Huanyin Chen

Exchange rings in which all regular elements are one-sided unit-regular

Czechoslovak Mathematical Journal, Vol. 58 (2008), No. 4, 899–910

Persistent URL: `http://dml.cz/dmlcz/140429`

Czechoslovak Mathematical Journal, 58 (133) (2008), 899–910

EXCHANGE RINGS IN WHICH ALL REGULAR ELEMENTS ARE ONE-SIDED UNIT-REGULAR

Huanyin Chen, Hangzhou

(Received September 2, 2006)

Abstract. Let R be an exchange ring in which all regular elements are one-sided unit-regular. Then every regular element in R is the sum of an idempotent and a one-sided unit. Furthermore, we extend this result to exchange rings satisfying related comparability.

Keywords: exchange ring, one-sided unit-regularity, idempotent

MSC 2010: 16E50, 16U99

An element $a \in R$ is clean (unit-regular) provided it is the sum (product) of an idempotent and a unit. A ring R is clean (unit-regular) if every element in R is clean (unit-regular). In [5, Theorem 1], Camillo and Khurana proved that every unit-regular ring is clean. Clean property has been extensively studied in literature; see, e.g., [8], [10] and [13]. An element $a \in R$ is one-sided unit-regular provided there exists a one-sided unit $u \in R$ such that $a = aua$. A regular $a \in R$ is one-sided unit-regular iff it is the product of an idempotent and a one-sided unit-regular. A regular ring R is one-sided unit-regular provided every element in R is one-sided unit-regular (see [6]). Let V be an infinite-dimensional vector space over a division ring D. Then $\mathrm{End}_D(V)$ is one-sided unit-regular, while it is not unit-regular. In [8, Theorem 1], the author extended [5, Theorem 1] and showed that every element in a one-sided unit-regular ring is the sum of an idempotent and a one-sided unit.

A ring R is an exchange ring if for every right R-module A and two decompositions $A = M \oplus N = \bigoplus\limits_{i\in I} A_i$, where $M_R \cong R$ and the index set I is finite, there exist submodules $A_i' \subseteq A_i$ such that $A = M \oplus \Big(\bigoplus\limits_{i\in I} A_i'\Big)$. A ring R is an exchange ring if and only if for any $x \in R$ there exists an idempotent $e \in Rx$ such that $1 - e \in R(1 - x)$ (cf. [11]). Clearly, regular rings, π-regular rings, semi-perfect rings, left or right continuous rings, clean rings and unit C^*-algebras of real rank zero (cf. [3, Theorem 7.2]) are all exchange rings.

As is well known, a single one-sided unit-regular element in a ring need not be written as a sum of an idempotent and a one-sided unit. A natural problem is how to extend [5, Theorem 1] and [8, Theorem 1] to exchange rings. Let R be an exchange ring in which every regular element is one-sided unit-regular. We have observed that every regular element in R is the sum of an idempotent and a non-zero divisor. In this note, we extend this result and prove that every regular element in R is the sum of an idempotent and a one-sided unit. Following the author, an exchange ring R is said to satisfy related comparability provided that $R = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong B_1$ implies that there exists a central idempotent $e \in R$ such that $B_1e \lesssim^{\oplus} B_2e$ and $B_2(1-e) \lesssim^{\oplus} B_1(1-e)$ (cf. [7]). Rings of this kind include exchange rings satisfying general comparability. Furthermore, we show that every regular element in an exchange ring satisfying related comparability is the sum of an idempotent and the product of two one-sided units. This generalizes [8, Theorem 3] as well.

Throughout the paper, every ring is associative with an identity. An element $x \in R$ is regular if there exists $y \in R$ such that $x = xyx$. A ring R is (one-sided unit) regular if every element in R is (one-sided unit) regular. $M \lesssim^{\oplus} N$ means that a right R-module M is isomorphic to a direct summand of a right R-module N.

Lemma 1. *Let R be an exchange ring. Then the following conditions are equivalent:*

(1) *Every regular element in R is one-sided unit-regular.*

(2) *Given any right R-module decompositions $R = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong A_2$, then $B_1 \lesssim^{\oplus} B_2$ or $B_2 \lesssim^{\oplus} B_1$.*

(2) *Every regular element in $M_n(R)$ ($n \in \mathbb{N}$) is one-sided unit-regular.*

Proof. (1) $\Leftrightarrow$ (2) is obvious by [15, Theorem 3.1].

