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PEXIDER TYPE OPERATORS AND THEIR NORMS IN X_λ SPACES

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Abstract. In this paper, we introduce Pexiderized generalized operators on certain special spaces introduced by Bielecki-Czerwik and investigate their norms.

Keywords: Pexiderized generalized operator, Pexiderized generalized Jensen operator

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1. Introduction

Let X and Y be complex normed spaces. After Czerwik [1], for a fixed non-negative real number λ, we denote by X_λ the linear space of all functions $f\colon X \to Y$ (with pointwise operations) for which there exists a constant $M_f \geqslant 0$ with

$$\|f(x)\| \leqslant M_f \mathrm{e}^{\lambda\|x\|}$$

for all $x \in X$. It is easy to show that the space X_λ with the norm

$$\|f\| := \sup_{x\in X}\{\mathrm{e}^{-\lambda\|x\|}\|f(x)\|\}$$

is a normed space. Let us we denote by X_λ^n the linear space of all functions φ: $\underbrace{X\times\ldots\times X}_{n\text{ times}} \to Y$ (with pointwise operations) for which there exists a constant $M_\varphi \geqslant 0$ with

$$\|\varphi(x_1,\ldots,x_n)\| \leqslant M_\varphi \mathrm{e}^{\lambda\sum\limits_{i=1}^{n}\|x_i\|}$$

for all $x_1,\ldots,x_n \in X$. It is not difficult to show that the space X_λ^n with the norm

$$\|\varphi\| := \sup_{x_1,\ldots,x_n\in X}\Big\{\|\varphi(x_1,\ldots,x_n)\|\mathrm{e}^{-\lambda\sum\limits_{i=1}^{n}\|x_i\|}\Big\}$$

is a normed space.

We denote by Z_λ^m the normed space $\bigoplus_{i=1}^{m} X_\lambda = \{(f_1,\dots,f_m)\colon f_1,\dots,f_m \in X_\lambda\}$ (with pointwise operations) together with the norm

$$\|(f_1,\dots,f_m)\| := \max\{\|f_1\|,\dots,\|f_m\|\}.$$

The norms of the Pexiderized Cauchy, quadratic and Jensen operators on function spaces X_λ have been investigated by Czerwik and Dlutek [1], [2]. In [4] Moslehian *et al.* have extended the results of [2] to the Pexiderized generalized Jensen and Pexiderized generalized quadratic operators on function spaces X_λ and provided more general results regarding their norms. In the proofs of our results we apply the ideas contained in the paper [2] (see also [1]).

In [3] S.-M. Jung investigated the norm of the cubic operator on the function spaces X_λ. In this paper, we introduce Pexiderized generalized operators and investigate their norms. The results extend the results of [2], [3], [4].

2. Main results

Definition 2.1. The operator $E_P^G\colon Z_\lambda^m \to X_\lambda^n$ defined by

$$E_P^G(f_1,\dots,f_m)(x_1,\dots,x_n) = \sum_{i=1}^{m} \alpha_i f_i\biggl(\sum_{j=1}^{n} \beta_{ij} x_j\biggr)$$

where $\alpha_i \in \mathbb{C}\setminus\{0\}, \beta_{ij} \in \mathbb{B}^1 := \{\mu \in \mathbb{C}\colon |\mu| \leqslant 1\}$ for all $1 \leqslant i \leqslant m$ and all $1 \leqslant j \leqslant n$, is called Pexiderized generalized operator.

The next theorem gives us the norm of E_P^G.

Theorem 2.2. *The operator E_P^G is a bounded linear operator with*

$$\|E_P^G\| = \sum_{i=1}^{m} |\alpha_i|.$$

P r o o f. First, we show that $\|E_P^G\| \leqslant \sum_{i=1}^{m} |\alpha_i|$. By the assumption we have

$$\max_{1\leqslant i\leqslant m} \biggl\{\biggl\|\sum_{j=1}^{n} \beta_{ij} x_j\biggr\|\biggr\} \leqslant \sum_{i=1}^{n} \|x_i\|$$

