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Applications of Mathematics, Vol. 54 (2009), No. 6, 465–489

Persistent URL: http://dml.cz/dmlcz/140379

Σ-CONVERGENCE OF NONLINEAR MONOTONE OPERATORS IN PERFORATED DOMAINS WITH HOLES OF SMALL SIZE

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(Received November 27, 2007, in revised version October 10, 2008)

Abstract. This paper is devoted to the homogenization beyond the periodic setting, of nonlinear monotone operators in a domain in $\mathbb{R}^N$ with isolated holes of size ε^2 ($\varepsilon > 0$ a small parameter). The order of the size of the holes is twice that of the oscillations of the coefficients of the operator, so that the problem under consideration is a reiterated homogenization problem in perforated domains. The usual periodic perforation of the domain and the classical periodicity hypothesis on the coefficients of the operator are here replaced by an abstract assumption covering a great variety of behaviors such as the periodicity, the almost periodicity and many more besides. We illustrate this abstract setting by working out a few concrete homogenization problems. Our main tool is the recent theory of homogenization structures.

Keywords: perforated domains, homogenization, reiterated

MSC 2010: 35B40, 46J10

1. Introduction

Let Ω be an open bounded set of $\mathbb{R}^N_x$ (the space $\mathbb{R}^N$ of variables $x = (x_1, \ldots, x_N)$, $N \geqslant 1$) with smooth boundary $\partial\Omega$, $Z = (-\frac{1}{2}, \frac{1}{2})^N$ the reference cell, $T \subset Z$ a compact set in $\mathbb{R}^N_z$ with smooth boundary ∂T and nonempty interior, and S an infinite subset of $\mathbb{Z}^N$ ($\mathbb{Z}$ denotes the integers).

For fixed $\varepsilon > 0$, we define the perforated domain Ω^ε as follows:

$$t^\varepsilon = \{k \in S\colon\ \varepsilon^2(k+T) \subset \Omega\},$$
$$T^\varepsilon = \bigcup_{k\in t^\varepsilon} \varepsilon^2(k+T)$$

and

$$\Omega^\varepsilon = \Omega \setminus T^\varepsilon \quad \text{(points in } \Omega \text{ lying off } T^\varepsilon).$$

The set t^ε is finite, since Ω is bounded. Hence T^ε is closed in $\mathbb{R}^N_x$ and so Ω^ε is open.

This being so, let $1 < p < \infty$. Let $(y, \lambda) \to a(y, \lambda)$ be a function from $\mathbb{R}^N \times \mathbb{R}^N$ to $\mathbb{R}^N$ with the properties:

(1.1) For each fixed $\lambda \in \mathbb{R}^N,$ the function $y \to a(y, \lambda)$ (denoted by $a(\cdot, \lambda)$) from $\mathbb{R}^N$ to $\mathbb{R}^N$ is measurable.

(1.2) $a(y, 0) = 0$ almost everywhere (a.e.) in $y \in \mathbb{R}^N$.

(1.3) There are four constants $c_1, c_2 > 0,\ 0 < \alpha_1 \leqslant \min(1, p-1)$ and $\alpha_2 \geqslant \max(p, 2)$ such that, a.e. in $y \in \mathbb{R}^N$ and for $\lambda, \mu \in \mathbb{R}^N$:
(i) $(a(y, \lambda) - a(y, \mu)) \cdot (\lambda - \mu) \geqslant c_1(|\lambda| + |\mu|)^{p-\alpha_2}|\lambda - \mu|^{\alpha_2}$
(ii) $|a(y, \lambda) - a(y, \mu)| \leqslant c_2(|\lambda| + |\mu|)^{p-1-\alpha_1}|\lambda - \mu|^{\alpha_1}$,
where the dot denotes the usual Euclidean inner product in $\mathbb{R}^N$ and $|\cdot|$ the associated norm.

For fixed $\varepsilon > 0$, let

$$\begin{aligned} -\operatorname{div} a\Big(\frac{x}{\varepsilon}, Du_\varepsilon\Big) &= f \quad \text{in } \Omega^\varepsilon, \\ a\Big(\frac{x}{\varepsilon}, Du_\varepsilon\Big) \cdot \nu &= 0 \quad \text{on } \partial T^\varepsilon, \\ u_\varepsilon &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.4}$$

where $f \in L^{p'}(\Omega; \mathbb{R})$ with $p' = p/(p-1)$, $\nu = (\nu_i)$ denotes the unit external normal vector to ∂T^ε with respect to Ω^ε, and D denotes the usual gradient operator with respect to x, i.e., $D = (D_{x_i})_{1 \leqslant i \leqslant N}$ with $D_{x_i} = \partial/\partial x_i$, and finally div denotes the usual divergence operator in Ω.

Once more, fix $\varepsilon > 0$, and fix $u \in W^{1,p}(\Omega^\varepsilon; \mathbb{R})$. In [24] the trace function $x \to a(x/\varepsilon, Du(x))$ from Ω^ε into $\mathbb{R}^N$ (denoted by $a^\varepsilon(\cdot, Du)$) is rigorously defined as an element of $L^{p'}(\Omega^\varepsilon; \mathbb{R})^N$ with the properties:

$$\begin{aligned} &(a^\varepsilon(\cdot, Du) - a^\varepsilon(\cdot, Dv)) \cdot D(u - v) \\ &\qquad \geqslant c_1(|Du| + |Dv|)^{p-\alpha_2}|Du - Dv|^{\alpha_2} \quad \text{a.e. in } \Omega^\varepsilon, \end{aligned} \tag{1.5}$$

$$\begin{aligned} &\|a^\varepsilon(\cdot, Du) - a^\varepsilon(\cdot, Dv)\|_{L^{p'}(\Omega^\varepsilon)^N} \\ &\qquad \leqslant c_2(\|Du\|_{L^p(\Omega^\varepsilon)^N} + \|Dv\|_{L^p(\Omega^\varepsilon)^N})^{p-1-\alpha_1}\|Du - Dv\|^{\alpha_1}_{L^p(\Omega^\varepsilon)^N}, \end{aligned} \tag{1.6}$$

for all $u, v \in W^{1,p}(\Omega^\varepsilon; \mathbb{R})$. Therefore, the problem (1.4) uniquely determines a function $u_\varepsilon \in V_\varepsilon = \{u \in W^{1,p}(\Omega^\varepsilon; \mathbb{R}) \colon\ u = 0 \text{ on } \partial\Omega\}$ (see, e.g., [17]). We endow the vector space V_ε with the norm (equivalent to the $W^{1,p}(\Omega^\varepsilon)$-norm) $\|u\|_{V_\varepsilon} = \|Du\|_{L^p(\Omega^\varepsilon)^N}$ $(u \in V_\varepsilon)$, which makes it a Banach space (see, e.g., [1]).

