) \sim N$ and $\langle b\rangle \sim C_G(b) \sim N$, so $\langle a\rangle \sim C_G(a) \sim N \sim C_G(b) \sim \langle b\rangle$. Then we have $\varrho(P,Q) \leqslant 4$. If $C_G(a) \cap N = 1$ or $C_G(b) \cap N = 1$, we may assume without loss of generality,that $C_G(a) \cap N = 1$, then $\langle a\rangle$ acts non-fixed point on the subgroup $N$. Thus $G = N : \langle a\rangle$ is a Frobenius group. Clearly, if $N$ is not a minimal normal subgroup of $G$, then we can choose a nontrivial normal subgroup $N_1$ of $N$ such that $N_1 \lhd G$. So we get a chain $\langle a\rangle \sim N_1\langle a\rangle \sim N_1\langle b\rangle \sim \langle b\rangle$, hence we have $\varrho(P,Q) \leqslant 3$. Certainly, if $N$ is a minimal normal subgroup of $G$, then $G$ satisfies the requirement (3).

(b) Case of $Q \leqslant N$. If $C_G(a) = G$ (or $C_G(b) = G$), then $P \lhd G$ (or $Q \lhd G$). Hence when $PQ \neq G$, we have a chain $P \sim PQ \sim Q$, and then $\varrho(P,Q) \leqslant 2$. Certainly, if $PQ = G$, then $G = P \times Q$ or $G = Q : P$ is a Frobenius group, and hence the intersection graph of $G$ is the empty graph on two or $q+1$ vertices.

Next, we consider the case of $C_G(a) \neq G$ and $C_G(b) \neq G$. If $C_G(a) \cap N \neq 1$, then $P \sim C_G(a) \sim N \sim Q$. Hence we have $\varrho(P,Q) \leqslant 3$. If $C_G(a) \cap N = 1$, then $P$ acts as the group $N$ of fixed point free automorphisms. Thus $G = N : \langle a\rangle$ is a Frobenius group. Similarly to the case (a), we have that $N$ is a minimal normal subgroup of $G$, hence $G$ satisfies the requirement (3).

Similarly, if $QN = G$, then we have the same results.

*Case 2.* $PN \neq G$ and $QN \neq G$.

$P$ and $Q$ can be joined by the chain $P \sim PN \sim QN \sim Q$. Thus $\varrho(P, Q) \leqslant 3$. $\square$

## 3. Proof of Theorem

By Lemma 5 it suffices to deal with the case of non-abelian simple groups. We use the following two assertions to complete the proof of Theorem.

**Assertion I.** *If $n > 4$, then the alternating group $A_n$ is connected and $\delta(A_n) \leqslant 4$.*

P r o o f. By Lemma 1 it suffices to prove that $\varrho(P, Q) \leqslant 4$ for any subgroups $P$ and $Q$ of prime order. Now we can assume that $P$ and $Q$ are contained in maximal subgroups $M_1$ and $M_2$, respectively. If $M_1 \cap M_2 \neq 1$, then $P \sim M_1 \sim M_2 \sim Q$, so that $\varrho(P, Q) \leqslant 3$. Next we will prove that the order of every maximal subgroup of $A_n$ with $n \geqslant 5$ is more than $n$. For the cases of $n = 5$ and 6, this is true by inspection. Now suppose that $n \geqslant 7$. Consider $A_n$ in its natural degree $n$ action. If a maximal subgroup $M$ is intransitive, say has an orbit of length $k$, then $|M| \geqslant k!(n-k)!/2 > n$. So $M$ is transitive. If $|M| = n$, then $M$ is regular. Each automorphism of $M$ is induced by conjugation with some element from $S_n$. Thus if $M$ is maximal in $A_n$, then the automorphism group of $M$ has order at most 2. Consider inner automorphisms, so the order of $M/Z(M)$ is less than or equal to 2, hence $M$ is abelian. From $|\mathrm{Aut}(M)| \leqslant 2$ we get $M = \mathbb{Z}_n$ with $n = 2, 3$ or 6, which is impossible. Now return to our question. If $M_1 \cap M_2 = 1$, we choose a largest maximal subgroup $M$ of $A_n$, then it follows that $M \cap M_1 \neq 1$ and $M \cap M_2 \neq 1$. Indeed, otherwise, if $M \cap M_1 = 1$, then $|MM_1| = |M||M_1|/|M \cap M_1| = |M||M_1| > n \cdot |A_{n-1}| = |A_n|$, a contradiction. Hence $P \sim M_1 \sim M \sim M_2 \sim Q$, and consequently $\varrho(P, Q) \leqslant 4$. $\square$

Assertion I is Theorem 2 of [5]. The above proof is different from that of Csákány and Pollák. Unfortunately, this method cannot be applied to the case of simple groups of Lie type, because the product of orders of the largest and smallest maximal subgroups may be less than $|G|$. This occurs e. g. with $L_2(p)$, where $p$ is prime and $p \equiv \pm 1 \pmod 8$, in which case $S_4$ is a maximal subgroup always. It seems that it only occurs in the case of small ranks or the maximal subgroups of $C_5$-type in the Aschbacher Theorem (see [1]). In the following, we prove that simple groups of Lie type and sporadic simple groups have a connected intersection graph.