for all $x_1, \dots, x_n \in X$. Therefore

$$\begin{aligned}
\|E_P^G(f_1, \dots, f_m)\| &= \sup_{x_1,\dots,x_n \in X} \left\{ \left\| \sum_{i=1}^m \alpha_i f_i\left(\sum_{j=1}^n \beta_{ij} x_j \right) \right\| \mathrm{e}^{-\lambda \sum\limits_{j=1}^n \|x_j\|} \right\} \\
&\leqslant \sup_{x_1,\dots,x_n \in X} \left\{ \sum_{i=1}^m |\alpha_i| \left\| f_i\left(\sum_{j=1}^n \beta_{ij} x_j \right) \right\| \mathrm{e}^{-\lambda \left\| \sum\limits_{j=1}^n \beta_{ij} x_j \right\|} \right\} \\
&\leqslant \sum_{i=1}^m |\alpha_i| \|f_i\| \leqslant \sum_{i=1}^m |\alpha_i| \max\{\|f_1\|, \dots, \|f_m\|\} \\
&= \sum_{i=1}^m |\alpha_i| \|(f_1, \dots, f_m)\|
\end{aligned}$$

for any $(f_1, \dots, f_m) \in Z_\lambda^m$. This implies that

$$\|E_P^G\| \leqslant \sum_{i=1}^m |\alpha_i|.$$

Now, let $\nu \in Y$ be such that $\|\nu\| = 1$ and let $\{\xi_k\}_k$ be a sequence of positive real numbers decreasing to 0. For each positive integer k, and each $1 \leqslant i \leqslant m$, we define

$$f_{ik}(x) = \begin{cases} \dfrac{\overline{\alpha}_i}{|\alpha_i|} \mathrm{e}^{n\lambda\xi_k} \nu & \text{if } \|x\| = \xi_k \left| \sum\limits_{j=1}^n \beta_{ij} \right|, \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in X$. Then we have

$$\mathrm{e}^{-\lambda\|x\|} \|f_{ik}(x)\| = \begin{cases} \mathrm{e}^{\lambda\xi_k \left(n - \left| \sum\limits_{j=1}^n \beta_{ij} \right| \right)} & \text{if } \|x\| = \xi_k \left| \sum\limits_{j=1}^n \beta_{ij} \right|, \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in X$, all positive integers k, and all $1 \leqslant i \leqslant m$. Thus $f_{ik} \in X_\lambda$ for all positive integers k and all $1 \leqslant i \leqslant m$ with

$$\|f_{ik}\| = \mathrm{e}^{\lambda\xi_k \left(n - \left| \sum\limits_{j=1}^n \beta_{ij} \right| \right)}. \tag{2.1}$$

Let $u \in X$ be such that $\|u\| = 1$ and take $x_1, \dots, x_n \in X$ as $x_1 = \ldots = x_n = \xi_k u$. Then

$$\begin{aligned}
\|E_P^G(f_{1k}, \dots, f_{mk})\| &= \sup_{x_1,\dots,x_n \in X} \left\{ \left\| \sum_{i=1}^m \alpha_i f_{ik}\left(\sum_{j=1}^n \beta_{ij} x_j \right) \right\| \mathrm{e}^{-\lambda \sum\limits_{j=1}^n \|x_j\|} \right\} \\
&\geqslant \mathrm{e}^{-n\lambda\xi_k} \left\| \sum_{i=1}^m |\alpha_i| \mathrm{e}^{n\lambda\xi_k} \nu \right\| = \sum_{i=1}^m |\alpha_i|.
\end{aligned} \tag{2.2}$$

If on the contrary $\|E_P^G\| < \sum_{i=1}^{m} |\alpha_i|$, then there exists a $\delta > 0$ such that

$$\|E_P^G(f_{1k}, \ldots, f_{mk})\| \leqslant \left(\sum_{i=1}^{m} |\alpha_i| - \delta\right) \|(f_{1k}, \ldots, f_{mk})\| \tag{2.3}$$

for all positive integers k. So it follows from (2.1), (2.2) and (2.3) that

$$\sum_{i=1}^{m} |\alpha_i| \leqslant \|E_P^G(f_{1k}, \ldots, f_{mk})\| \leqslant \left(\sum_{i=1}^{m} |\alpha_i| - \delta\right) \max_{1 \leqslant i \leqslant m} \|f_{ik}\| \tag{2.4}$$

for all positive integers k. Since $\lim_{k \to \infty} \|f_{ik}\| = 1$ for all $1 \leqslant i \leqslant m$, the right hand side of (2.4) tends to $\sum_{i=1}^{m} |\alpha_i| - \delta$ as $k \to \infty$, whence $\sum_{i=1}^{m} |\alpha_i| \leqslant \sum_{i=1}^{m} |\alpha_i| - \delta$, which is a contradiction. Hence, $\|E_P^G\| = \sum_{i=1}^{m} |\alpha_i|$. $\square$

Theorem 2.6 of [4] is an alternative result for the following corollary.

