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BLASCHKE PRODUCT GENERATED COVERING SURFACES

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Abstract. It is known that, under very general conditions, Blaschke products generate branched covering surfaces of the Riemann sphere. We are presenting here a method of finding fundamental domains of such coverings and we are studying the corresponding groups of covering transformations.

Keywords: Blaschke product, covering surface, covering transformation, fundamental domain, Cantor set

MSC 2010: 30D50, 14H30

1. Introduction

Every finite or infinite sequence (a_k), $a_k \in D := \{z \in \mathbb{C};\ |z| < 1\}$ defines a Blaschke product

$$w = B(z) = \prod_{k=1}^{n\leqslant\infty} b_k(z), \tag{1.1}$$

where

$$b_k(z) = \frac{\bar{a}_k}{|a_k|}\frac{a_k - z}{1 - \bar{a}_k z}. \tag{1.2}$$

It is known (see for example [8]) that if $n = \infty$ then the condition $\sum\limits_{n=1}^{\infty}(1-|a_n|) < \infty$ is sufficient for the product (1) to converge uniformly on compact subsets of $\mathbb{C}\setminus(E\cup A)$, where E is the set of cluster points of zeros a_k of B, $E \subset \partial D$, and $A = \{z \in \mathbb{C};\ z = 1/\bar{a}_k,\ k = 1, 2, 3, \ldots\}$. We have studied in [3] and [4] Blaschke products for which E is a generalized Cantor set (see [7]). The word generalized used here has the meaning that we allow more liberty of choice for the open arcs to be removed. In particular we allow some of them to have common ends and therefore the corresponding closed interval between them to be a unique point. In other words, E is allowed to have isolated points. We will also allow E to have only isolated points or to be the empty set. Obviously, in this last case B is a finite Blaschke product. The statements which follow in the subsequent sections do not depend on what type of generalized Cantor set is E.

We have shown (see [3] and [4]) that for a Blaschke product whose cluster set E of zeros is a generalized Cantor set (for more about this concept, see for example [6]), there is a partition of $W = \widehat{\mathbb{C}} \setminus E$ into regions (fundamental "domains") Ω_k which are mapped continuously and bijectively by B onto $\widehat{\mathbb{C}}$. The mapping is conformal in the interior of every Ω_k. The local injectivity of B is violated just on a set of points (branch points) which is finite in the case of a finite Blaschke product, and which has its cluster points in E in the case of an infinite Blaschke product. These properties define (W, B) as a (branched) covering surface of $\widehat{\mathbb{C}}$ (see [2]). The regions Ω_k accumulate at every point $\mathrm{e}^{\mathrm{i}\alpha} \in E$.

For a Blaschke product of order n, there are exactly n regions Ω_k. When $B(z) = [\bar{a}/|a| \cdot (a - z)/(1 - \bar{a}z)]^n$, these are regions bounded by the arcs

$$\Big\{z = z_k(\lambda)\,;\; z_k(\lambda) = \frac{\omega_k\lambda - r}{\omega_k\lambda r - 1}\mathrm{e}^{\mathrm{i}\theta},\ \lambda \geqslant 0\Big\}, \quad k = 0, 1, 2, \ldots, n-1,$$

where $a = r\mathrm{e}^{\mathrm{i}\theta}$ and ω_k are the roots of order n of unity.

Since $z_k(\lambda)$ are Moebius transformations and λ varies in an interval, these regions are bounded by arcs of a circle or by lines, as can be seen in Figures 1 and 2, where $a = 0.5$ and $a = 0.8$ respectively. We notice how some of the regions get smaller as a becomes closer to the unit circle.