Perforated media are nowadays widely used in various domains such as the aerospace industry, civil engineering, and have a lot of applications in physics, chemistry and geology, in particular. That is why the homogenization of partial differential equations in perforated domains is actually a very attractive field. We refer, e.g., to [8], [6], [7], [9], [11], [12] and the references therein, for some bibliographical links.

In most of the previous works, the results were established for the case that the holes are periodically distributed and the size of the holes is of the same order as that of the period of oscillations of the coefficients of the operator under consideration. Cioranescu and Murat [10] have for the first time envisaged the situation, where the order of the size of the holes is different from that of the period of oscillations of the coefficients of the operator, but the perforation is still a periodic one.

Recently Nguetseng [23] has considered the more general situation where the periodic perforation is replaced by an abstract hypothesis covering a range of concrete behaviors such as the periodic perforation, the almost periodic perforation and the concentration of the holes in a neighborhood of some hyperplane. But here again, the size of the holes and the period of oscillations are of the same order. Similarly, see, e.g., [2], [27].

In the present work, we consider the situation where the order of the size of the holes is twice that of the period of oscillations of the coefficients of the operator in (1.4), and at the same time, we replace the periodic perforation by a general assumption similar to that in [23]. This seems to be, to our knowledge, a true advance and in all points of view, new.

More precisely, we investigate the asymptotic behavior of u_ε (the solution of (1.4)) as $\varepsilon \to 0$, under an abstract assumption on $a(y, \lambda)$ (for fixed λ) and an hypothesis characterizing the manner in which the holes are distributed. That abstract assumption is made, on the one hand, on the function $a(y, \lambda)$ with respect to the variable $y = x/\varepsilon$, and on the other hand, on the characteristic function $\chi_G(z)$ of the set $G = \mathbb{R}^N_z \setminus \Big(\bigcup_{k \in S} (k + T)\Big)$ with respect to the variable $z = x/\varepsilon^2$. Thus, the problem under consideration is a reiterated homogenization problem in perforated domains. Each of these abstract hypotheses covers a great variety of concrete behaviors including, in particular, the classical periodicity hypothesis and the almost periodicity hypothesis.

The layout of the paper is as follows. In Section 2 we give preliminary notions and results about reiterated Σ-convergence, and we state and solve the abstract homogenization problem for (1.4). Section 3 deals with concrete homogenization problems for (1.4).

Unless otherwise specified, the vector spaces throughout are assumed to be complex vector spaces, and the scalar functions are assumed to attain values in $\mathbb{C}$ (the complex numbers). This permits us to make use of basic tools provided by the classical Banach algebras theory. We shall always assume that the numerical space $\mathbb{R}^m$ (integer $m \geqslant 1$) and its open sets are each provided with the Lebesgue measure denoted by $\mathrm{d}x = \mathrm{d}x_1 \ldots \mathrm{d}x_m$. Finally, let $F(X)$ be a given function space. We shall denote by $F(X;\mathbb{R})$ or $F_{\mathbb{R}}(X)$ the subspace of $F(X)$ consisting of real valued functions.

2. The abstract homogenization problem for (1.4)

2.1. Reiterated Σ-convergence

We first state some fundamentals of homogenization structures beyond the classical two-scale setting.

Let $\mathcal{H} = (H_\varepsilon)_{\varepsilon>0}$ be one of the following two actions of $\mathbb{R}_+^*$ (the multiplicative group of positive real numbers) on the numerical space $\mathbb{R}^N$, defined as follows:

$$H_\varepsilon(x) = \frac{x}{\varepsilon} \quad (x \in \mathbb{R}^N), \tag{2.1}$$

$$H_\varepsilon(x) = \frac{x}{\varepsilon^2} \quad (x \in \mathbb{R}^N). \tag{2.2}$$

It is an easy task to see that each of these two actions satisfies the following properties:

(H)$_1$ Each H_ε maps continuously $\mathbb{R}^N$ into itself;

(H)$_2$ $\lim\limits_{\varepsilon\to 0} |H_\varepsilon(x)| = +\infty$ for any $x \in \mathbb{R}^N$ with $x \neq 0$;

(H)$_3$ The Lebesgue measure λ on $\mathbb{R}^N$ is quasi-invariant under $\mathcal{H}$, i.e., to each $\varepsilon > 0$ there is attached some $\gamma(\varepsilon) > 0$ such that $H_\varepsilon(\lambda) = \gamma(\varepsilon)\lambda$.

Now, given $\varepsilon > 0$, let

$$u^\varepsilon(x) = u(H_\varepsilon(x)) \quad (x \in \mathbb{R}^N)$$

for $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N_y)$ (as usual, $\mathbb{R}^N_y$ denotes the numerical space $\mathbb{R}^N$ of variables $y = (y_1, \ldots, y_N)$). In view of (H)$_3$, u^ε lies in $L^1_{\mathrm{loc}}(\mathbb{R}^N_x)$. More generally, if u lies in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ (resp. $L^p(\mathbb{R}^N)$), $1 \leqslant p < +\infty$, then so also does u^ε.

We denote by $\Pi^\infty(\mathbb{R}^N_y;\mathcal{H})$, or simply Π^∞ when there is no danger of confusion, the space of those functions $u \in \mathcal{B}(\mathbb{R}^N_y)$ (the space of bounded continuous complex functions on $\mathbb{R}^N_y$) for which a complex number $M(u)$ exists such that $u^\varepsilon \to M(u)$ in $L^\infty(\mathbb{R}^N_x)$-weak* as $\varepsilon \to 0$. There is no difficulty in verifying that Π^∞ is a closed vector subspace of $\mathcal{B}(\mathbb{R}^N)$ (provided with the supremum norm) and hence a Banach space under the supremum norm. Also, Π^∞ contains the constants and further Π^∞ is closed under complex conjugation. On the other hand, the mapping $u \longmapsto M(u)$ of Π^∞ into $\mathbb{C}$, denoted below by M, is a positive linear form on Π^∞ attaining the value 1 on the constant function 1. Hence, M is continuous on Π^∞; more precisely $|M(u)| \leqslant \|u\|_\infty$ for all $u \in \Pi^\infty$. The mapping M is called the mean value on $\mathbb{R}^N$ for $\mathcal{H}$.

For the benefit of the reader we summarize below a few basic notions and results about the homogenization structures in a general setting [26].

First, by a *structural representation* on $\mathbb{R}^N$ for the action $\mathcal{H}$ is meant any countable set $\Gamma \subset \Pi^\infty$ with the properties:

(HS1) Γ is a group under multiplication in $\mathcal{B}(\mathbb{R}^N_y)$,

(HS2) Γ is closed under complex conjugation.