(1) $\Leftrightarrow$ (3) In the proof of [7, Lemma 5] we choose $e = 0$ or 1, and then the result follows. $\square$

Many authors investigated exchange rings in which every regular element is one-sided unit-regular. Let R be an exchange ring. Then every regular element in R is one-sided unit-regular iff for every $a \in R$ there exist an idempotent $e \in R$ and a one-sided unit $u \in R$ such that $a = eu$ (cf. [15, Theorem 3.4]). Following Ara et al. (cf. [3] and [11]), a ring R is a separative ring if the following condition holds for all finitely generated projective right R-modules A, B: $A \oplus A \cong A \oplus B \cong B \oplus B \Longrightarrow A \cong B$. In [7, Theorem 3], the author proved that such an exchange ring is separative; hence, its stable rank is $1, 2$ or ∞.

Theorem 2. *Let R be an exchange ring in which every regular element is one-sided unit-regular. Then every regular element in R is the sum of an idempotent and a one-sided unit.*

Proof. Let $a \in R$ be regular. Then we have $x \in R$ such that $a = axa$. So $R = \operatorname{Im} a \oplus (1 - ax)R = xaR \oplus \operatorname{Ker} a$. Since R is an exchange ring, there exist right R-modules X_1, Y_1 such that $R = \operatorname{Im} a \oplus X_1 \oplus Y_1$ with $X_1 \subseteq \operatorname{Ker} a$ and $Y_1 \subseteq xaR$. Clearly, $\operatorname{Ker} a = \operatorname{Ker} a \cap (X_1 \oplus \operatorname{Im} a \oplus Y_1) = X_1 \oplus X_2$, where $X_2 = \operatorname{Ker} a \cap (\operatorname{Im} a \oplus Y_1)$. Likewise, we have a right R-module Y_2 such that $xaR = Y_1 \oplus Y_2$. Clearly,

$$R = \operatorname{Im} a \oplus X_1 \oplus Y_1 = X_1 \oplus X_2 \oplus xaR.$$

In addition, $\theta\colon \operatorname{Im} a = aR \cong xaR$ given by $\theta(ar) = xar$ for any $r \in R$. Thus, $\operatorname{Im} a \oplus X_1 \cong X_1 \oplus xaR$. In view of Lemma 1, we get $X_2 \lesssim^{\oplus} Y_1$ or $Y_1 \lesssim^{\oplus} X_2$. Thus, there exist right R-morphisms $\psi\colon X_2 \to Y_1$ and $\varphi\colon Y_1 \to X_2$ such that $\varphi\psi = 1_{X_2}$ or $\psi\varphi = 1_{Y_1}$. Let

$$\begin{aligned} &k\colon X_1 \oplus X_2 \to X_1 \oplus Y_1;\ x_1 + x_2 \mapsto x_1 + \psi(x_2), \forall x_1 \in X_1, x_2 \in X_2;\\ &l\colon X_1 \oplus Y_1 \to X_1 \oplus Y_2;\ x_1 + y_1 \mapsto x_1 + \varphi(y_1), \forall x_1 \in X_1, y_1 \in Y_1. \end{aligned}$$

Let

$$\begin{aligned} h\colon\ & R = X_1 \oplus X_2 \oplus Y_1 \oplus Y_2 \to X_1 \oplus Y_1 \oplus X_2 \oplus Y_2 = R;\\ & x_1 + x_2 + y_1 + y_2 \mapsto k(x_1 + x_2) + y_1, \forall x_1 \in X_1, x_2 \in X_2, y_1 \in Y_1, y_2 \in Y_2;\\ v\colon\ & X_1 \oplus Y_1 \oplus X_2 \oplus Y_2 \to X_1 \oplus X_2 \oplus Y_1 \oplus Y_2;\\ & x_1 + y_1 + x_2 + y_2 \mapsto l(x_1 + y_1) + \psi(x_2), \forall x_1 \in X_1, y_1 \in Y_1, x_2 \in X_2, y_2 \in Y_2. \end{aligned}$$

For any $x_1 \in X_1$, $y_1 \in Y_1$, $x_2 \in X_2$, $y_2 \in Y_2$, it is easy to verify that

$$\begin{aligned} hvhv(x_1 + y_1 + x_2 + y_2) &= hvh\big(l(x_1 + y_1) + \psi(x_2)\big) = hv\big(kl(x_1 + y_1) + \psi(x_2)\big)\\ &= h\big(lkl(x_1 + y_1) + \varphi\psi(x_2)\big) = klkl(x_1 + y_1) + \psi\varphi\psi(x_2)\\ &= kl(x_1 + y_1) + \psi(x_2) = hv(x_1 + y_1 + x_2 + y_2). \end{aligned}$$

Hence $(hv)^2 = hv$. Let $e = hv$. Then $e = e^2 \in \operatorname{End}_R(R)$.

