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MEAN VALUE THEOREMS FOR DIVIDED DIFFERENCES AND APPROXIMATE PEANO DERIVATIVES

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Abstract. Several mean value theorems for higher order divided differences and approximate Peano derivatives are proved.

Keywords: mean value, higher order divided difference, approximate Peano derivative, n-convex function

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1. Introduction

The mean value theorems involving derivative are well known. But mean value theorems for divided differences of a function are useful particularly when the derivative of the function does not exist. We prove a mean value theorem for higher order divided differences in a general setting independent of the concept of derivative. We also prove a result which can reduce an nth order divided difference of a function f to an $(n-r)$th order divided difference of the rth order approximate Peano derivative $f_{(r)\,\mathrm{ap}}$ of f, $0 \leqslant r \leqslant n$, if $f_{(r)\,\mathrm{ap}}$ exists. Some consequences are studied.

Let a function f be defined in a neighbourhood of a point x_0. If there exist numbers $\alpha_1, \alpha_2, \ldots, \alpha_r$ depending on x_0 but not on h such that

$$\lim_{h\to 0} \mathrm{ap}\, \frac{r!}{h^r}\Big\{ f(x_0+h) - f(x_0) - \sum_{k=1}^{r} \frac{h^k}{k!}\,\alpha_k \Big\} = 0$$

where 'lim ap' denotes the approximate limit then α_r is called the approximate Peano derivative of f at x_0 of order r and is denoted by $f_{(r)\,\mathrm{ap}}(x_0)$. We shall write $f_{(0)\,\mathrm{ap}}(x_0) = f(x_0)$. The kth divided difference of f at $k+1$ distinct points

$x_0, x_1, \dots, x_k$ is defined by

$$Q_k(f; x_0, x_1, \dots, x_k) = \sum_{i=0}^{k} \frac{f(x_i)}{\omega'(x_i)}$$

where

$$\omega(x) = \prod_{i=0}^{k} (x - x_i).$$

So, the divided difference $Q_k(f; x_0, x_1, \dots, x_k)$ does not depend on the order of the points $x_0, x_1, \dots, x_k$.

Note that if $Q_n(f; x_0, x_1, \dots, x_n) \geqslant 0$ for all choices of $n+1$ distinct points $x_0, x_1, \dots, x_n$ in $[a, b]$, then f is called n-convex in $[a, b]$. Clearly 1-convex is just nondecreasing. If $-f$ is n-convex in $[a, b]$ then f is called n-concave in $[a, b]$.

Unless otherwise stated we consider functions from $\mathbb{R}$ to $\mathbb{R}$. If a function f has the Darboux property we write $f \in \mathcal{D}$ and if f is in Baire class 1, we write $f \in \mathcal{B}_1$.

2. Auxiliary lemmas and theorems

Theorem 2.1. *If $f \in \mathcal{D} \cap \mathcal{B}_1$ and g is continuous then $f + g \in \mathcal{D} \cap \mathcal{B}_1$ and $fg \in \mathcal{D} \cap \mathcal{B}_1$.*

For a proof see [1; p. 14, Theorem 3.2].

Theorem 2.2. *If $f \in \mathcal{D} \cap \mathcal{B}_1$ and if $x_1 < x_2 < \dots < x_m$ then $\varphi \in \mathcal{D} \cap \mathcal{B}_1$ in every closed subinterval of each of the intervals $(-\infty, x_1)$; $(x_1, x_2); \dots; (x_{m-1}, x_m)$; (x_m, ∞), where $\varphi(x) = Q_m(f; x, x_1, \dots, x_m)$.*

Proof. Let $[\xi_1, \xi_2]$ be any interval which does not contain any of the points $x_1, x_2, \dots, x_m$ Then since

$$\varphi(x) = \frac{f(x)}{\prod\limits_{i=1}^{m} (x - x_i)} + \sum_{i=1}^{m} \frac{f(x_i)}{(x_i - x) \prod\limits_{\substack{k=1 \\ k \neq i}}^{m} (x_i - x_k)}$$

and since $\Big(\prod\limits_{i=1}^{m} (x - x_i)\Big)^{-1}$ and $(x_i - x)^{-1}$, $1 \leqslant i \leqslant m$, are all continuous in $[\xi_1, \xi_2]$, the result follows from Theorem 2.1. □