Moreover, we have found that the invariants of B, i.e. the mappings $U_k\colon \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ with the property that $B \circ U_k = B$, are Moebius transformations of the form

$$U_k(z) = \frac{a(1 - \omega_k) - (|a|^2 - \omega_k)z}{1 - |a|^2\omega_k - \bar{a}\,(1 - \omega_k)z}.$$

They form a cyclic group of order n with respect to composition, where $U_k \circ U_{k'} = U_{k+k' \pmod n}$. In particular, $U_k \circ U_{n-k} = U_0$, where $U_0(z) = z$. In this case U_k are cyclically permuting the regions Ω_k in the sense that every $\Omega_{k'}$ is mapped conformally onto an $\Omega_{k+k' \pmod n}$. Indeed, it can be easily checked that

$$U_k(z_{k'}(\lambda)) = z_{k+k' \pmod n}(\lambda) \quad \text{for every} \quad k,\ k' \in \{0, 1, \ldots, n-1\}.$$

This means that for every $k' \in \{0, 1, \ldots, n-1\}$ the arcs $\{z\,;\ z = z_{k'}(\lambda),\ \lambda \geqslant 0\}$ and $\{z\,;\ z = z_{k'+1 \pmod n}(\lambda), \lambda \geqslant 0\}$ are mapped bijectively by U_k onto the arcs

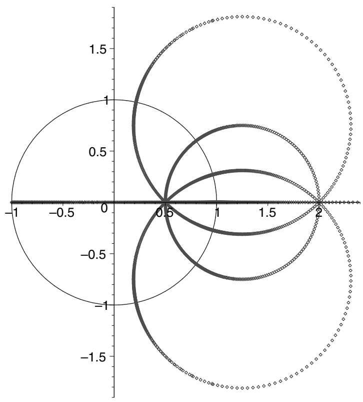

Figure 1

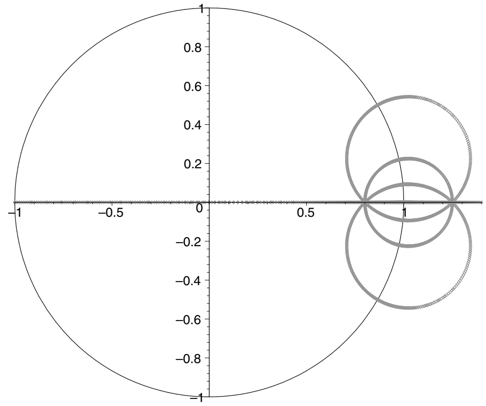

Figure 2

$\{z;\ z = z_{k+k' \pmod n}(\lambda), \lambda \geqslant 0\}$ and $\{z;\ z = z_{k+k'+1 \pmod n}(\lambda), \lambda \geqslant 0\}$ respectively. Then, by the conformal correspondence theorem (see [8], page 154), the domain $\Omega_{k'}$ bounded by the first two arcs is mapped by U_k conformally onto the domain $\Omega_{k+k' \pmod n}$ bounded by the other two arcs.

We can say even more, namely that all U_k have the fixed points a and $1/\bar{a}$. Moreover, for every $z \in \widehat{\mathbb{C}}$ and every $k = 0, 1, 2, \ldots, n-1$, we have $U_k(1/\overline{z}) = 1/\overline{U_k(z)}$; in particular, all U_k map ∂D onto ∂D. As a consequence, every $z \in \widehat{\mathbb{C}}$ has exactly n pre-images by B, if we consider that a and $1/\bar{a}$ are values taken with multiplicity n (indeed, they belong to every one of the n fundamental domains Ω_k).

We were trying to draw similar conclusions for Blaschke products of a more general form.

The technique of simultaneous continuation, which has been described in [3], allowed us to prove the existence of fundamental domains Ω_k for any Blaschke product whose cluster set of zeros E is a generalized Cantor set.

Let us repeat here the main features of the technique. If B is a Blaschke product of degree n, then there are exactly n distinct solutions $\mathrm{e}^{\mathrm{i}\alpha_k}$ of the equation $B(z) = 1$. They determine a partition of ∂D into n half-open arcs Γ_k. Let $b_1, b_2, \ldots, b_q$ be the solutions of the equation $B'(z) = 0$ situated in D and let $w_j = B(b_j)$, $j = 1, 2, \ldots, q$. We might have $w_j = w_{j'}$ even if $b_j \neq b_{j'}$ In particular, $w_j = 0$ for all multiple zeros a_j of B, for which we have obviously $B'(a_j) = 0$.