Assume that $\varphi\psi = 1_{X_2}$. Let

$$\begin{aligned} \varphi\colon\ & R = \operatorname{Im} a \oplus X_1 \oplus Y_1 \to R;\\ & z + x_1 + y_1 \mapsto xz + x_1 + \varphi(y_1),\ \forall z \in \operatorname{Im} a, x_1 \in X_1, y_1 \in Y_1. \end{aligned}$$

Then

$$\begin{aligned}
&\varphi(a-hv)(x_1+y_1+x_2+y_2)=\varphi\big(a(y_1+y_2)-kl(x_1+y_1)-\psi(x_2)\big)\\
&\quad=\varphi\big(a(y_1+y_2)-x_1-\psi\varphi(y_1)-\psi(x_2)\big)=xa(y_1+y_2)-x_1-\varphi\psi\varphi(y_1)-\varphi\psi(x_2)\\
&\quad=y_1+y_2-x_1-\varphi(y_1)-x_2, \forall x_1\in X_1, y_1\in Y_1, x_2\in X_2, y_2\in Y_2.
\end{aligned}$$

Furthermore, we get

$$\begin{aligned}
&\varphi(a-hv)\varphi(a-hv)(x_1+y_1+x_2+y_2)\\
&\quad=\varphi(a-hv)\big(y_1+y_2-x_1-\varphi(y_1)-x_2\big)\\
&\quad=y_1+y_2+x_1-\varphi(y_1)+\varphi(y_1)+x_2\\
&\quad=x_1+x_2+y_1+y_2, \forall x_1\in X_1, y_1\in Y_1, x_2\in X_2, y_2\in Y_2.
\end{aligned}$$

This implies that $\varphi(a-hv)\varphi(a-hv)=1_R$, and so $a-hv\in \operatorname{End}_R(R)$ is left invertible.

Assume that $\psi\varphi=1_{Y_1}$. Given any $t\in \operatorname{Im} a, x_1\in X_1, y_1\in Y_1$, we have $t\in aR=a\big(Y_1\oplus Y_2\big)$. So we can find $y_1'\in Y_1$ and $y_2'\in Y_2$ such that $t=a(y_1'+y_2')$. Choose $x_1'=-x_1\in X_1$ and $x_2'=-\varphi(y_1+y_1')\in X_2$. Then

$$\begin{aligned}
(a-hv)(x_1'+x_2'+y_1'+y_2')&=a(y_1'+y_2')-x_1'-\psi\varphi(y_1')-\psi(x_2')\\
&=t+x_1-y_1'-\psi(x_2')=t+x_1+y_1.
\end{aligned}$$

This means that $a-hv\in \operatorname{End}_R(R)$ is a right R-epimorphism. As R is a projective right R-module, $a-hv\in \operatorname{End}_R(R)$ is right invertible. Let $u=a-hv$. Thus, $a=e+u$, where $u\in R$ is right or left invertible. The proof is complete. $\square$

Recall that an exchange ring R satisfies the comparability axiom provided that for any regular $x,y\in R$, either $xR\lesssim^{\oplus} yR$ or $yR\lesssim^{\oplus} xR$. Let R be an exchange ring satisfying the comparability axiom. We claim that every regular element in R is the sum of an idempotent and a one-sided unit. Since every regular element in such an exchange ring is one-sided unit-regular, we are done by Theorem 2.

Corollary 3. *Let R be an exchange ring in which every regular element is one-sided unit-regular. Then every regular matrix $A\in M_n(R)$ ($n\in\mathbb{N}$) is the sum of an idempotent matrix and a right or left invertible matrix.*

Proof. In view of Lemma 1, $M_n(R)$ is an exchange ring in which every regular matrix is one-sided unit-regular. Therefore the assertion is true by Theorem 2. $\square$

Corollary 4. *Let R be an exchange ring in which every regular element is one-sided unit-regular. If $a \in R$ is regular, then $2a$ is the sum of two right or left invertible elements.*

Proof. Let $a \in R$. In view of Theorem 2, there exists an idempotent $e \in R$ such that $a - e \in R$ is right or left invertible. As in the proof of Theorem 2, we also have that $a + e \in R$ is right or left invertible. Therefore $2a = (a - e) + (a + e)$, as required. $\square$

Let R be an exchange ring in which every regular element is one-sided unit-regular. If $\frac{1}{2} \in R$, then every regular $a \in R$ is the sum of two one-sided units.

Theorem 2. *Let A be a right R-module having the finite exchange property, and let $E = \mathrm{End}_R(A)$. Suppose that A is expressible as a direct sum of isomorphic indecomposable submodules. Then every regular element in E is the sum of an idempotent and an one-sided unit.*