Theorem 2.3. *If f is n-convex in $[a, b]$ and for some $n+1$ distinct points x_i, $0 \leqslant i \leqslant n$, $a \leqslant x_0 < x_1 < \dots < x_n \leqslant b$ we have $Q_n(f; x_0, x_1, \dots, x_n) = 0$ then f is a polynomial of degree at most $(n-1)$ on $[x_0, x_n]$.*

Proof is in [3; Theorem 5].

Lemma 2.4. *If* $\varphi(x) = f(ax+b)$ *and* $f_{(n)\,\mathrm{ap}}$ *exists then* $\varphi_{(n)\,\mathrm{ap}}$ *exists and* $\varphi_{(n)\,\mathrm{ap}}(x) = a^n f_{(n)\,\mathrm{ap}}(ax+b)$.

This can be proved by induction.

Lemma 2.5. *If* f *and* g *are defined at the points* $x_0, x_1, \ldots, x_n$, *then*

$$Q_n(fg; x_0, x_1, \ldots, x_n) = \sum_{i=0}^{n} Q_i(g; x_0, \ldots, x_i) Q_{n-i}(f; x_i, \ldots, x_n)$$

where $Q_0(g; x_0) = g(x_0)$.

This is known as the Leibniz rule for divided difference and can be proved using induction.

3. Mean value theorems for divided differences

Theorem 3.1. *Let* $f \in \mathcal{D} \cap \mathcal{B}_1$. *If* $Q_n(f; x_0, x_1, \ldots, x_n)$ *and* $Q_n(f; y_0, y_1, \ldots, y_n)$ *are of opposite signs for two sets of points* $\{x_0, x_1, \ldots, x_n\}$ *and* $\{y_0, y_1, \ldots, y_n\}$, *then there is a set of points* $\{\xi_0, \xi_1, \ldots, \xi_n\}$ *such that* $\min[x_0, \ldots, x_n, y_0, \ldots, y_n] \leqslant \xi_i \leqslant \max[x_0, \ldots, x_n, y_0, \ldots, y_n]$ *for all* i *and* $Q_n(f; \xi_0, \xi_1, \ldots, \xi_n) = 0$.

Proof. Since the divided difference does not depend on the ordering of the points we may suppose that $x_0 < x_1 < \ldots < x_n$ and $y_0 < y_1 < \ldots < y_n$. We may further suppose that $x_0 < y_0$, for otherwise the procedure that follows would start from $x_s < y_s$ such that $x_i = y_i$ for $i = 0, 1, \ldots, s-1$ (if necessary, by interchanging and renaming the sets $\{x_i;\ 0 \leqslant i \leqslant n\}$ and $\{y_i;\ 0 \leqslant i \leqslant n\}$). Suppose that the theorem is not true. Then for every set of points $\{z_0, z_1, \ldots, z_n\}$ with $\min\{x_0, y_0\} \leqslant z_i \leqslant \max\{x_n, y_n\}$ for $0 \leqslant i \leqslant n$, $Q_n(f; z_0, z_1, \ldots, z_n)$ is not zero. In our argument we will repeatedly use the following observation which follows from Theorem 2.2.

For any fixed set of points $a_0 < a_1 < \ldots < a_n$ the function $F(t) = Q_n(f; a_0, a_1, \ldots, a_s, t, a_{s+2}, \ldots, a_n)$ has the Darboux property on (a_s, a_{s+2}) for $0 \leqslant s < n-1$ and the functions $F_1(t) = Q_n(f; a_0, a_1, \ldots, a_{n-1}, t)$ and $F_2(t) = Q_n(f; t, a_1, \ldots, a_n)$ have the Darboux property in (a_{n-1}, ∞) and in $(-\infty, a_1)$, respectively.