**Assertion II.** *If $G$ is a simple group of Lie type or a sporadic simple group, then its intersection graph is connected.*

Proof. Suppose that $G$ has a disconnected intersection graph. Let the order of $G$ be $p_1^{e_1}p_2^{e_2}\dots p_n^{e_n}$ and let $B_1, B_2, \dots, B_k$ be the blocks of $G$. Now we choose a series of numbers $b_1, b_2, \dots, b_k$ such that $p_l^{e_l} \parallel b_i$ if and only if there is an element of order $p_l$ in $B_i$ for $l = 1, 2, \dots, n$ and $i = 1, 2, \dots, k$. By Lemma 4, if some $B_i$ is a subgroup, then $B_i$ is a maximal subgroup and $B_i^g$ is also a block of $G$ for every $g \in G$. On the other hand, $N_G(B_i) = B_i$ since $B_i$ is maximal and $B_i$ is not a normal subgroup. It follows that $N_G(B_i^g) = N_G(B_i)^g = B_i^g$, and hence $B_i \cap B_i^g = 1$ for all $g \in G \setminus B_i$. Thus $G$ has a non-trivial normal subgroup by the well-known Frobenius theorem (see 8.5.5 of [7]), which contradicts the fact that $G$ is a simple group. So every $B_i$ is a normal subset of $G$ by Lemma 3. Next, we will prove $(b_i, b_j) = 1$ for $i \neq j$. If for some $1 \leqslant l \leqslant n$ and $1 \leqslant i, j \leqslant k$ there exists $p_l$ such that $p_l \mid (b_i, b_j)$, then there are $a \in B_i$, $b \in B_j$ satisfying $o(a) = o(b) = p_l$. Obviously, there exist Sylow $p_l$-subgroups $P_1$ and $P_2$ of $G$ containing $a$ and $b$, respectively. Since $P_1$ and $P_2$ are conjugate, we set $P_1^h = P_2$, then $P_2$ is contained in $B_i$ by Lemma 3, and hence $B_i$ and $B_j$ are connected, a contradiction. Therefore, $|G| = b_1b_2\dots b_k$ and $a \in B_i$ if and only if $o(a) \mid b_i$ for any $a \in G$.

Choose $M_i$ to be a maximal subgroup of $G$ in the block $B_i$ for $i = 1, 2, \dots, k$. By the above arguments we have $(|M_i|, |M_j|) = 1$ for $i \neq j$. Hence for every prime pairs $p_i, p_j$, where $p_i \mid b_i$ and $p_j \mid b_j$ for $i \neq j$, we have that $G$ has no element of order $p_ip_j$. Now we define another graph $\Lambda(G)$ of $G$ called the prime graph of $G$, whose vertices set is $\pi(G) = \{p\colon\ p \text{ is a prime divisor of } |G|\}$, vertices $p$ and $q$ in $\pi(G)$ are joined by an edge if and only if there exists an element of order $pq$ (see [8]). The classification of disconnected prime graphs of non-abelian simple groups is due to Williams and Kondrat'ev [6]. Now let $\pi(b_i) = \{p\colon\ p \text{ is a prime divisor of } b_i\}$, then $\pi(b_i)$ is a prime graph component of $G$ for $i = 1, 2, \dots, k$. Assume that 2 is contained in $\pi(b_1)$. If $G$ is a simple group of Lie type except $A_1(q)$, then $M_i$ is a maximal torus of $G$ for $i \geqslant 2$, and hence $N_G(M_i)/M_i \cong W$, which is the Weyl group of the corresponding simple group (see Chapter 8 of [3]). The structure of Weyl groups of simple groups of Lie type is determined completely. It is easy to see that their orders are all even, then $N_G(M_i) \cap B_1 \neq 1$, hence $M_i$ is connected to $M_1$, a contradiction. If $G$ is $A_1(q)$ with $q$ odd, set $\pi(b_2) = \pi(q) = p$, then $M_2$ is a elementary abelian $p$-group and $M_2$ is a Sylow $p$-subgroup of $G$, and we have $N_G(M_2) \neq M_2$ by the well-known Burnside theorem which states that a finite group $G$ satisfying $N_G(P) = C_G(P)$ for some abelian Sylow $p$-subgroup $P$ is $p$-nilpotent. Thus $M_2$ is not a maximal subgroup of $G$, a contradiction. For the remaining cases when $M_i$ of $A_1(q)$ for $i \geqslant 2$ is a maximal torus, we will get similar results. If $G$ is a sporadic simple group or

${}^2F_4(2)'$, the prime graph component's vertices $\pi(b_i)$ with $i \geqslant 2$ form a single point set $\{p\}$ and $M_i$ is a cyclic Sylow $p$-subgroup of $G$ by the result of [8]. Clearly, $M_i$ is not a maximal subgroup by the well-known Thompson theorem which asserts that a finite group having an odd order nilpotent maximal subgroup must be solvable. Therefore, $G$ is connected. $\square$

By Lemma 5 and Assertion I we know that the diameter $\delta(G) \leqslant 4$ if $G$ is an alternating group $A_n$ or another non-simple group. The problems arise: whether the diameters of non-abelian simple groups have an upper bound? Whether or not the best upper bound is 4? These problems are still open.

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