Proof. Assume that $A = A_1 \oplus B = A_2 \oplus C$ with $A_1 \cong A_2$, then $A = A_1 \oplus B = \bigoplus\limits_{i \in I} Y_i$, where each Y_i is isomorphic to an indecomposable submodule Y of A. In view of [16, Lemma 28.1], A_1 has the finite exchange property. Thus, we have some $Y_i' \subseteq Y_i$ such that $A = A_1 \oplus \Big(\bigoplus\limits_{i \in I} Y_i'\Big)$. It is easy to verify that $Y_i' \subseteq^{\oplus} Y_i$ for all $i \in I$. As each Y_i is indecomposable, we see that either $Y_i' = 0$ or $Y_i' = Y_i$. Thus, there is a subset H_1 of I such that $B \cong \bigoplus\limits_{i \in H_1} Y_i$. Likewise, there is a subset H_2 of I such that $C \cong \bigoplus\limits_{i \in H_2} Y_i$. Clearly, $|H_1| \leqslant |H_2|$ or $|H_2| \leqslant |H_1|$, whence either $B \lesssim^{\oplus} C$ or $C \lesssim^{\oplus} B$. As in the proof of Lemma 1, every regular element in E is one-sided unit-regular. Therefore we complete the proof by Theorem 2. $\square$

Separativity plays a key role in the direct sum decomposition theory of exchange rings. It is conceivable that all exchange rings are separative (see [11]).

Corollary 6. *Let R be a simple separative exchange ring. Then every regular element in R is the sum of an idempotent and a one-sided unit.*

Proof. If R is directly finite, R has stable range one from [3, Theorem 3.4]. It follows by Corollary 5 that every regular element in R is the sum of an idempotent and a unit. If R is directly infinite, then $R \oplus D \cong R$ for some nonzero right R-module D. Let $x, y \in R$ be regular. If $x = 0$ or $y = 0$, then $xR \lesssim^{\oplus} yR$ or $yR \lesssim^{\oplus} xR$. Now we assume that $x \neq 0, y \neq 0$. Since R is simple, there exists $n \in \mathbb{N}$ such that $xR \lesssim^{\oplus} nD$. Thus $xR \oplus R \lesssim^{\oplus} nD \oplus R \cong R$, and so $xR \oplus R \lesssim^{\oplus} R \lesssim^{\oplus} yR \oplus R$. Hence, $R \oplus \big(xR \oplus E\big) \cong R \oplus yR$ for a right R-module E. As xR and yR are both nonzero, we have $R \lesssim^{\oplus} s(xR) \lesssim^{\oplus} s\big(xR \oplus E\big)$ and $R \lesssim^{\oplus} t(yR)$ for some $s, t \in \mathbb{N}$. Applying [3, Lemma 2.1], $xR \lesssim^{\oplus} xR \oplus E \cong yR$. In view of [15, Theorem 3.1], every regular element in R is one-sided unit-regular. Therefore the result follows from Theorem 2. $\square$

An element $e \in R$ is infinite if there exist orthogonal idempotents $f, g \in R$ such that $e = f + g$ while $eR \cong fR$ and $g \neq 0$. A simple ring is said to be purely infinite if every nonzero right ideal of R contains an infinite idempotent (cf. [2]). It is well known that a ring R is a purely infinite, simple ring iff it is not a division ring and for any nonzero $a \in R$ there exist $s, t \in R$ such that $sat = 1$ (see [4, Theorem 1.6]). The class of purely infinite simple regular rings is rather large (cf. [4, Example 1.3]).

Theorem 7. *Let R be a purely infinite, simple ring. Then every regular element in R is the sum of an idempotent and a one-sided unit.*

Proof. In view of [2, Theorem 1.1], R is an exchange ring. Let $x, y \in R$ be regular. If $x = 0$, then $xR \lesssim^{\oplus} yR$. If $x \neq 0$, then there exist $s, t \in R$ such that $sxt = 1$. Construct a right R-morphism $\varphi\colon xR \to R$ given by $\varphi(xr) = sxr$ for any $r \in R$. For any $r \in R$, we see that $r = sxtr = \varphi(xtr)$, and so φ is an R-epimorphism. Since R is a projective right R-module, we get a split exact sequence:

$$0 \to \operatorname{Ker}\varphi \hookrightarrow xR \xrightarrow{\varphi} R \to 0.$$

Thus, $R \lesssim^{\oplus} R \oplus \operatorname{Ker}\varphi \cong xR$. As $y \in R$ is regular, $yR \lesssim^{\oplus} R$. As a result, $yR \lesssim^{\oplus} xR$. According to [15, Theorem 3.1], every regular element in R is one-sided unit-regular, and therefore we complete the proof from Theorem 2. $\square$

We claim that every purely infinite, simple ring is separative. As in the proof of Theorem 7, every regular element in a purely infinite, simple ring is one-sided unit-regular, and we are through by [7, Theorem 3]. Use $2R$ to denote the set $\{2a \in R;\ a \in R\}$. Now we derive

Corollary 8. *Let R be a purely infinite, simple ring. Then $2R$ is generated by six one-sided units.*

Proof. By virtue of [2, Theorem 1.1], R is an exchange ring. Let $a \in R$. Then there exists an idempotent $e \in aR$ such that $1-e \in (1-a)R$. Thus, $ea, (1-e)(1-a) \in R$ are both regular. According to Theorem 7, we have an idempotent $f \in R$ and a one-sided unit $u \in R$ such that $ea = f + u$. Similarly to Corollary 4, we also have an one-sided unit $v \in R$ such that $ea = -f + v$. Thus, $2ea = (f + u) + (-f + v) = u + v$.