Since the function $\varphi_1(t) = Q_n(f; x_0, t, x_2, \ldots, x_n)$ has the Darboux property in (x_0, x_2), $\varphi_1(t)$ has the same sign in (x_0, x_2) as $\varphi_1(x_1)$, for otherwise the theorem would be true. Choose $\bar{x}_1$ such that $x_0 < \bar{x}_1 < \min\{x_1, y_0\}$. Since $\bar{x}_1 \in (x_0, x_2)$, $\varphi_1(x_1)$ and $\varphi_1(\bar{x}_1)$ have the same sign and so $Q_n(f; x_0, x_1, x_2, \ldots, x_n)$ and $Q_n(f; x_0, \bar{x}_1, x_2, \ldots, x_n)$ have the same sign on this interval. Applying the above argument and choosing $\bar{x}_2$, $\bar{x}_1 < \bar{x}_2 < \min\{x_2, y_0\}$ we conclude that

$Q_n(f;x_0,x_1,\dots,x_n)$ and $Q_n(f;x_0,\bar{x}_1,\bar{x}_2,x_3,\dots,x_n)$ have the same sign. Continuing this process we get points $\bar{x}_1,\bar{x}_2,\dots,\bar{x}_{n-1},x_0<\bar{x}_1<\bar{x}_2<\dots<\bar{x}_{n-1}<\min\{x_{n-1},y_0\}$ such that $Q_n(f;x_0,x_1,\dots,x_n)$ and $Q_n(f;x_0,\bar{x}_1,\dots,\bar{x}_{n-1},x_n)$ have the same sign. Consider $\varphi_n(t)=Q_n(f;x_0,\bar{x}_1,\dots,\bar{x}_{n-1},t)$. Since $\varphi_n(t)$ has the Darboux property in $(\bar{x}_{n-1},\infty)$, $\varphi_n(x_n)$ and $\varphi_n(y_n)$ have the same sign and so $Q_n(f;x_0,\bar{x}_1,\dots,\bar{x}_{n-1},y_n)$ and $Q_n(f;x_0,x_1,\dots,x_n)$ have the same sign. Let $\psi_{n-1}(t)=Q_n(f;x_0,\bar{x}_1,\dots,\bar{x}_{n-2},t,y_n)$. Since $\psi_{n-1}(y_{n-1})$ and $\psi_{n-1}(\bar{x}_{n-1})$ have the same sign, $Q_n(f;x_0,\bar{x}_1,\dots,\bar{x}_{n-2},y_{n-1},y_n)$ and $Q_n(f;x_0,x_1,x_2,\dots,x_n)$ have the same sign. Continuing this process we conclude that $Q_n(f;x_0,y_1,\dots,y_n)$ and $Q_n(f;x_0,x_1,\dots,x_n)$ have the same sign. Finally we apply the argument to the function $Q_n(f;t,y_1,\dots,y_n)$ over the interval $(-\infty,y_1)$ to deduce that $Q_n(f;y_0,y_1,\dots,y_n)$ has the same sign as $Q_n(f;x_0,x_1,\dots,x_n)$, which is a contradiction. This completes the proof. □

Corollary 3.2. *Let $f\in\mathcal{D}\cap\mathcal{B}_1$. If $Q_m(f;z_0,z_1,\dots,z_m)<\alpha<Q_m(f;y_0,y_1,\dots,y_m)$ for any two sets of points $\{z_0,z_1,\dots,z_m\}$ and $\{y_0,y_1,\dots,y_m\}$ then there is a set of points $\{\xi_0,\xi_1,\dots,\xi_m\}$ such that $\min[z_0,\dots,z_m,y_0,\dots,y_m]\leqslant\xi_i\leqslant\max[z_0,\dots,z_m,y_0,\dots,y_m]$ for all i, and $Q_m(f;\xi_0,\xi_1,\dots,\xi_m)=\alpha$.*

Proof. Consider $g(x)=f(x)-\alpha x^m$ and apply Theorem 2.1 and Theorem 3.1 to g. □