We connect $w = 1$ and all the points w_j by a polygonal line η with no self-intersection, then we perform continuations γ_k over η from every point $\mathrm{e}^{\mathrm{i}\alpha_k}$. The arcs γ_k and Γ_k determine a partition of D into sets A_k whose interiors are mapped conformally by B onto slit unit discs. The fundamental domains Ω_k are $A_k \cup \hat{A}_k$, where $\hat{A}_k$ is symmetric to A_k with respect to the unit circle.

The case of an infinite Blaschke product is reduced to the finite case by using an exhaustion sequence of $\mathbb{C} \setminus E$.

Figure 3 above exhibits computer generated fundamental domains for the Blaschke product defined by the triple zero $0.4+0.3\mathrm{i}$ and the double zero $0.5-0.6\mathrm{i}$. In Figure 4 we added the triple zero $0.9-0.2\mathrm{i}$ in order to show that, due to its closeness to the unit circle, the addition of this zero affects in a visible way only the domain containing it. This and the previous examples might contribute to a better understanding of the geometry of infinite Blaschke product mappings.

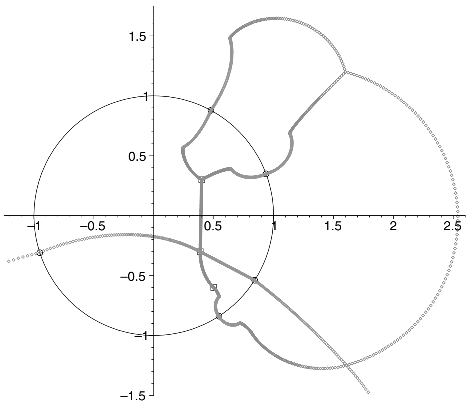

Figure 3

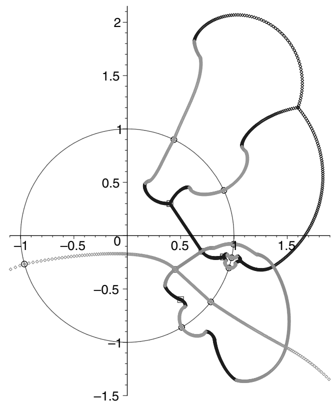

Figure 4

2. Invariants of B on ∂D

We introduce the notation needed in order to account for all the removed open arcs. At every stage m we remove 2^{m-1} open arcs.

Let us put all these arcs in a sequence

$$I_n = \{z = \mathrm{e}^{\mathrm{i}\theta};\ \theta_n < \theta < \theta'_n\},\ n = 1, 2, \ldots$$

and set

$$E = \partial D \setminus \bigcup_{n=1}^{\infty} I_n.$$

We have also

$$\partial D \setminus E = \bigcup_{n=1}^{\infty} I_n.$$

For every such removed open arc there are infinitely many disjoint half-open sub-arcs (see [3]):

$$\Gamma_{nj} = \{z = \mathrm{e}^{\mathrm{i}\theta};\ \alpha_{n,j} \leqslant \theta < \alpha_{n,j+1}\},\ j \in \mathbb{Z},\ \lim_{j\to+\infty} \alpha_{n,j} = \theta'_n,\ \lim_{j\to-\infty} \alpha_{n,j} = \theta_n.$$

The sub-arcs $\Gamma_{n,j}$ are mapped by B continuously and bijectively onto the unit circle in the w-plane with $B(\mathrm{e}^{\mathrm{i}\alpha_{n,j}}) = 1$.

If B is a finite Blaschke product of order n, then $E = \emptyset$ and ∂D is the disjoint union of exactly n arcs Γ_k which are mapped by B continuously and bijectively onto ∂D (see [5]). Let us denote by $\Psi_{n,j}$ the inverse mappings of $B|_{\Gamma_{n,j}}$ and associate with every bijection $\chi\colon \mathbb{N}^* \times \mathbb{Z} \longrightarrow \mathbb{N}^* \times \mathbb{Z}$ a mapping $U_\chi\colon \partial D \setminus E \longrightarrow \partial D \setminus E$ defined in the following way. If $(n', j') = \chi(n, j)$, then for every $\Gamma_{n,j}$ we set

$$U_\chi|_{\Gamma_{n,j}} = \Psi_{n',j'} \circ \Psi_{n,j}^{-1}|_{\Gamma_{n,j}}. \tag{2.1}$$