Likewise, we have two one-sided units $u', v' \in R$ such that $2(1-e)(1-a) = u' + v'$. Thus

$$\begin{aligned} 2a &= 2(1-e) + 2(eax) - 2(1-e)(1-a) \\ &= 1 + (1-2e) + u + v - u' - v' \\ &= 1 + (1-2e)^{-1} + u + v - u' - v'. \end{aligned}$$

Therefore $2R$ is generated by six one-sided units. □

Lemma 9. *Let R be an exchange ring. Then the following conditions are equivalent:*

(1) *R satisfies related comparability.*

(2) *Given any right R-module decompositions $R = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong A_2$, there exists a central idempotent $e \in R$ such that $B_1e \lesssim^{\oplus} B_2e$ and $B_2(1-e) \lesssim^{\oplus} B_1(1-e)$.*

(3) *$M_n(R)(n \in \mathbb{N})$ satisfies related comparability.*

P r o o f. See [7, Theorem 6] and [9, Lemma 5]. □

Theorem 10. *Let R be an exchange ring satisfying related comparability. Then every regular element in R is the sum of an idempotent and the product of two one-sided units.*

P r o o f. Let $a \in R$ be regular. Then we have $x \in R$ such that $a = axa$. As in the proof of Theorem 2, we can find right R-modules X_1, X_2, Y_1, Y_2 such that $R = X_1 \oplus X_2 \oplus Y_1 \oplus Y_2$. In addition,

$$R = \operatorname{Im} a \oplus X_1 \oplus Y_1 = X_1 \oplus X_2 \oplus xaR \quad \text{with } \operatorname{Im} a \oplus X_1 \cong X_1 \oplus xaR.$$

Since R satisfies related comparability, there exists a central idempotent $e \in R$ such that $X_2e \lesssim^{\oplus} Y_1e$ and $Y_1(1-e) \lesssim^{\oplus} X_2(1-e)$. Thus, there exist right R-morphisms $\psi\colon X_2e \to Y_1e$ and $\varphi\colon Y_1e \to X_2e$ such that $\varphi\psi = 1_{X_2e}$. Let

$$\begin{aligned} &k\colon X_1e \oplus X_2e \to X_1e \oplus Y_1e; \\ &\quad x_1 + x_2 \mapsto x_1 + \psi(x_2), \forall x_1 \in X_1e, x_2 \in X_2e; \\ &l\colon X_1e \oplus Y_1e \to X_2e \oplus Y_2e; \\ &\quad x_1 + y_1 \mapsto x_1 + \varphi(y_1),\ \forall x_1 \in X_1e, y_1 \in Y_1e. \end{aligned}$$

Let

$$\begin{aligned} &h\colon \operatorname{Re} = X_1e \oplus X_2e \oplus Y_1e \oplus Y_2e \to X_1e \oplus Y_1e \oplus X_2e \oplus Y_2e = \operatorname{Re}, \\ &\quad x_1 + x_2 + y_1 + y_2 \mapsto k(x_1 + x_2) + y_1, \\ &v\colon \operatorname{Re} = X_1e \oplus Y_1e \oplus X_2e \oplus Y_2e \to X_1e \oplus X_2e \oplus Y_1e \oplus Y_2e = \operatorname{Re}, \\ &\quad x_1 + y_1 + x_2 + y_2 \mapsto l(x_1 + y_1) + \psi(x_2), \end{aligned}$$

$\forall x_1 \in X_1 e$, $y_1 \in Y_1 e$, $x_2 \in X_2 e$, $y_2 \in Y_2 e$. As in the proof of Theorem 2, we get that $hv \in \mathrm{End}_R(\mathrm{Re})$ is an idempotent. Let

$$\begin{aligned} \varphi\colon\ & \mathrm{Re} = (\operatorname{Im} a)e \oplus X_1 e \oplus Y_1 e \to \mathrm{Re}; \\ & z + x_1 + y_1 \mapsto xz + x_1 + \varphi(y_1), \forall z \in (\operatorname{Im} a)e, x_1 \in X_1 e, y_1 \in Y_1 e. \end{aligned}$$

Then one simply checks that $\varphi(ae-hv)\varphi(ae-hv) = 1_{\mathrm{Re}}$, and so $ae-hv \in \mathrm{End}_R(\mathrm{Re})$ is left invertible.