Theorem 3.3. *If $f_{(m)\,\mathrm{ap}}$ exists then for any set of $n+1$ distinct points x_i with $x_0<x_1<\dots<x_n$, $n\geqslant m$, there is δ, $0<\delta<1$, such that*

$$m!\,Q_n(f;x_0,\dots,x_n) = \delta^{n-m}\sum_{i=0}^{n-1}Q_i((x-x_n)^{m-1};y_0,\dots,y_i)\,Q_{n-1-i}(f_{(m)\,\mathrm{ap}};y_i,\dots,y_{n-1})$$

where $y_k=x_n+(x_k-x_n)\delta$, $0\leqslant k\leqslant n$.

Proof. Let $a<x_0<x_1<\dots<x_n<b$ be fixed. Let

$$\psi(t)=\sum_{i=0}^{n}\frac{f(x_n+(x_i-x_n)t)}{\prod\limits_{\substack{j=0\\ j\neq i}}^{n}(x_i-x_j)}=\sum_{i=0}^{n-1}\frac{f(x_n+(x_i-x_n)t)}{\prod\limits_{\substack{j=0\\ j\neq i}}^{n}(x_i-x_j)}+\frac{f(x_n)}{\prod\limits_{j=0}^{n-1}(x_n-x_j)}.$$

Then by Lemma 2.4

$$\psi_{(r)\,\mathrm{ap}}(t)=\sum_{i=0}^{n-1}(x_i-x_n)^r f_{(r)\,\mathrm{ap}}(x_n+(x_i-x_n)t)\Bigg/\prod_{\substack{j=0\\ j\neq i}}^{n}(x_i-x_j)$$

$$= \sum_{i=0}^{n-1} (x_i - x_n)^{r-1} f_{(r)\,\mathrm{ap}}(x_n + (x_i - x_n)t) \Big/ \prod_{\substack{j=0\\ j\neq i}}^{n-1} (x_i - x_j).$$

Hence

$$\psi_{(r)\,\mathrm{ap}}(0) = f_{(r)\,\mathrm{ap}}(x_n) \sum_{i=0}^{n-1} (x_i - x_n)^{r-1} \Big/ \prod_{\substack{j=0\\ j\neq i}}^{n-1} (x_i - x_j)$$
$$= f_{(r)\,\mathrm{ap}}(x_n) Q_{n-1}((x - x_n)^{r-1}; x_0, x_1, \dots, x_{n-1}).$$

So $\psi_{(r)\,\mathrm{ap}}(0) = 0$ if $r < m$. Hence by the mean value theorem [5; Theorem 1] there is δ, $0 < \delta < 1$, such that

$$Q_n(f; x_0, \dots, x_n) = \psi(1) = \psi(1) - \psi(0) - \dots - \frac{1}{(m-1)!}\,\psi_{(m-1)\,\mathrm{ap}}(0)$$
$$= \frac{1}{m!}\,\psi_{(m)\,\mathrm{ap}}(\delta) = \frac{1}{m!} \sum_{i=0}^{n-1} (x_i - x_n)^{m-1} f_{(m)\,\mathrm{ap}}(x_n + (x_i - x_n)\delta) \Big/ \prod_{\substack{j=0\\ j\neq i}}^{n-1} (x_i - x_j)$$
$$= \frac{1}{m!} \sum_{i=0}^{n-1} \delta^{n-m} (y_i - y_n)^{m-1} f_{(m)\,\mathrm{ap}}(y_i) \Big/ \prod_{\substack{j=0\\ j\neq i}}^{n-1} (y_i - y_j)$$
$$= \frac{\delta^{n-m}}{m!} Q_{n-1}((x - y_n)^{m-1} f_{(m)\,\mathrm{ap}}(x); y_0, y_1, \dots, y_{n-1}).$$