Corollary 2.3. *The Pexiderized generalized Jensen operator* $J_P^{r,s,t}\colon Z_\lambda^3 \to X_\lambda^2$ *given by*

$$J_P^{r,s,t}(f,g,h)(x,y) := f\left(\frac{sx+ty}{r}\right) - \frac{s}{r}g(x) - \frac{t}{r}h(y)$$

where $r, s, t \in \mathbb{C}$ *with* $r \neq 0$ *and* $\max\{|s|,|t|\} \leqslant |r|$, *is a bounded linear operator such that*

$$\|J_P^{r,s,t}\| = \frac{|r|+|s|+|t|}{|r|}.$$

Corollary 2.4 [4]**.** *The Pexiderized quadratic operator* $Q_P^G\colon Z_\lambda^4 \to X_\lambda^2$ *given by*

$$Q_P^G(f,g,h,k)(x,y) := f(x+y) + g(x) - 2h(x) - 2k(y)$$

is a bounded linear operator with $\|Q_P^G\| = 6$.

Corollary 2.5 [3]**.** *The Pexiderized cubic operator* $C_P^G\colon Z_\lambda^5 \to X_\lambda^2$ *given by*

$$C_P^G(f,g,h,k,l)(x,y) := f(x+y) + g(x-y) - 2h\left(\frac{1}{2}x+y\right) - 2k\left(\frac{1}{2}x-y\right) - 12l\left(\frac{1}{2}x\right)$$

is a bounded linear operator with $\|C_P^G\| = 18$.

Theorem 2.6. *The generalized operator $E^G\colon X_\lambda \to X_\lambda^n$ given by*

$$E^G(f)(x_1,\ldots,x_n) = \sum_{i=1}^m \alpha_i f\biggl(\sum_{j=1}^n \beta_{ij}x_j\biggr)$$

where $\alpha_i \in \mathbb{C}\setminus\{0\}$, $\beta_{ij} \in \mathbb{B}^1 := \{\mu \in \mathbb{C}\colon |\mu| \leqslant 1\}$ for all $1 \leqslant i \leqslant m$ and all $1 \leqslant j \leqslant n$, is a bounded linear operator. Moreover, if $(\beta_{i1},\ldots,\beta_{in}) \neq (\beta_{k1},\ldots,\beta_{kn})$ and there exist $x_1^*,\ldots,x_n^* \in X$ such that $\sum\limits_{j=1}^n \beta_{ij}x_j^* \neq \sum\limits_{j=1}^n \beta_{kj}x_j^*$ for all $1 \leqslant i,k \leqslant m$ with $i \neq k$, then

$$\|E^G\| = \sum_{i=1}^m |\alpha_i|.$$

P r o o f. Similarly to the proof of Theorem 2.2, we have

$$\begin{aligned}
\|E^G(f)\| &= \sup_{x_1,\ldots,x_n\in X}\biggl\{\biggl\|\sum_{i=1}^m \alpha_i f\biggl(\sum_{j=1}^n \beta_{ij}x_j\biggr)\biggr\|\mathrm{e}^{-\lambda\sum\limits_{j=1}^n\|x_j\|}\biggr\}\\
&\leqslant \sup_{x_1,\ldots,x_n\in X}\biggl\{\sum_{i=1}^m |\alpha_i|\biggl\|f\biggl(\sum_{j=1}^n \beta_{ij}x_j\biggr)\biggr\|\mathrm{e}^{-\lambda\|\sum\limits_{j=1}^n\beta_{ij}x_j\|}\biggr\}\\
&\leqslant \sum_{i=1}^m |\alpha_i|\|f\|
\end{aligned}$$

for any $f \in X_\lambda$. So we have

$$\|E^G\| \leqslant \sum_{i=1}^m |\alpha_i|.$$

Now, suppose that $\sum\limits_{j=1}^n \beta_{ij}x_j^* \neq \sum\limits_{j=1}^n \beta_{kj}x_j^*$ for all $1 \leqslant i,k \leqslant m$ with $i \neq k$. Let $\nu \in Y$ be such that $\|\nu\| = 1$ and let $\{\xi_k\}_k$ be a sequence of positive real numbers decreasing to 0. For each positive integer k, we define