It can be easily checked that for every $\Gamma_{n,j}$,

$$B \circ U_\chi|_{\Gamma_{n,j}}(\mathrm{e}^{\mathrm{i}\theta}) = B(\Psi_{n',j'}(\Psi_{n,j}^{-1}(\mathrm{e}^{\mathrm{i}\theta}))) = \Psi_{n,j}^{-1}(\mathrm{e}^{\mathrm{i}\theta}) = B(\mathrm{e}^{\mathrm{i}\theta}). \tag{2.2}$$

Therefore

$$B \circ U_\chi = B \text{ on } \partial D \setminus E. \tag{2.3}$$

We notice that U_χ are continuous functions in every $\Gamma_{n,j}$, but they may fail to be continuous at the points $\mathrm{e}^{\mathrm{i}\alpha_{n,j}}$ if the images by U_χ of $\Gamma_{n,j-1}$ and $\Gamma_{n,j}$ are not adjacent arcs. On the other hand, all U_{χ_k} with χ_k of the form $\chi_k(n, j) = (n, j + k)$, $k \in \mathbb{Z}$, are continuous in every I_n. Our purpose is to extend analytically U_χ, therefore we will deal in the next section only with functions of the type U_{χ_k}. Nonetheless, the following theorem may be of some interest.

Theorem 1. *The set of mappings $\{U_\chi\}$ is a group with respect to composition. If B is a Blaschke product of degree n, then U_χ realizes a permutation of the n half-open arcs and $\{U_{\chi_k}\}$ is a cyclic subgroup of order n. Moreover, in the infinite case, $\{U_{\chi_k};\ \chi_k(n, j) = (n', j + k)\}$ are infinite cyclic subgroups for every given bijection $n \to n'$ of $\mathbb{N}^*$. Here k varies in $\mathbb{Z}$.*

Proof. Indeed, with $(n', j') = \chi(n, j)$ and $(n'', j'') = \chi'(n', j')$, there is χ'' such that $(n'', j'') = \chi' \circ \chi(n, j) = \chi''(n, j)$; therefore

$$\begin{aligned} U_{\chi''}|_{\Gamma_{n,j}} &= \Psi_{n'',j''} \circ \Psi_{n,j}^{-1}|_{\Gamma_{n,j}} = \Psi_{n'',j''} \circ \Psi_{n',j'}^{-1}|_{\Gamma_{n',j'}} \circ \Psi_{n',j'} \circ \Psi_{n,j}^{-1}|_{\Gamma_{n,j}} \\ &= U_{\chi'}|_{\Gamma_{n',j'}} \circ U_\chi|_{\Gamma_{n,j}}. \end{aligned}$$

In other words,

$$U_{\chi'} \circ U_\chi = U_{\chi' \circ \chi}. \tag{2.4}$$

In particular, if χ_0 is the identity mapping, then

$$U_\chi \circ U_{\chi_0} = U_{\chi_0} \circ U_\chi = U_\chi \tag{2.5}$$

for every χ, i.e. U_{χ_0} is the identity element of the group.

If $\chi_k(n,j) = (n, j+k)$, then $\chi_{k'} \circ \chi_k(n,j) = (n, j+k+k') = \chi_{k+k'}(n,j)$, hence

$$U_{\chi_k} \circ U_{\chi_{k'}} = U_{\chi_{k+k'}},\ k \in \mathbb{Z}.$$

3. Analytic extensions of the functions U_{χ_k}

Theorem 6 of [6] can be easily extended to the case when E is a generalized Cantor set. More exactly, we have

Theorem 2. *Let K be a compact subset of $\partial D \setminus E$ (in the topology of ∂D). Then there is a neighborhood V of K (in $\mathbb{C}$) such that every function U_{χ_k} can be extended analytically to V. The extended functions still verify the identity $B \circ U_{\chi_k} = B$.*

Proof. For every $z \in W$, the derivative of B is given by

$$B'(z) = -B(z) \sum_{n=1}^{\infty} \frac{1-|a_n|^2}{(a_n - z)(1-\bar{a}_n z)}. \tag{3.1}$$