Hence by Lemma 2.5

$$m!\, Q_n(f; x_0, \dots, x_n)$$
$$= \delta^{n-m} \sum_{i=0}^{n-1} Q_i((x - y_n)^{m-1}; y_0, \dots, y_i)\, Q_{n-1-i}(f_{(m)\,\mathrm{ap}}; y_i, \dots, y_{n-1})$$
$$= \delta^{n-m} \sum_{i=0}^{n-1} Q_i((x - x_n)^{m-1}; y_0, \dots, y_i)\, Q_{n-1-i}(f_{(m)\,\mathrm{ap}}; y_i, \dots, y_{n-1}),$$

completing the proof. □

Corollary 3.4. *If f'_{ap} exists then for every set of points $x_i, x_0 < x_1 < \dots < x_n$, there is δ, $0 < \delta < 1$, such that*

$$Q_n(f; x_0, \dots, x_n) = \delta^{n-1} Q_{n-1}(f'_{\mathrm{ap}}; y_0, \dots, y_{n-1}) \quad \textit{where } y_k = x_n + (x_k - x_n)\delta.$$

Proof. Putting $m = 1$ in Theorem 3.3 the result follows. □

Corollary 3.5. *If f'_{ap} exists and is k-convex in (a,b) then f is $(k+1)$-convex in (a,b).*

P r o o f. The result follows from Corollary 3.4. □

Corollary 3.6. *If f is k-convex in (a,b) and if for a fixed $c \in (a,b)$,*

$$F(x) = \int_c^x f(t)\,\mathrm{d}t, \quad x \in (a,b),$$

then F is $(k+1)$-convex in (a,b).

P r o o f. The result follows from Corollary 3.5. □

Theorem 3.7. *Let $f_{(r)\,\mathrm{ap}}$ exist in (a,b) where $r \geqslant 1$. Then for any $n \geqslant r$ and for any $(n+1)$ distinct points x_i, $a < x_0 < x_1 < \ldots < x_n < b$, there are distinct points $\xi_0, \xi_1, \ldots, \xi_{n-r}$ in (x_0, x_n) such that*

$$n!\,Q_n(f; x_0, \ldots, x_n) = (n-r)!\,Q_{n-r}(f_{(r)\,\mathrm{ap}}; \xi_0, \ldots, \xi_{n-r}).$$

P r o o f. Since f is approximately continuous in (a,b), we have $f \in \mathcal{D} \cap \mathcal{B}_1$ in (a,b). Let $g(x) = f(x) - x^n Q_n(f; x_0, \ldots, x_n)$. Then $g_{(r)\,\mathrm{ap}}$ exists in (a,b) and by Theorem 2.1, $g \in \mathcal{D} \cap \mathcal{B}_1$ in (a,b). Also

$$g_{(r)\,\mathrm{ap}}(x) = f_{(r)\,\mathrm{ap}}(x) - \frac{x^{n-r}}{(n-r)!}\,n!\,Q_n(f; x_0, \ldots, x_n). \tag{1}$$

Suppose $n > r$. Let $Q_{n-r}(g_{(r)\,\mathrm{ap}}; y_0, \ldots, y_{n-r}) \geqslant 0$ for every set of $n-r+1$ points $y_0, y_1, \ldots, y_{n-r}$ in (x_0, x_n). Then $g_{(r)\,\mathrm{ap}}$ is $(n-r)$-convex in (x_0, x_n). Also $g_{(r)\,\mathrm{ap}}$ is continuous in (x_0, x_n). In fact, if $n-r = 1$, then $g_{(r)\,\mathrm{ap}}$ is non decreasing and therefore, since $g_{(r)\,\mathrm{ap}} \in \mathcal{D}$, $g_{(r)\,\mathrm{ap}}$ is continuous in (x_0, x_n). If $n-r=2$, then $g_{(r)\,\mathrm{ap}}$ is convex in (x_0, x_n) and so it is continuous in (x_0, x_n) and if $n-r > 2$ then $g_{(r)\,\mathrm{ap}}$ has finite derivative in (x_0, x_n) [2, Theorem 7(a)] and the assertion follows.