$$f_k(x) = \begin{cases} \dfrac{\overline{\alpha}_i}{|\alpha_i|}\mathrm{e}^{\lambda\theta\xi_k}\nu & \text{if } x = \xi_k \sum\limits_{j=1}^n \beta_{ij}x_j^* \quad \text{for} \ \ 1 \leqslant i \leqslant m,\\ 0 & \text{otherwise,}\end{cases}$$

for all $x \in X$, where $\theta := \sum\limits_{j=1}^n \|x_j^*\|$. Hence we have

$$\mathrm{e}^{-\lambda\|x\|}\|f_k(x)\| = \begin{cases} \mathrm{e}^{\lambda\xi_k\left(\theta-\left\|\sum\limits_{j=1}^n\beta_{ij}x_j^*\right\|\right)} & \text{if } x = \xi_k \sum\limits_{j=1}^n \beta_{ij}x_j^* \quad \text{for} \ \ 1 \leqslant i \leqslant m,\\ 0 & \text{otherwise,}\end{cases}$$

for all $x \in X$ and all positive integers k. So that $f_k \in X_\lambda$ for all positive integers k with

$$\|f_k\| = \max_{1\leqslant i\leqslant m} \mathrm{e}^{\lambda\xi_k\left(\theta - \left\|\sum\limits_{j=1}^{n}\beta_{ij}x_j^*\right\|\right)}. \tag{2.5}$$

Similarly to the proof of Theorem 2.2, take $x_1,\ldots,x_n \in X$ as $x_j = \xi_k x_j^*$ for all $1 \leqslant j \leqslant n$. Then we have

$$\begin{aligned}\|E^G(f_k)\| &= \sup_{x_1,\ldots,x_n\in X}\left\{\left\|\sum_{i=1}^{m}\alpha_i f_k\left(\sum_{j=1}^{n}\beta_{ij}x_j\right)\right\|\mathrm{e}^{-\lambda\sum\limits_{j=1}^{n}\|x_j\|}\right\}\\ &\geqslant \mathrm{e}^{-\lambda\theta\xi_k}\left\|\sum_{i=1}^{m}|\alpha_i|\mathrm{e}^{\lambda\theta\xi_k}\nu\right\| = \sum_{i=1}^{m}|\alpha_i|.\end{aligned}$$

If on the contrary $\|E^G\| < \sum\limits_{i=1}^{m}|\alpha_i|$, then there exists a $\delta > 0$ such that

$$\|E^G(f_k)\| \leqslant \left(\sum_{i=1}^{m}|\alpha_i| - \delta\right)\|f_k\|$$

for all positive integers k. Similarly to the proof of Theorem 2.2, we have

$$\sum_{i=1}^{m}|\alpha_i| \leqslant \|E^G(f_k)\| \leqslant \left(\sum_{i=1}^{m}|\alpha_i| - \delta\right)\|f_k\| \tag{2.6}$$

for all positive integers k. Since $\lim\limits_{k\to\infty}\xi_k = 0$, it follows from (2.5) that $\lim\limits_{k\to\infty}\|f_k\| = 1$. Then the right hand side of (2.6) tends to $\sum\limits_{i=1}^{m}|\alpha_i| - \delta$ as $k \to \infty$, whence $\sum\limits_{i=1}^{m}|\alpha_i| \leqslant \sum\limits_{i=1}^{m}|\alpha_i| - \delta$, which is a contradiction. Hence, $\|E^G\| = \sum\limits_{i=1}^{m}|\alpha_i|$. □

Corollary 2.7. *Let $E^G\colon X_\lambda \to X_\lambda^n$ be an operator given by*

$$E^G(f)(x_1,\ldots,x_n) = \sum_{i=1}^{m}\alpha_i f\left(\sum_{j=1}^{n}\beta_{ij}x_j\right)$$

where $\alpha_i \in \mathbb{C}\setminus\{0\}, \beta_{ij} \in \mathbb{B}^1 := \{\mu \in \mathbb{C}\colon |\mu| \leqslant 1\}$ for all $1 \leqslant i \leqslant m$ and all $1 \leqslant j \leqslant n$. If $(\beta_{i1},\ldots,\beta_{in}) \neq (\beta_{k1},\ldots,\beta_{kn})$ and $\sum\limits_{j=1}^{n}\beta_{ij} \neq \sum\limits_{j=1}^{n}\beta_{kj}$ for all $1 \leqslant i,k \leqslant m$ with $i \neq k$, then