Also we have two right R-morphisms $\psi'\colon\ X_2(1-e) \to Y_1(1-e)$ and $\varphi'\colon\ Y_1(1-e) \to X_2(1-e)$ such that $\psi\varphi = 1_{Y_1(1-e)}$. Let

$$\begin{aligned} k'\colon\ & X_1(1-e) \oplus X_2(1-e) \to X_1(1-e) \oplus Y_1(1-e); \\ & x_1 + x_2 \mapsto x_1 + \psi(x_2), \forall x_1 \in X_1(1-e), x_2 \in X_2(1-e); \\ l'\colon\ & X_1(1-e) \oplus Y_1(1-e) \to X_2(1-e) \oplus Y_2(1-e); \\ & x_1 + y_1 \mapsto x_1 + \varphi(y_1), \forall x_1 \in X_1(1-e), y_1 \in Y_1(1-e). \end{aligned}$$

Let

$$\begin{aligned} h'\colon\ & R(1-e) = X_1(1-e) \oplus X_2(1-e) \oplus Y_1(1-e) \oplus Y_2(1-e) \to \\ & X_1(1-e) \oplus Y_1(1-e) \oplus X_2(1-e) \oplus Y_2(1-e) = R(1-e), \\ & x_1 + x_2 + y_1 + y_2 \mapsto k(x_1 + x_2) + y_1, \\ v'\colon\ & R(1-e) = X_1(1-e) \oplus Y_1(1-e) \oplus X_2(1-e) \oplus Y_2(1-e) \to \\ & X_1(1-e) \oplus X_2(1-e) \oplus Y_1(1-e) \oplus Y_2(1-e) = R(1-e); \\ & x_1 + y_1 + x_2 + y_2 \mapsto l(x_1 + y_1) + \psi(x_2), \end{aligned}$$

$\forall x_1 \in X_1(1-e)$, $y_1 \in Y_1(1-e)$, $x_2 \in X_2(1-e)$, $y_2 \in Y_2(1-e)$. Similarly, we see that $h'v' \in \mathrm{End}_R\big(R(1-e)\big)$ is an idempotent. Given any $t \in (\operatorname{Im} a)(1-e)$, $x_1 \in X_1(1-e)$, $y_1 \in Y_1(1-e)$, we have $t \in aR(1-e) = a\big(Y_1 \oplus Y_2\big)(1-e)$. So we can find $y_1' \in Y_1(1-e)$ and $y_2' \in Y_2(1-e)$ such that $t = a(y_1' + y_2')$. Choose $x_1' = -x_1 \in X_1(1-e)$ and $x_2' = -\varphi(y_1 + y_1') \in X_2(1-e)$. Then we have

$$\big(a(1-e) - hv\big)(x_1' + x_2' + y_1' + y_2') = t + x_1 + y_1.$$

As $R(1-e)$ is a projective right R-module, $a(1-e) - h'v' \in \mathrm{End}_R\left(R(1-e)\right)$ is right invertible. As a result, we deduce that

$$\begin{aligned} a &= ae + a(1-e) = \big(hv + h'v'\big) + \big((ae - hv) + (a(1-e) - h'v')\big) \\ &= \big(hv + h'v'\big) + \big((ae - hv) + 1_{R(1-e)}\big)\big(1_{\mathrm{Re}} + (a(1-e) - h'v')\big), \end{aligned}$$

where

$$
\begin{aligned}
hv + h'v' \colon\ & R = \mathrm{Re} + R(1-e) \to \mathrm{Re} + R(1-e) = R,\\
& x + y \mapsto hv(x) + h'v'(y),\\
(ae - hv) + 1_{R(1-e)} \colon\ & R = \mathrm{Re} + R(1-e) \to \mathrm{Re} + R(1-e) = R,\\
& x + y \mapsto (ae - hv)(x) + y,\\
1_{\mathrm{Re}} + \big(a(1-e) - h'v'\big) \colon\ & R = \mathrm{Re} + R(1-e) \to \mathrm{Re} + R(1-e) = R,\\
& x + y \mapsto x + \big(a(1-e) - h'v'\big)(y),
\end{aligned}
$$

$\forall x \in \mathrm{Re}$, $y \in R(1-e)$. It is easy to verify that $hv + h'v' \in \mathrm{End}_R(R)$ is an idempotent, $(ae - hv) + 1_{R(1-e)} \in \mathrm{End}_R(R)$ is left invertible and $1_{\mathrm{Re}} + \big(a(1-e) - h'v'\big) \in \mathrm{End}_R(R)$ is right invertible. Thus the result follows. $\square$

Corollary 11. *Let R be an exchange ring satisfying related comparability. Then every regular matrix $A \in M_n(R)$ $(n \in \mathbb{N})$ is the sum of an idempotent matrix and the product of two right or left invertible matrices.*

Proof. In view of Lemma 9, $M_n(R)$ is an exchange ring satisfying related comparability, and therefore the assertion is true by Theorem 10. $\square$