If $\zeta = \mathrm{e}^{\mathrm{i}\theta} \in \partial D \setminus E$, then $(a_n - \zeta)(1 - \bar{a}_n\zeta) = -\zeta(a_n - \zeta)(\bar{a}_n - \overline{\zeta}) = -\zeta|a_n - \zeta|^2$ and $|B(\zeta)| = 1$. Thus

$$|B'(\zeta)| = \sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|a_n - \zeta|^2} > 0. \tag{3.2}$$

Consequently, the local inverse theorem (see [1], p. 131) can be applied at the point ζ and we conclude that there is a neighborhood V_ζ of ζ, $V_\zeta \subset W$ such that B maps V_ζ conformally onto a domain W_ζ from the w-plane. Therefore, there is an analytic local inverse $\varphi_\zeta \colon W_\zeta \to V_\zeta$ of B. On the other hand, there is a couple (n,j) such that $\zeta \in \Gamma_{n,j}$. On $\Gamma_{n,j}$ the inverse of B is $\Psi_{n,j}$, in other words φ_ζ is an analytic extension of $\Psi_{n,j}|_{V_\zeta \cap \Gamma_{n,j}}$. There is a finite covering $\{V_{\zeta_1}, V_{\zeta_2}, \ldots, V_{\zeta_p}\}$ of K and $\Psi_{n,j}|_{V_\zeta \cap \Gamma_{n,j}}$ can be extended analytically in each of them, therefore it can be extended analytically in a neighborhood of K. The same is true for U_{χ_k} defined by the formula (2.1). Let us denote by φ_n the analytic extension of $U_{\chi_k}|_{I_n}$ in a neighborhood in $\mathbb{C}$ of $K \cap I_n$ and let φ_K be defined on K by $\varphi_K(z) = \varphi_n(z)$ for every n, where $z \in K \cap I_n$. Then φ_K is an analytic extension of U_{χ_k} in a neighborhood V of K and for every $\zeta \in \Gamma_{n,j} \subset K \cap I_n$ we have

$$B(\varphi_K(\zeta)) = B(U_{\chi_k}(\zeta)) = B(\zeta). \tag{3.3}$$

By virtue of the functional relations theorem (see [1], p. 288), the identity (3.3) is true in V. □

4. The covering transformations of (W, B)

For simplicity of notation, we drop in this section the subscript k in χ_k. The analytic functions φ_χ extending $U_\chi|_{\Gamma_{n,j}}$ are defined in a neighborhood of every arc $\Gamma_{n,j}$. Let us denote by $\gamma_{n,j}$ the arcs obtained as in [3] by simultaneous continuation and let $A_{n,j}$ be the sets bounded by $\Gamma_{n,j} + \gamma_{n,j} - \gamma_{n,j+1}$ to which the arcs $\Gamma_{n,j}$ and $\gamma_{n,j}$, considered as point sets, are added. If $\hat{A}_{n,j}$ is symmetric to $A_{n,j}$ with respect to the unit circle, we denote $\Omega_{n,j} = A_{n,j} \cup \hat{A}_{n,j}$. Let $D_{n,j}$ be the interior of $\Omega_{n,j}$.

Theorem 3. *The functions φ_χ can be extended analytically to W. The extended functions verify the identity $B \circ \varphi_\chi = B$, therefore they are cover transformations of (W, B). Moreover, they are the only cover transformations of (W, B) and they represent conformally every $D_{n,j}$ on a $D_{n,j'}$.*

Proof. We extend first φ_χ to every $A_{n,j}$ in the following way. Given arbitrary $\varepsilon > 0$, we denote $O_\varepsilon = \bigcup D(b_k, \varepsilon)$, where $D(b_k, \varepsilon)$ are open discs of radius ε centered at every branch point of (W, B).