Now, in the collection of all structural representations on $\mathbb{R}^N_y$ (for $\mathcal{H}$) we consider the equivalence relation $\sim$ defined as: $\Gamma \sim \Gamma'$ if and only if $\mathrm{CLS}(\Gamma) = \mathrm{CLS}(\Gamma')$, where $\mathrm{CLS}(\Gamma)$ denotes the closed vector subspace of $\mathcal{B}(\mathbb{R}^N_y)$ spanned by Γ. By an *H-structure* on $\mathbb{R}^N_y$ for $\mathcal{H}$ (H stands for *homogenization*) is understood any equivalence class modulo $\sim$.

An H-structure is fully determined by its image. Specifically, let Σ be an H-structure on $\mathbb{R}^N$. Put $A = \mathrm{CLS}(\Gamma)$, where Γ is any equivalence class representative of Σ (such a Γ is termed a *representation* of Σ). The space A is a so-called H-algebra on $\mathbb{R}^N_y$ for $\mathcal{H}$, that is, a closed algebra contained in Π^∞ with:

(HA1) A contains the constants,

(HA2) A is closed under complex conjugation.

It is to be noted that by the definition of A, endowed with the supremum norm, A is a separable Banach algebra. Moreover, A depends only on Σ and not on the chosen representation Γ of Σ; so that we may set $A = \mathcal{J}(\Sigma)$ (the *image* of Σ), which yields a one-to-one mapping $\Sigma \to \mathcal{J}(\Sigma)$ that carries the collection of all H-structures (for $\mathcal{H}$) onto the collection of all H-algebras on $\mathbb{R}^N_y$ (for $\mathcal{H}$) (see [19, Theorem 3.1]).

Let A be an H-algebra on $\mathbb{R}^N_y$. Clearly A (with the supremum norm) is a commutative $\mathcal{C}^*$-algebra with identity (the involution is here the usual one of complex conjugation). We denote by $\Delta(A)$ the spectrum of A and by $\mathcal{G}$ the Gelfand transformation on A. We recall that $\Delta(A)$ (a subset of the topological dual A') is the set of all nonzero multiplicative linear forms on A, and $\mathcal{G}$ is the mapping of A into $\mathcal{C}(\Delta(A))$ such that $\mathcal{G}(u)(s) = \langle s, u\rangle$ ($s \in \Delta(A)$), where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between A' (the topological dual of A) and A. The topology on $\Delta(A)$ is the relative weak* topology on A'. With this topology, $\Delta(A)$ is a metrizable compact space, and the Gelfand transformation is an isometric isomorphism of the $\mathcal{C}^*$-algebra A onto the $\mathcal{C}^*$-algebra $\mathcal{C}(\Delta(A))$. For further details concerning the Banach algebras theory we refer to [16]. The basic measure on $\Delta(A)$ is the so-called M-measure for A, namely the positive Radon measure β (of total mass 1) on $\Delta(A)$ such that $M(u) = \int_{\Delta(A)} \mathcal{G}(u)\,\mathrm{d}\beta$ for $u \in A$ (see [19, Proposition 2.1]).

For the benefit of the reader, we recall the following well-known fact: If Σ is the periodic H-structure $\Sigma_{\mathbb{Z}^N}$ represented by the network $\mathbb{Z}^N$ of $\mathbb{R}^N$, then $\Delta(A)$ can be identified with the period $Y = [-1/2, 1/2]^N$. In the case when Σ is an almost periodic H-structure $\Sigma_{\mathcal{R}}$ on $\mathbb{R}^N$ represented by some countable subgroup $\mathcal{R}$ of $\mathbb{R}^N$, then $\Delta(A)$ is a compact topological group homeomorphic to the dual group $\hat{\mathcal{R}}$ of $\mathcal{R}$ consisting of the characters γ_k ($k \in \mathcal{R}$) of $\mathbb{R}^N$, defined by $\gamma_k(y) = \exp(2\mathrm{i}\pi k \cdot y)$ ($y \in \mathbb{R}^N$) (see [22, Propositions 2.2 and 2.6] for details).

The partial derivative of index i ($1 \leqslant i \leqslant N$) on $\Delta(A)$ is defined to be the mapping $\partial_i = \mathcal{G} \circ D_{y_i} \circ \mathcal{G}^{-1}$ (usual composition) of $\mathcal{D}^1(\Delta(A)) = \{\varphi \in \mathcal{C}(\Delta(A))\colon\ \mathcal{G}^{-1}(\varphi) \in A^1\}$ into $\mathcal{C}(\Delta(A))$, where $A^1 = \{\psi \in \mathcal{C}^1(\mathbb{R}^N)\colon\ \psi, D_{y_i}\psi \in A\ (1 \leqslant i \leqslant N)\}$ with $D_{y_i}\psi = \partial\psi/\partial y_i$. Higher order derivatives are defined analogously. At the same time, let A^∞ be the space of $\psi \in \mathcal{C}^\infty(\mathbb{R}^N_y)$ such that $D^\alpha_y\psi = \partial^{|\alpha|}\psi/(\partial y_1^{\alpha_1} \ldots \partial y_N^{\alpha_N}) \in A$ for every $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$, and let $\mathcal{D}(\Delta(A)) = \{\varphi \in \mathcal{C}(\Delta(A))\colon\ \mathcal{G}^{-1}(\varphi) \in A^\infty\}$. Endowed with a suitable locally convex topology (see [19]), A^∞ (resp. $\mathcal{D}(\Delta(A))$) is a Fréchet space and further, $\mathcal{G}$ viewed as defined on A^∞ is a topological isomorphism of A^∞ onto $\mathcal{D}(\Delta(A))$.

Any continuous linear form on $\mathcal{D}(\Delta(A))$ is referred to as a distribution on $\Delta(A)$. The space of all distributions on $\Delta(A)$ is then the dual, $\mathcal{D}'(\Delta(A))$, of $\mathcal{D}(\Delta(A))$. We endow $\mathcal{D}'(\Delta(A))$ with the strong dual topology. If we assume that A^∞ is dense in A (this condition is always fulfilled in practice), which amounts to assuming that $\mathcal{D}(\Delta(A))$ is dense in $\mathcal{C}(\Delta(A))$, then $L^p(\Delta(A)) \subset \mathcal{D}'(\Delta(A))$ ($1 \leqslant p \leqslant \infty$) with continuous embedding (see [19] for more details). Hence we may define

$$W^{1,p}(\Delta(A)) = \{u \in L^p(\Delta(A))\colon\ \partial_i u \in L^p(\Delta(A)) \quad (1 \leqslant i \leqslant N)\}$$

where the derivative $\partial_i u$ is taken in the distribution sense on $\Delta(A)$ (exactly as the Schwartz derivative is taken in the classical case). We equip $W^{1,p}(\Delta(A))$ with the norm