Corollary 12. *Let R be an exchange ring satisfying related comparability. If $a \in R$ is regular, then $2a$ is the sum of two products of two right or left invertible elements.*

Proof. According to Theorem 9, there exists an idempotent $e \in R$ and two one-sided units $u, v \in R$ such that $a - e = uv$. As in the proof in Theorem 9, we also have $a + e = u'v'$ for some one-sided units $u', v' \in R$. As a result, we deduce that $2a = (a - e) + (a + e) = uv + u'v'$, as asserted. $\square$

Theorem 13. *Let R be an exchange ring and let $a \in R$ be regular. If $\mathrm{End}_R(aR)$ satisfies related comparability, then $a \in R$ is the sum of an idempotent and the product of two one-sided units.*

Proof. Let $a \in R$ be regular. Then we have $x \in R$ such that $a = axa$. Let $f = ax$. Then $\mathrm{End}_R(aR) \cong fRf$, and so fRf is an exchange ring satisfying related comparability. By virtue of Theorem 10, there exist an idempotent $g \in fRf$ and two one-sided units $u, v \in fRf$ such that $af = g + uv$. Thus,

$$a = af + a(1-f) = g + uv + a(1-f) = (g + 1 - f) + \big(uv + a(1-f) - (1-f)\big).$$

Let $e = g+(1-f)$. Then $e \in R$ is an idempotent. In addition, $uv+a(1-f)-(1-f) = \big(u + a(1-f) - (1-f)\big)\big(v + (1-f)\big)$. Assume that there exist $s, t \in fRf$ such that $su = f = vt$. So we get

$$\big(s + a(1-f) - (1-f)\big)\big(u + a(1-f) - (1-f)\big) = 1 - a(1-f) + sa(1-f).$$

This implies that

$$\begin{aligned}&\big(1 + a(1-f) - sa(1-f)\big)\big(s + a(1-f) - (1-f)\big)\big(u + a(1-f) - (1-f)\big)\\ &\quad = \big(1 + a(1-f) - sa(1-f)\big)\big(1 - a(1-f) + sa(1-f)\big) = 1.\end{aligned}$$

That is, $u + a(1-f) - (1-f) \in R$ is left invertible. Furthermore, we see that

$$\big(v + (1-f)\big)\big(t + (1-f)\big) = f + (1-f) = f,$$

i.e., $v + (1-f) \in R$ is right invertible. Therefore $a \in R$ is the sum of an idempotent $e \in R$ and the product of two one-sided units $u + a(1-f) - (1-f)$ and $v + (1-f)$, as required. $\square$

Recall that an ideal I of an exchange ring R satisfies general comparability provided that for any regular $x, y \in I$ there exists a central idempotent $e \in R$ such that $x\,\mathrm{Re} \lesssim^{\oplus} y\,\mathrm{Re}$ and $yR(1-e) \lesssim^{\oplus} xR(1-e)$. Let I be an ideal of an exchange ring. If I satisfies 1-comparability, then it satisfies general comparability (cf. [12]). Let e be a primitive idempotent in a regular ring R. By virtue of [12, Example 1.2], $\mathrm{Re}\,R$ is an ideal satisfying general comparability.

Corollary 14. *Let I be an ideal of an exchange ring R and let $a \in I$ be regular. If I satisfies general comparability, then $a \in I$ is the sum of an idempotent and the product of two one-sided units.*

P r o o f. Since $a \in I$ is regular, we can find $e = e^2 \in I$ such that $aR = eR$; hence $\mathrm{End}_R(aR) \cong e\,\mathrm{Re}$. Given $a'x' + b' = e$ in $e\,\mathrm{Re}$, then $(a' + 1 - e)(x' + 1 - e) + b' = 1$ in R. Let $a'' = a' + 1 - e$, $x'' = x' + 1 - e$ and $b'' = b'$. Then $a''x'' + b'' = 1$ with $a'', x'' \in 1 + I, b'' \in I$. As R is an exchange ring, we have an idempotent $f \in R$ such that $f = b''s$ and $1-f = (1-b'')t$ for some $s, t \in R$. Thus, $(1-f)a''x''t + f = 1$. Let $a = (1-f)a''$, $x = x''t$ and $b = f$. Then $ax+b = 1$ in R. Clearly, $a \in 1+I$ is regular; hence, there exists $c \in R$ such that $a = aca$. In addition, $c \in 1 + I$. By assumption, $(1-ac)Rg \lesssim^{\oplus} (1-ca)Rg$ and $(1-ca)R(1-g) \lesssim^{\oplus} (1-ac)R(1-g)$ for a central idempotent $g \in R$. Obviously, we have $\varphi\colon\ aRg \cong caRg$ given by $\varphi(arg) = carg$ for any $r \in R$. Clearly, there exists a split R-monomorphism $\psi\colon\ (1-ac)Rg \to (1-ca)Rg$. Construct an R-morphism $\varphi\colon\ Rg = aRg \oplus (1-ac)Rg \to caRg \oplus (1-ca)Rg = Rg$