$$\|E^G\| = \sum_{i=1}^{m}|\alpha_i|.$$

Corollary 2.8. *The cubic operator* $C^G\colon X_\lambda \to X_\lambda^2$ *given by*

$$C^G(f)(x,y) := f(x+y) + f(x-y) - 2f\Big(\frac{1}{2}x + y\Big) - 2f\Big(\frac{1}{2}x - y\Big) - 12f\Big(\frac{1}{2}x\Big)$$

is a bounded linear operator with $\|C_P^G\| = 18$.

Theorem 2.9. *Let* $\alpha_i, \beta_i \in \mathbb{B}^1 := \{\mu \in \mathbb{C}\colon |\mu| \leqslant 1\}$ *for all* $1 \leqslant i \leqslant 4$ *such that* $\alpha_4 = \beta_3 = 0$ *and* $(\alpha_i, \beta_i) \neq (\alpha_j, \beta_j)$ *for all* $1 \leqslant i < j \leqslant 4$. *Let* $\Lambda\colon X_\lambda \to X_\lambda^2$ *be an operator given by*

$$\Lambda(f)(x,y) = \sum_{i=1}^{4} \gamma_i f\big(\alpha_i x + \beta_i y\big)$$

where $\gamma_i \in \mathbb{C} \setminus \{0\}$ *for all* $1 \leqslant i \leqslant 4$. *Then*

$$\|\Lambda\| = \sum_{i=1}^{4} |\gamma_i|.$$

Proof. Similarly to the proof of Theorem 2.6, we have

$$\|\Lambda\| \leqslant \sum_{i=1}^{4} |\gamma_i|.$$

Let $\eta \in \mathbb{C}$ be such that $\eta(\beta_i - \beta_j) \neq \alpha_j - \alpha_i$ for all $1 \leqslant i, j \leqslant 4$ with $i \neq j$. Let $u \in X, \nu \in Y$ be such that $\|u\| = \|\nu\| = 1$ and let $\{\xi_k\}_k$ be a sequence of positive real numbers decreasing to 0. For each positive integer k, we define

$$f_k(x) = \begin{cases} \dfrac{\overline{\gamma}_i}{|\gamma_i|} \mathrm{e}^{\lambda(1+|\eta|)\xi_k} \nu & \text{if } x = \xi_k(\alpha_i + \eta\beta_i)u \text{ for } 1 \leqslant i \leqslant 4, \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in X$. Therefore we have

$$\mathrm{e}^{-\lambda\|x\|}\|f_k(x)\| = \begin{cases} \mathrm{e}^{\lambda\xi_k(1+|\eta|-|\alpha_i+\eta\beta_i|)} & \text{if } x = \xi_k(\alpha_i + \eta\beta_i)u \text{ for } 1 \leqslant i \leqslant 4, \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in X$ and all positive integers k. Thus $f_k \in X_\lambda$ for all positive integers k with

$$\|f_k\| = \max_{1 \leqslant i \leqslant 4} \mathrm{e}^{\lambda\xi_k(1+|\eta|-|\alpha_i+\eta\beta_i|)}. \tag{2.7}$$

Similarly to the proof of Theorem 2.6, take $x, y \in X$ as $x = \xi_k u$ and $y = \xi_k \eta u$. Then it is clear that

$$\|\Lambda(f_k)\| \geqslant \sum_{i=1}^{4} |\gamma_i|$$

for all positive integers k. The rest of the proof is similar to the proof of Theorem 2.6.

□

Corollary 2.10. *The generalized Jensen operator* $J^{r,s,t}\colon\ X_\lambda \to X_\lambda^2$ *given by*

$$J^{r,s,t}(f)(x,y) := f\Big(\frac{sx+ty}{r}\Big) - \frac{s}{r}f(x) - \frac{t}{r}f(y)$$

where $r,s,t \in \mathbb{C}$ *with* $r \neq 0$ *and* $\max\{|s|,|t|\} \leqslant |r|$, *is a bounded linear operator such that*

$$\|J^{r,s,t}\| = \frac{|r|+|s|+|t|}{|r|}.$$

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