$$\|u\|_{W^{1,p}(\Delta(A))} = \left[\|u\|^p_{L^p(\Delta(A))} + \sum_{i=1}^N \|\partial_i u\|^p_{L^p(\Delta(A))}\right]^{1/p} \quad (u \in W^{1,p}(\Delta(A))),$$

which makes it a Banach space. However, we will be mostly concerned with the space

$$W^{1,p}(\Delta(A))/\mathbb{C} = \left\{u \in W^{1,p}(\Delta(A))\colon\ \int_{\Delta(A)} u(s)\,\mathrm{d}\beta(s) = 0\right\}$$

provided with the seminorm

$$\|u\|_{W^{1,p}(\Delta(A))/\mathbb{C}} = \left(\sum_{i=1}^N \|\partial_i u\|^p_{L^p(\Delta(A))}\right)^{1/p} \quad (u \in W^{1,p}(\Delta(A))/\mathbb{C}).$$

With this seminorm, $W^{1,p}(\Delta(A))/\mathbb{C}$ is in general nonseparated and noncomplete. We denote by $W_{\#}^{1,p}(\Delta(A))$ the separated completion of $W^{1,p}(\Delta(A))/\mathbb{C}$ and by J the canonical mapping of $W^{1,p}(\Delta(A))/\mathbb{C}$ into its separated completion. $W_{\#}^{1,p}(\Delta(A))$ is a Banach space and $W_{\#}^{1,2}(\Delta(A))$ is a Hilbert space. Furthermore, as pointed out in [19], the distribution derivative ∂_i viewed as a mapping of $W^{1,p}(\Delta(A))/\mathbb{C}$ into $L^p(\Delta(A))$ extends to a unique continuous linear mapping, still denoted by ∂_i, of $W_{\#}^{1,p}(\Delta(A))$ into $L^p(\Delta(A))$ such that $\partial_i J(v) = \partial_i v$ for $v \in W^{1,p}(\Delta(A))/\mathbb{C}$ and

$$\|u\|_{W_{\#}^{1,p}(\Delta(A))} = \left(\sum_{i=1}^{N} \|\partial_i u\|_{L^p(\Delta(A))}^p\right)^{1/p} \quad \text{for } u \in W_{\#}^{1,p}(\Delta(A)).$$

To enhance the comprehension of the space $W_{\#}^{1,p}(\Delta(A))$, let us note that in the case of a periodic H-structure $\Sigma_{\mathbb{Z}^N}$, $W_{\#}^{1,p}(\Delta(A))$ stands for the well-known space $W_{\#}^{1,p}(Y)$ ($Y = (-1/2, 1/2)^N$) of Y-periodic functions in $W_{\mathrm{loc}}^{1,p}(\mathbb{R}^N)$ of zero mean value.

However, the notion of a product H-structure (see [19]) needs a few further details. To that end, we define the product action $\mathcal{H}^*$ of the preceding two actions (2.1) and (2.2) by

$$\mathcal{H}^* = (H_\varepsilon^*)_{\varepsilon>0}\colon\ H_\varepsilon^*(x, x') = \left(\frac{x}{\varepsilon}, \frac{x'}{\varepsilon^2}\right) \quad ((x, x') \in \mathbb{R}^N \times \mathbb{R}^N).$$

There is no difficulty in checking that the action $\mathcal{H}^*$ has the properties $(\mathrm{H})_1$–$(\mathrm{H})_3$. In the sequel, the action (2.2) will be denoted by $\mathcal{H}' = (H'_\varepsilon)_{\varepsilon>0}$, that is, $H'_\varepsilon(x) = x/\varepsilon^2$ ($x \in \mathbb{R}^N$).

Now, if Σ_1 (resp. Σ_2) is an H-structure on $\mathbb{R}^N$ (resp. $\mathbb{R}^N$) for $\mathcal{H}$ (resp. $\mathcal{H}'$), then [19, Proposition 3.1] carries over to the present setting, so that the product $\Sigma = \Sigma_1 \times \Sigma_2$ is well defined as an H-structure on $\mathbb{R}^{2N}$ for the product action $\mathcal{H}^*$. Moreover, Proposition 3.2, Theorem 3.2 and Corollaries 3.1–3.2 of [19] carry over mutatis mutandis to the present context.

In the sequel, we will denote by the same letter M the mean value on $\mathbb{R}^N$ for $\mathcal{H}$ and for $\mathcal{H}'$, and on $\mathbb{R}^{2N}$ for $\mathcal{H}^*$ as well. An H-structure Σ (of image A) will be termed of class $\mathcal{C}^\infty$ if A^∞ is dense in A.

This being so, let Σ_y and Σ_z be two H-structures of class $\mathcal{C}^\infty$ on $\mathbb{R}_y^N$ (for $\mathcal{H}$) and $\mathbb{R}_z^N$ (for $\mathcal{H}'$), respectively, and let $\Sigma = \Sigma_y \times \Sigma_z$ be their product, which is an H-structure of class $\mathcal{C}^\infty$ on $\mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N$ for the product action $\mathcal{H}^*$. We introduce their respective images $A_y = \mathcal{J}(\Sigma_y)$, $A_z = \mathcal{J}(\Sigma_z)$ and $A = \mathcal{J}(\Sigma)$, and we use the same letter, $\mathcal{G}$, to denote the Gelfand transformation on A_y, A_z and A, as well. Points in $\Delta(A_y)$ (resp. $\Delta(A_z)$) are denoted by s (resp. r). The compact space $\Delta(A_y)$ (resp. $\Delta(A_z)$) is equipped with the M-measure β_y (resp. β_z) for A_y (resp. A_z). It is important to recall that $\Delta(A) = \Delta(A_y) \times \Delta(A_z)$ (Cartesian product) and further the M-measure for A, with which $\Delta(A)$ is equipped, is precisely the product measure $\beta = \beta_y \otimes \beta_z$ (see [19]). We have the following proposition.

Proposition 2.1. *For each ψ in A we have, as $\varepsilon \to 0$,*

$$\psi^\varepsilon \to M(\psi) \quad \text{in } L^\infty(\mathbb{R}^N_x)\text{-weak*},$$

where ψ^ε is defined in an obvious way by $\psi^\varepsilon(x) = \psi(x/\varepsilon, x/\varepsilon^2)$ for $x \in \mathbb{R}^N$ and M is the mean value on $\mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N$ for the product action $\mathcal{H}^*$.