Let us denote by H_D the set of branch points of (W, B) situated in D. It is known (see [3]) that all the points of H_D belong to the arcs $\gamma_{n,j}$ and $H_D \cap K$ is a finite set for every compact set $K \subset \widehat{\mathbb{C}} \setminus E$. Every set $\overline{A}_{n,j} \setminus O_\varepsilon$ is compact, and therefore it can be covered by a finite number $q = q(n,j)$ of open discs $V_1, V_2, \ldots, V_q$ in which B is injective. For $z \in V_r$, let $\varphi_{n,j,r}(z) = B^{-1}|_{A_{n',j'}}(B(z))$, where $(n', j') = \chi(n,j)$. It is obvious that if $z \in V_r \cap V_{r'}$, then $\varphi_{n,j,r}(z) = \varphi_{n,j,r'}(z)$; therefore there is a unique analytic function $\varphi_{n,j}$ defined in $V = \bigcup_{r=1}^{q} V_r$, which coincides with $\varphi_{n,j,r}$ in every V_r. As $z \in V$ implies $z \in V_r$ for some r, we have

$$B \circ \varphi_{n,j}(z) = B \circ \varphi_{n,j,r}(z) = B \circ B^{-1}|_{A_{n',j'}}(B(z)) = B(z). \tag{4.1}$$

Since ε is arbitrary, $\varphi_{n,j}$ are in fact defined on $A_{n,j} \setminus H_D$ and $B \circ \varphi_{n,j}(z) = B(z)$ for every $(n,j) \in \mathbb{N}^* \times \mathbb{Z}$. The functions $\varphi_{n,j}$ are analytic continuations in $A_{n,j}$ of the functions U_χ from Section 2, since they are defined by a formula similar to (2.1). Consequently, if U_χ are continuous in I_n and V, V' are neighborhoods of $A_{n,j}$ and $A_{n,j+1}$ respectively, then $\varphi_{n,j}(z) = \varphi_{n,j+1}(z)$ for $z \in V \cap V'$. Consequently, there is a unique function φ_χ defined on $\bigcup\limits_{j=-\infty}^{\infty} A_{n,j}$ such that

$$B \circ \varphi_\chi(z) = B(z). \tag{4.2}$$

This happens for sure if $\chi(n,j) = (n, j+k)$, $k \in \mathbb{Z}$. We can extend φ_χ by symmetry to $\widehat{\mathbb{C}} \setminus E$ and it will continue to verify the identity (4.2).

Let us suppose now that U is an arbitrary covering transformation of (W, B) over $\mathbb{C}$, i.e. an analytic function $U\colon W \longrightarrow W$ such that

$$B(U(z)) = B(z) \text{ for every } z \in W. \tag{4.3}$$

In particular, the identity (4.3) is true for $|z| = 1$, $z \notin E$. It is known that $|B(z)| = 1$ if and only if $|z| = 1$ and $z \notin E$. Therefore $|B(U(z))| = 1$ if and only if $|U(z)| = 1$ and $U(z) \notin E$. Consequently $|U(z)| = 1$ if and only if $|z| = 1$.

On the other hand, with the notation of Section 2, $B(U(z)) = 1$ if and only if $z = \mathrm{e}^{\mathrm{i}\alpha_{n,j}}$. Therefore, if $\alpha_{n,j}$ is given, then $U(\mathrm{e}^{\mathrm{i}\alpha_{n,j}}) = \mathrm{e}^{\mathrm{i}\alpha_{n',j'}}$ for some $(n', j') \in \mathbb{N}^* \times \mathbb{Z}$. We cannot have $(n', j') = (n, j)$, since covering transformations have no fixed points. Therefore U realizes a permutation of the points $\mathrm{e}^{\mathrm{i}\alpha_{n,j}}$, hence of the arcs $\Gamma_{n,j}$. Since U is continuous on $\partial D \setminus E$, the permutation must be a cyclic one. Then

$$B|_{\Gamma_{n',j'}}(U|_{\Gamma_{n,j}}(z)) = B|_{\Gamma_{n,j}(z)} \tag{4.4}$$

and therefore

$$U|_{\Gamma_{n,j}}(z) = B^{-1}|_{\Gamma_{n',j'}}(B|_{\Gamma_{n,j}(z)}). \tag{4.5}$$

In other words,

$$U|_{\Gamma_{n,j}}(z) = \varphi_{n,j}|_{\Gamma_{n,j}}(z). \tag{4.6}$$

By virtue of the functional relations theorem (see [1], p. 288), we have that $U(z) = \varphi_\chi(z)$ throughout W for $\chi(n,j) = (n', j')$.