Hence $g_{(r)\,\mathrm{ap}}$ is the continuous rth derivative of g in every closed subinterval of (x_0, x_n) [5], and since $g_{(r)\,\mathrm{ap}}$ is $(n-r)$ convex, by repeated application of Corollary 3.6 we obtain that g is n-convex in (x_0, x_n). Hence $\lim\limits_{t \to x_0} g(t)$ and $\lim\limits_{t \to x_n} g(t)$ exist and so by property $\mathcal{D}$, g is continuous in $[x_0, x_n]$ and therefore g is n-convex in $[x_0, x_n]$. Since

$$Q_n(g; x_0, x_1 \ldots, x_n) = Q_n(f; x_0, x_1, \ldots, x_n) - Q_n(f; x_0, x_1, \ldots, x_n) = 0$$

Theorem 2.3 implies that g is a polynomial of degree at most $n-1$ and so $g_{(r)\,\mathrm{ap}}$ is a polynomial of degree at most $n-r-1$. Hence $Q_{n-r}(g_{(r)\,\mathrm{ap}}; y_0, \ldots, y_{n-r}) = 0$ for any set of $n-r+1$ points $y_0,\dots,y_{n-r}$, which gives by (1)

$$\frac{n!}{(n-r)!}\,Q_n(f;x_0,\dots,x_n)=Q_{n-r}(f_{(r)\,\mathrm{ap}};y_0,\dots,y_{n-r}),$$

proving the theorem. Similarly, if $Q_{n-r}(g_{(r)\,\mathrm{ap}};y_0,\dots,y_{n-r})\leqslant 0$ for every set of $n-r+1$ points $y_0,\dots,y_{n-r}$ in (x_0,x_n) then the proof follows. So we suppose that there is a set of points $y_0,\dots,y_{n-r}$ and a set of points $z_0,\dots,z_{n-r}$ in (x_0,x_n) such that

$$Q_{n-r}(g_{(r)\,\mathrm{ap}};y_0,\dots,y_{n-r})>0>Q_{n-r}(g_{(r)\,\mathrm{ap}};z_0,\dots,z_{n-r}).$$

Since $g_{(r)\,\mathrm{ap}}\in\mathcal{B}_1$, [4] and $g_{(r)\,\mathrm{ap}}\in\mathcal{D}$, [5], by Theorem 3.1 there is a set of points $\xi_0,\dots,\xi_{n-r}$ in (x_0,x_n) such that $Q_{n-r}(g_{(r)\,\mathrm{ap}};\xi_0,\dots,\xi_{n-r})=0$, which by (1) proves the result in this case.

Finally, we consider $n=r$. Then writing g as above we have

$$g_{(n)\,\mathrm{ap}}(x)=f_{(n)\,\mathrm{ap}}(x)-n!\,Q_n(f;x_0,\dots,x_n).$$

If $g_{(n)\,\mathrm{ap}}(x)\geqslant 0$ for every $x\in(x_0,x_n)$ then $g_{(n-1)\,\mathrm{ap}}$ is non decreasing [5] and so g is n-convex in $[x_0,x_n]$ and so as above g is a polynomial of degree at most $n-1$ in $[x_0,x_n]$ and hence $g_{(n)\,\mathrm{ap}}(x)=0$ for all $x\in[x_0,x_n]$, that is $f_{(n)\,\mathrm{ap}}(x)=n!\,Q_n(f;x_0,x_1,\dots,x_n)$ for all $x\in[x_0,x_n]$, proving the result. Similarly, if $g_{(n)\,\mathrm{ap}}(x)\leqslant 0$ for all $x\in(x_0,x_n)$ the result follows. So suppose that there are $\xi_1,\xi_2\in(x_0,x_n)$ such that $g_{(n)\,\mathrm{ap}}(\xi_1)>0>g_{(n)\,\mathrm{ap}}(\xi_2)$. Then by the property $\mathcal{D}$ of $g_{(n)\,\mathrm{ap}}$, [5], there is $\xi\in(x_0,x_n)$ such that $g_{(n)\,\mathrm{ap}}(\xi)=0$, that is $f_{(n)\,\mathrm{ap}}(\xi)=n!\,Q_n(f;x_0,\dots,x_n)$. This completes the proof. □

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