Proof. One needs to prove the following:

$$\text{for each } \psi \in A,\ \psi^\varepsilon \to M(\psi) \text{ in } L^\infty(\mathbb{R}^N_x)\text{-weak*},$$
$$\text{where } \psi^\varepsilon(x) = \psi(x/\varepsilon, x/\varepsilon^2) \text{ for } x \in \mathbb{R}^N.$$

It is sufficient to check this for $\psi = u \otimes v$, where $u \in A_y$ and $v \in A_z$. To that end, let $\eta > 0$. For such ψ and for a fixed φ in $L^1(\mathbb{R}^N_x)$ we have

$$\begin{aligned}
&\left|\int (u^\varepsilon v^\varepsilon \varphi - M(u \otimes v)\varphi)\,\mathrm{d}x\right| \\
&\qquad \leqslant \left|\int [u^\varepsilon \varphi - M(u)\varphi] v^\varepsilon\,\mathrm{d}x\right| + \left|\int [v^\varepsilon \varphi - M(v)\varphi] M(u)\,\mathrm{d}x\right| \\
&\qquad \leqslant \|v\|_\infty \left|\int [u^\varepsilon \varphi - M(u)\varphi]\,\mathrm{d}x\right| + |M(u)| \left|\int [v^\varepsilon \varphi - M(v)\varphi]\,\mathrm{d}x\right|,
\end{aligned}$$

where $u^\varepsilon(x) = u(x/\varepsilon)$ and $v^\varepsilon(x) = v(x/\varepsilon^2)$ for $x \in \mathbb{R}^N$. The result follows at once by the convergence results $\int [u^\varepsilon \varphi - M(u)\varphi]\,\mathrm{d}x \to 0$ and $\int [v^\varepsilon \varphi - M(v)\varphi]\,\mathrm{d}x \to 0$. $\square$

We can now introduce the concepts of reiterated weak and strong Σ-convergence. The letter E throughout will denote a family of positive real numbers admitting 0 as an accumulation point. In particular if $E = (\varepsilon_n)$ (with integers $n \geqslant 0$) with $0 < \varepsilon_n \leqslant 1$ and $\varepsilon_n \to 0$ as $n \to \infty$, then E is referred to as a *fundamental sequence*.

Definition 2.1. A sequence $(u_\varepsilon)_{\varepsilon \in E} \subset L^p(\Omega)$ $(1 \leqslant p < \infty)$ is said to:

(i) *weakly Σ-converge reiteratively* in $L^p(\Omega)$ to some $u_0 \in L^p(\Omega \times \Delta(A))$ if as $E \ni \varepsilon \to 0$, we have

$$\int_\Omega u_\varepsilon v^\varepsilon\,\mathrm{d}x \to \iint_{\Omega \times \Delta(A)} u_0 \hat{v}\,\mathrm{d}x\,\mathrm{d}\beta \tag{2.3}$$

for every $v \in L^{p'}(\Omega; A)$ $(1/p' = 1 - 1/p)$, where v^ε is given by $v^\varepsilon(x) = v(x, x/\varepsilon, x/\varepsilon^2)$ for $x \in \Omega$ (see [18]), and $\hat{v} = \mathcal{G} \circ v$;

(ii) *strongly Σ-converge reiteratively* in $L^p(\Omega)$ to some $u_0 \in L^p(\Omega \times \Delta(A))$ if the following condition is fulfilled:

$$\text{(SC)} \qquad \begin{aligned} &\text{given } \eta > 0 \text{ and } v \in L^p(\Omega; A) \text{ with } \|u_0 - \hat{v}\|_{L^p(\Omega\times\Delta(A))} \leqslant \tfrac{1}{2}\eta, \\ &\text{there is some } \alpha > 0 \text{ such that } \|u_\varepsilon - v^\varepsilon\|_{L^p(\Omega)} \leqslant \eta \text{ provided} \\ &E \ni \varepsilon \leqslant \alpha. \end{aligned}$$

We express this by writing $u_\varepsilon \to u_0$ *reit. in* $L^p(\Omega)$*-weak* Σ in case (i), and $u_\varepsilon \to u_0$ *reit. in* $L^p(\Omega)$*-strong* Σ in case (ii).

There is no difficulty in verifying the following results.

(1) Suppose $u_0 = \hat{v}_0$ with $v_0 \in L^p(\Omega; A)$. Then $u_\varepsilon \to u_0$ reit. in $L^p(\Omega)$-strong Σ if and only if $\|u_\varepsilon - v_0^\varepsilon\|_{L^p(\Omega)} \to 0$ as $E \ni \varepsilon \to 0$.

(2) For $u \in L^p(\Omega; A)$ we have $u^\varepsilon \to \hat{u}$ reit. in $L^p(\Omega)$-strong Σ.

(3) If $u_\varepsilon \to u$ in $L^p(\Omega)$ (strong) as $E \ni \varepsilon \to 0$, then $u_\varepsilon \to u$ reit. in $L^p(\Omega)$-strong Σ.

Also, the proof of the next proposition is a simple exercise left to the reader.

Proposition 2.2. *Suppose a sequence $(u_\varepsilon)_{\varepsilon\in E} \subset L^p(\Omega)$ ($1 \leqslant p < \infty$) weakly Σ-converges reiteratively in $L^p(\Omega)$ to some $u_0 \in L^p(\Omega \times \Delta(A))$. Define $u_0^* \in L^p(\Omega \times \Delta(A_y))$ as $u_0^*(x,s) = \int_{\Delta(A_z)} u_0(x,s,r)\,\mathrm{d}\beta_z(r)$ ($x \in \Omega$, $s \in \Delta(A_y)$), and $\tilde{u} \in L^p(\Omega)$ as $\tilde{u}(x) = \int_{\Delta(A_z)}\int_{\Delta(A_y)} u_0(x,s,r)\,\mathrm{d}\beta_y(s)\,\mathrm{d}\beta_z(r)$ ($x \in \Omega$). Then, as $E \ni \varepsilon \to 0$,*

(i) *$u_\varepsilon \to u_0^*$ in $L^p(\Omega)$-weak Σ_y* [19, Definition 4.1],

(ii) *$u_\varepsilon \to \tilde{u}$ in $L^p(\Omega)$-weak.*

The results of the Σ-convergence setting [19] carry over mutatis mutandis, together with their proofs, to the present setting. Let us state the most important of such results.

Proposition 2.3. *Assume that $1 < p < \infty$. Given a fundamental sequence E and a sequence $(u_\varepsilon)_{\varepsilon\in E}$ which is bounded in $L^p(\Omega)$, a subsequence E' can be extracted from E such that the sequence $(u_\varepsilon)_{\varepsilon\in E'}$ weakly Σ-converges reiteratively in $L^p(\Omega)$.*

Proposition 2.4. *Suppose a sequence $(u_\varepsilon)_{\varepsilon\in E}$ strongly Σ-converges reiteratively in $L^p(\Omega)$ to some $u_0 \in L^p(\Omega \times \Delta(A))$. Then as $E \ni \varepsilon \to 0$,*

(i) *$u_\varepsilon \to u_0$ reit. in $L^p(\Omega)$-weak Σ,*

(ii) $\|u_\varepsilon\|_{L^p(\Omega)} \to \|u_0\|_{L^p(\Omega\times\Delta(A))}$.