This theorem tells us that the set $G = \{\varphi_\chi\}$ of conformal self-mappings of W represents the whole group of cover transformations of (W, B) over $\mathbb{C}$.

Obviously, this is true for every type of Blaschke product, regardless of whether it is finite or infinite. □

In the first case G is a finite cyclic group. In the infinite case, with E discrete, the group G is finitely generated and has a finite number of infinite cyclic subgroups. The fundamental domains of G are the domains $\Omega_{n,j}$ previously described. Obviously, $\Omega_{n,j}$ are not uniquely determined, since they all depend on the initial choice of the arcs $\Gamma_{n,j}$, as well as on the arcs $\gamma_{n,j}$.

If we start with an equation of the form $B(z) = \lambda, |\lambda| = 1$ instead of $B(z) = 1$, we might arrive at different domains $\Omega'_{n,j}$ and different mappings $\psi_{n,j}$ between them, generating invariants for B. However, assuming that for some corresponding half-open arcs $\Gamma'_{n',j'}$ in this new situation we have $\Gamma'_{n',j'} \cap \Gamma_{n,j} \neq \emptyset$, and having in view the way $\varphi_{n,j}$ and $\psi_{n',j'}$ have been constructed, they must coincide on $\Gamma'_{n',j'} \cap \Gamma_{n,j}$, and then, by virtue of the functional relations, they coincide throughout W.

Consequently, the cover transformations of (W, B) are independent (as they should be!) of the construction we used. Moreover, we can reformulate the final result in [3] as follows:

Theorem 4. *Suppose that B is an infinite Blaschke product whose cluster point set E of zeros is a generalized Cantor set. Then B generates a branched covering surface (W, B) of the Riemann sphere, where $W = \mathbb{C} \setminus E$. The fundamental domains of the group of covering transformations of (W, B) over $\widehat{\mathbb{C}}$ accumulate at every point of E, i.e. for every $\zeta \in E$ and every neighborhood V_ζ of ζ there are infinitely many sets $\Omega_{n,j} \subset V_\zeta$ which are mapped by B continuously and bijectively onto $\widehat{\mathbb{C}}$. The mappings are conformal in the interior of every $\Omega_{n,j}$. Moreover, if $K \subset \mathbb{C} \setminus E$ is a compact set, then there is a finite covering of K with sets $\Omega_{n,j}$, hence every $w = B(z)$, $z \in K$ has a finite number of pre-images by B.*

The novelty here resides in the fact that it is transparent how these cover transformations intrinsically depend on B. The question arises whether this property is specific to Blaschke products and if not, what other classes of non-univalent functions may display something similar. It would be also interesting to characterize proper subgroups of the group G when E is a Cantor set.

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References

[1] *L. V. Ahlfors*: Complex Analysis. International Series in Pure and Applied Mathematics, Mc Graw-Hill Company, Düsseldorf, 1979.

[2] *L. V. Ahlfors, L. Sario*: Riemann Surfaces. Princeton University Press, Princeton N.J., 1960.

[3] *I. Barza, D. Ghisa*: The Geometry of Blaschke Product Mappings (H. G. W. Begehr, A. O. Celebi, R. P. Gilbert, eds.). Further Progress in Analysis, World Scientific, 2008.

[4] *I. Barza, D. Ghisa*: Blaschke Self-Mappings of the Real Projective Plane. The Procedings of the 6-th Congress of Romanian Mathematiciens, Bucharest, 2007.

[5] *G. Cassier, I. Chalendar*: The group of invariants of a finite Blaschke product. Complex Variables, Theory Appl. *42* (2000), 193–206.

[6] *T. Cao-Huu, D. Ghisa*: Invariants of infinite Blaschke products. Matematica *45* (2007), 1–8.

[7] *C. Constantinescu, et al.*: Integration Theory, Vol. 1. John Wiley & Sons, New York, 1985.

[8] *J. B. Garnett*: Bounded Analytic Functions. Academic Press, New York, 1981.

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