given by $\varphi(s+t) = \varphi(s) + \psi(t)$ for any $s \in aRg$, $t \in (1-ac)Rg$. Then $a\varphi(g)a = ag$ and $\varphi(g) \in R$ is left invertible in gRg. Likewise, we have $a\varphi(1-g)a = a(1-g)$ and $\varphi(1-g) \in R$ is right invertible in $(1-g)R(1-g)$. Thus, $a = aua$, where $ug \in gR$ is left invertible and $u(1-g) \in R(1-g)$ is right invertible. Set $h = ua$. Then $hx + ub = u$, and so $h(x+ub) + (1-h)ub = u$. Clearly, there is a $y \in R$ such that $(1-h)ub = (1-h)uby(1-h)ub$. Let $k = (1-h)uby(1-h)$. Then $h(x+ub)+kub = u$, $h = h^2$, $k = k^2$ and $hk = kh = 0$. As a result, $h(x+ub) = hu$ and $kub = ku$. Hence, $(h+k)u = u$, and so $\bigl(h(1-huby(1-h)) + uby(1-h)\bigr)u = u$. Furthermore, we get

$$\begin{aligned} &u\bigl(a + by(1-h)(1+huby(1-h))\bigr)\bigl(1-huby(1-h)\bigr)u \\ &\quad = \bigl(h + uby(1-h)(1+huby(1-h))\bigr)\bigl(1-huby(1-h)\bigr)u = u. \end{aligned}$$

Let $z = y(1-h)\bigl(1+huby(1-h)\bigr)$. Then we get that $g(a+by) \in gR$ is left invertible and $(1-g)(a+by) \in (1-g)R$ is right invertible. This implies that $e\,\mathrm{Re}$ satisfies related comparability. According to Theorem 13 we complete the proof. $\square$

Corollary 15. *Let R be an exchange ring and let $a \in R$ be regular. If RaR satisfies general comparability, then $a \in R$ is the sum of an idempotent and the product of two one-sided units.*

P r o o f. Let $I = RaR$. Then I satisfies general comparability. As $a \in I$, the result follows from Corollary 14. $\square$

References

[1] *P. Ara*: Strongly π-regular rings have stable range one. Proc. Amer. Math. Soc. *124* (1996), 3293–3298.
[2] *P. Ara*: The exchange property for purely infinite simple rings. Proc. Amer. Math. Soc. *132* (2004), 2543–2547.
[3] *P. Ara, K. R. Goodearl, K. C. O'Meara and E. Pardo*: Separative cancellation for projective modules over exchange rings. Israel J. Math. *105* (1998), 105–137.
[4] *P. Ara, K. R. Goodearl and E. Pardo*: K_0 of purely infinite simple regular rings. K-Theory *26* (2002), 69–100.
[5] *V. P. Camillo and D. A. Khurana*: Characterization of unit regular rings. Comm. Algebra *29* (2001), 2293–2295.
[6] *H. Chen*: Elements in one-sided unit regular rings. Comm. Algebra *25* (1997), 2517–2529.
[7] *H. Chen*: Related comparability over exchange rings. Comm. Algebra *27* (1999), 4209–4216.
[8] *H. Chen*: Regular rings, idempotents and products of one-sided units. Comm. Algebra *34* (2006), 737–741.
[9] *H. Chen and F. Li*: Exchange rings satisfying related comparability. Collect. Math. *53* (2002), 157–164.
[10] *D. Khurana and T. Y. Lam*: Clean matrices and unit-regular matrices. J. Algebra *280* (2004), 683–698.

[11] *T. Y. Lam*: A crash course on stable range, cancellation, substitution and exchange. J. Algebra Appl. *3* (2004), 301–343.
[12] *D. Lu, Q. Li and W. Tong*: Comparability, stability, and completions of ideals. Comm. Algebra *32* (2004), 2617–2634.
[13] *W. K. Nicholson and Y. Zhou*: Clean rings: A survey, Advances in Ring Theory. Proceedings of the 4th China-Japan-Korea International Conference, 2004, pp. 181–198.
[14] *E. Pardo*: Comparability, separativity, and exchange rings. Comm. Algebra *24* (1996), 2915–2929.
[15] *J. Wei*: Unit-regularity and stable range conditions. Comm. Algebra *33* (2005), 1937–1946.
[16] *A. A. Tuganbaev*: Rings Close to Regular. Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.

Author's address: Huanyin Chen, Hangzhou Normal University, Hangzhou 310036, People's Republic of China, e-mail: huanyinchen@yahoo.cn.