Conversely, if $p = 2$ and if the assertions (i)–(ii) *hold, then $u_\varepsilon \to u_0$ reit. in $L^p(\Omega)$-strong Σ.*

Proposition 2.5. *Suppose the three real numbers $\sigma, p, q \geqslant 1$ are such that $1/\sigma = 1/p + 1/q \leqslant 1$. Let $u_0 \in L^p(\Omega \times \Delta(A))$ and $v_0 \in L^q(\Omega \times \Delta(A))$, and let $u_\varepsilon \in L^p(\Omega)$ and $v_\varepsilon \in L^q(\Omega)$ for $\varepsilon \in E$. Finally, assume that $u_\varepsilon \to u_0$ reit. in $L^p(\Omega)$-strong Σ and $v_\varepsilon \to v_0$ reit. in $L^p(\Omega)$-weak Σ. Then $u_\varepsilon v_\varepsilon \to u_0 v_0$ reit. in $L^\sigma(\Omega)$-weak Σ.*

The notion of a $W^{1,p}(\Omega)$-proper H-structure will play a fundamental role in the present study.

Definition 2.2. The H-structure $\Sigma = \Sigma_y \times \Sigma_z$ is termed $W^{1,p}(\Omega)$-*proper* (where p is a given real number with $p \geqslant 1$) if the following two conditions are satisfied.

(PR1) For $\zeta \in \{y, z\}$, Σ_ζ is total for p, i.e., $\mathcal{D}(\Delta(A_\zeta))$ is dense in $W^{1,p}(\Delta(A_\zeta))$.

(PR2) Given a fundamental sequence E and a sequence $(u_\varepsilon)_{\varepsilon \in E}$ which is bounded in $W^{1,p}(\Omega)$, there are a subsequence E' from E and three functions $u_0 \in W^{1,p}(\Omega)$, $u_1 \in L^p(\Omega; W^{1,p}_\#(\Delta(A_y)))$ and $u_2 \in L^p(\Omega; L^p(\Delta(A_y); W^{1,p}_\#(\Delta(A_z))))$, such that, as $E' \ni \varepsilon \to 0$,

$$u_\varepsilon \to u_0 \quad \text{in } W^{1,p}(\Omega)\text{-weak},$$
$$\frac{\partial u_\varepsilon}{\partial x_j} \to \frac{\partial u_0}{\partial x_j} + \partial_j u_1 + \partial_j u_2 \quad \text{reit. in } L^p(\Omega)\text{-weak } \Sigma \quad (1 \leqslant j \leqslant N).$$

We give here below a few examples of $W^{1,p}(\Omega)$-proper H-structures.

Example 2.1. Let $\Sigma_{\mathbb{Z}^N}$ be the periodic H-structure on $\mathbb{R}^N$ represented by $\mathbb{Z}^N$ [19, Example 3.2]. Then $\Sigma_{\mathbb{Z}^N}$ is an H-structure on $\mathbb{R}^N$ for $\mathcal{H}$ and $\mathcal{H}'$. It can be shown that Σ is $W^{1,p}(\Omega)$-proper for any arbitrary real $p > 1$ [18, Proposition 2.8].

Example 2.2. Let $\mathcal{R}$ be a countable subgroup of $\mathbb{R}^N$, and let $\Sigma_\mathcal{R}$ be the almost periodic H-structure on $\mathbb{R}^N$ represented by $\mathcal{R}$ [19, Example 3.3]. Then $\Sigma_\mathcal{R}$ is an H-structure on $\mathbb{R}^N$ for $\mathcal{H}$ and $\mathcal{H}'$. This being so, let $\mathcal{R}_y$ and $\mathcal{R}_z$ be two countable subgroups of $\mathbb{R}^N$. Then the H-structure $\Sigma = \Sigma_{\mathcal{R}_y} \times \Sigma_{\mathcal{R}_z}$ is $W^{1,2}(\Omega)$-proper [18, Proposition 2.9].

Example 2.3. Let Σ_∞ be the H-structure of the convergence at infinity on $\mathbb{R}^N$ [19, Example 3.4]. This is an H-structure on $\mathbb{R}^N$ for $\mathcal{H}$ and $\mathcal{H}'$. The H-structure $\Sigma = \Sigma_{\mathcal{R}_y} \times \Sigma_\infty$ where $\mathcal{R}_y$ and $\Sigma_{\mathcal{R}_y}$ are as in Example 2.2, is $W^{1,2}(\Omega)$-proper [18, Example 2.10].

Example 2.4. Let $\Sigma = \Sigma_{\infty,\mathcal{R}_y} \times \Sigma_{\mathcal{R}_z}$ where $\mathcal{R}_y$ and $\mathcal{R}_z$ are as in Example 2.2, and $\Sigma_{\infty,\mathcal{R}_y}$ is the H-structure on $\mathbb{R}^N_y$ for $\mathcal{H}$ defined in [19, Example 3.5]. Then the H-structure Σ is $W^{1,2}(\Omega)$-proper. If in particular, $\mathcal{R}_y = \mathbb{Z}^N = \mathcal{R}_z$, then $\Sigma = \Sigma_{\infty,\mathbb{Z}^N} \times \Sigma_{\mathbb{Z}^N}$ is $W^{1,p}(\Omega)$-proper for any real $p > 1$ [18, Example 2.9].

2.2. Abstract structure hypothesis and preliminary results

Let $1 \leqslant p < \infty$. We denote by $\Xi^p(\mathbb{R}^N_y;\mathcal{B}(\mathbb{R}^N_z))$, or simply Ξ^p when there is no danger of confusion, the space of those functions $u \in L^p_{\mathrm{loc}}(\mathbb{R}^N_y;\mathcal{B}(\mathbb{R}^N_z))$ for which

$$\|u\|_{\Xi^p} = \sup_{0<\varepsilon\leqslant 1}\left(\int_{B_N}\left(\text{ess sup}_z u\left(\frac{x}{\varepsilon},z\right)\right)^p \mathrm{d}x\right)^{1/p} < \infty,$$

B_N being the unit ball in $\mathbb{R}^N_x$. In this norm Ξ^p is a Banach space.

Now, let $\Sigma = \Sigma_y \times \Sigma_z$ be an H-structure on $\mathbb{R}^N_y \times \mathbb{R}^N_z$ for $\mathcal{H}^*$. We define $\mathfrak{X}^p_\Sigma(\mathbb{R}^N_y;\mathcal{B}(\mathbb{R}^N_z))$ or simply, $\mathfrak{X}^p_\Sigma$ if there is no danger of confusion, to be the closure of $A = \mathcal{J}(\Sigma)$ in Ξ^p. Provided with the Ξ^p-norm, $\mathfrak{X}^p_\Sigma$ is a Banach space. All of the results collected in the framework of [19] are still valid in the present context. Let us especially draw attention to the following two fundamental results:

1) The mean value M on $\mathbb{R}^N\times\mathbb{R}^N$ for $\mathcal{H}^*$, viewed as defined on $A = \mathcal{J}(\Sigma)$, extends by continuity to a positive continuous linear form (still denoted by M) on $\mathfrak{X}^p_\Sigma$. Furthermore, for each $u \in \mathfrak{X}^p_\Sigma$, we have $u^\varepsilon \to M(u)$ in $L^p(\Omega)$-weak as $\varepsilon \to 0$, where $u^\varepsilon \in L^p(\Omega)$ is defined by $u^\varepsilon(x) = u(x/\varepsilon, x/\varepsilon^2)$ for $x \in \Omega$ [18, Section 2], and Ω is a bounded open set in $\mathbb{R}^N_x$.
2) The Gelfand transformation $\mathcal{G}\colon A \to \mathcal{C}(\Delta(A))$ extends by continuity to a unique continuous linear mapping, still denoted by $\mathcal{G}$, of $\mathfrak{X}^p_\Sigma$ into $L^p(\Delta(A))$.

Before we can state the abstract problem, let us, however, state another definition: we define the space $\mathfrak{X}^p_{\Sigma_z}(\mathbb{R}^N_z)$ to be the closure of $A_z = \mathcal{J}(\Sigma_z)$ in the Banach space $\Xi^p(\mathbb{R}^N_z)$ of functions $u \in L^p_{\mathrm{loc}}(\mathbb{R}^N_z)$ such that

$$\|u\|_{\Xi^p(\mathbb{R}^N_z)} = \sup_{0<\varepsilon\leqslant 1}\left(\int_{B_N}\left|u\left(\frac{x}{\varepsilon^2}\right)\right|^p \mathrm{d}x\right)^{1/p} < \infty.$$

Let $G = \mathbb{R}^N_z \setminus \Theta$ with $\Theta = \bigcup\limits_{k\in S}(k+T)$. Since T is compact, the set Θ is closed and therefore G is an open set in $\mathbb{R}^N_z$. We denote by χ_G the characteristic function of G in $\mathbb{R}^N_z$.

We are now in a position to state the so-called *abstract homogenization problem* for (1.4). For that, let Σ be as above, $A = \mathcal{J}(\Sigma)$ its image and $A_\mathbb{R} = A \cap \mathcal{C}(\mathbb{R}^N \times \mathbb{R}^N;\mathbb{R})$. We denote by a_i $(1 \leqslant i \leqslant N)$ the ith component of the function a. Our goal is to investigate the limiting behavior of u_ε (the solution of (1.4) for fixed $\varepsilon > 0$), when $\varepsilon \to 0$, under the hypotheses

$$\chi_G \in \mathfrak{X}^r_{\Sigma_z}(\mathbb{R}^N_z) \quad \text{with } r \geqslant \max(p,p'), \tag{2.4}$$

$$M(\chi_G) > 0, \tag{2.5}$$

$$a_i(\cdot,\Psi) \in \mathfrak{X}^{p'}_\Sigma(\mathbb{R}^N_y;\mathcal{B}(\mathbb{R}^N_z)) \quad \text{for all } \Psi \in (A_\mathbb{R})^N \quad (1 \leqslant i \leqslant N), \tag{2.6}$$

where the function $a_i(\cdot, \Psi)$ is defined in the sense of [18] by $a_i(\cdot, \Psi)(y, z) = a_i(y, \Psi(y, z))$ $(y, z \in \mathbb{R}^N)$. This problem is solvable provided we give some preliminary results.

Lemma 2.6. *Under the hypothesis* (2.4)*, there exists a β_z-measurable set $\hat{G} \subset \Delta(A_z)$ such that $\chi_{\hat{G}} = \hat{\chi}_G$ a.e. in $\Delta(A_z)$, where $\hat{\chi}_G = \mathcal{G}(\chi_G)$, and $\chi_{\hat{G}}$ denotes the characteristic function of $\hat{G}$ in $\Delta(A_z)$.*

Proof. Observe that $\chi_G \in \mathfrak{X}^1_{\Sigma_z}(\mathbb{R}^N_z)$, since $\chi_G \in \mathfrak{X}^p_{\Sigma_z}(\mathbb{R}^N_z) \cap \mathfrak{X}^{p'}_{\Sigma_z}(\mathbb{R}^N_z)$ (recall that $r \geqslant \max(p, p')$) and proceed exactly as in the proof of [23, Lemma 2.1]. $\square$

Remark 2.1. According to Lemma 2.6, we have $\chi^\varepsilon_G \to \hat{\chi}_G$ reit. in $L^p(\Omega)$-weak Σ as $\varepsilon \to 0$, where $1 < p < \infty$, $\chi^\varepsilon_G(x) = \chi_G(x/\varepsilon^2)$ $(x \in \Omega)$. Furthermore, $\beta_z(\hat{G}) = \int_{\Delta(A_z)} \hat{\chi}_G(r)\, \mathrm{d}\beta_z(r) = M(\chi_G)$ (see [19, Section 2.3]).

Now, let $Q^\varepsilon = \Omega \setminus \varepsilon^2\Theta$. This is an open set in $\mathbb{R}^N$. We have the following result (see [23]).

Lemma 2.7. *Let $K \subset \Omega$ be a compact set independent of ε. There is some $\varepsilon_0 > 0$ such that $\Omega^\varepsilon \setminus Q^\varepsilon \subset \Omega \setminus K$ for any $0 < \varepsilon \leqslant \varepsilon_0$.*

The next classical extension result will prove very important in the homogenization process.

Proposition 2.8. *For each real $\varepsilon > 0$, there exists an operator P_ε of V_ε (the space defined in Section* 1*) into $W^{1,p}_0(\Omega; \mathbb{R})$ with the following properties:*

(i) *P_ε sends continuously and linearly V_ε into $W^{1,p}_0(\Omega; \mathbb{R})$,*

(ii) *$(P_\varepsilon v)|_{\Omega^\varepsilon} = v$ for all $v \in V_\varepsilon$,*

(iii) *$\|D(P_\varepsilon v)\|_{L^p(\Omega)^N} \leqslant c\|Dv\|_{L^p(\Omega^\varepsilon)^N}$ for all $v \in V_\varepsilon$,*

where the constant $c > 0$ is independent of ε, and D denotes the usual gradient operator with respect to x.

The proof of the preceding proposition being classical, is therefore omitted.

Now, let $\mathfrak{X}^{p,\infty}_\Sigma \equiv \mathfrak{X}^p_\Sigma \cap L^\infty(\mathbb{R}^N_y; \mathcal{B}(\mathbb{R}^N_z))$ be provided with the $L^\infty(\mathbb{R}^N_y; \mathcal{B}(\mathbb{R}^N_z))$-norm. Then, for $u \in \mathfrak{X}^{p,\infty}_\Sigma$, we have $\mathcal{G}(u) \in L^\infty(\Delta(A))$ and $\|\mathcal{G}(u)\|_{L^\infty(\Delta(A))} \leqslant \|u\|_{L^\infty(\mathbb{R}^N_y; \mathcal{B}(\mathbb{R}^N_z))}$; see [18].

Let $1 \leqslant i \leqslant N$ be fixed. For $\boldsymbol{\varphi} = (\varphi_j)_{1 \leqslant j \leqslant N}$ in $\mathcal{C}_\mathbb{R}(\Delta(A))^N$ let

$$b_i(\boldsymbol{\varphi}) = \mathcal{G}(a_i(\cdot, \mathcal{G}^{-1}\boldsymbol{\varphi})), \tag{2.7}$$

where $\mathcal{G}^{-1}\boldsymbol{\varphi} = (\mathcal{G}^{-1}\varphi_j)_{1 \leqslant j \leqslant N}$. Then, by the hypothesis (2.6), we see that $a_i(\cdot, \mathcal{G}^{-1}\boldsymbol{\varphi})$ lies in $\mathfrak{X}^{p',\infty}_\Sigma$, and hence (2.7) defines a mapping b_i of $\mathcal{C}_\mathbb{R}(\Delta(A))^N$ into $L^\infty(\Delta(A))$. As in [18] we have the following proposition and corollary.

Proposition 2.9. *Let $1 < p < \infty$. Let Ω be a bounded open set in $\mathbb{R}^N_x$. Suppose* (2.6) *holds. Then the following assertions are true:*

(i) *Let the index $1 \leqslant i \leqslant N$ be fixed. For $\Psi = (\psi_j)_{1\leqslant j\leqslant N}$ in $\mathcal{C}(\overline{\Omega}; (A_\mathbb{R})^N)$, the function $b_i \circ \hat{\Psi}$ (usual composition) of $\overline{\Omega}$ into $L^\infty(\Delta(A))$ lies in $\mathcal{C}(\overline{\Omega}; L^\infty(\Delta(A)))$ and further $a_i^\varepsilon(\cdot, \Psi^\varepsilon) \to b_i \circ \hat{\Psi}$ reit. in $L^{p'}(\Omega)$-weak Σ as $E \ni \varepsilon \to 0$, where $\Psi^\varepsilon = (\psi_j^\varepsilon)_{1\leqslant j\leqslant N}$ (with ψ_j^ε defined by $\psi_j^\varepsilon(x) = \psi_j(x, x/\varepsilon, x/\varepsilon^2)$ $(x \in \Omega)$).*

(ii) *The mapping $\Phi \to b(\Phi) = (b_i \circ \Phi)_{1\leqslant i\leqslant N}$ of $\mathcal{C}(\overline{\Omega}; (A_\mathbb{R})^N)$ into $L^{p'}(\Omega \times \Delta(A))^N$ extends by continuity to a mapping, still denoted by b, of $L^p(\Omega; L^p(\Delta(A); \mathbb{R})^N)$ into $L^{p'}(\Omega \times \Delta(A))^N$ such that*

$$\|b(\mathbf{u}) - b(\mathbf{v})\|_{L^{p'}(\Omega\times\Delta(A))^N} \leqslant c_1 \||\mathbf{u}| + |\mathbf{v}|\|^{p-1-\alpha_1}_{L^p(\Omega\times\Delta(A))} \|\mathbf{u} - \mathbf{v}\|^{\alpha_1}_{L^p(\Omega;L^p(\Delta(A))^N)}$$

and

$$(b(\mathbf{u}) - b(\mathbf{v})) \cdot (\mathbf{u} - \mathbf{v}) \geqslant (|\mathbf{u}| + |\mathbf{v}|)^{p-\alpha_2} |\mathbf{u} - \mathbf{v}|^{\alpha_2} \tag{2.8}$$

for all $\mathbf{u}, \mathbf{v} \in L^p(\Omega; L^p(\Delta(A); \mathbb{R})^N)$.

Corollary 2.10. *Let the hypotheses be those of Proposition* 2.9. *For each real $\varepsilon > 0$, let $\Phi_\varepsilon \in \mathcal{D}_\mathbb{R}(\Omega) = \mathcal{C}_0^\infty(\Omega; \mathbb{R})$ be given by $\Phi_\varepsilon = \psi_0 + \varepsilon\psi_1^\varepsilon + \varepsilon^2\psi_2^\varepsilon$, i.e.,*

$$\Phi_\varepsilon(x) = \psi_0(x) + \varepsilon\psi_1\Big(x, \frac{x}{\varepsilon}\Big) + \varepsilon^2\psi_2\Big(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\Big) \quad (x \in \Omega) \tag{2.9}$$

with $\psi_0 \in \mathcal{D}_\mathbb{R}(\Omega)$, $\psi_1 \in \mathcal{D}_\mathbb{R}(\Omega) \otimes {}_\mathbb{R}A_y^\infty$, $\psi_2 \in \mathcal{D}_\mathbb{R}(\Omega) \otimes {}_\mathbb{R}A_y^\infty \otimes {}_\mathbb{R}A_z^\infty$, where ${}_\mathbb{R}A_y^\infty = A_y^\infty \cap \mathcal{C}(\mathbb{R}^N; \mathbb{R})$ and a similar definition for ${}_\mathbb{R}A_z^\infty$. Let the index $1 \leqslant i \leqslant N$ be fixed. Then as $\varepsilon \to 0$,

$$a_i^\varepsilon(\cdot, D\Phi_\varepsilon) \to b_i(D\psi_0 + \partial_s\hat{\psi}_1 + \partial_r\hat{\psi}_2) \quad \textit{reit. in } L^{p'}(\Omega)\textit{-weak } \Sigma,$$

where: $\partial_s\hat{\psi}_1 = (\partial_j\hat{\psi}_1)_{1\leqslant j\leqslant N}$ with $\partial_j\hat{\psi}_1 = \partial_j \circ \hat{\psi}_1$, the partial derivative ∂_j being here taken on $\Delta(A) = \Delta(A_y) \times \Delta(A_z)$ with respect to $\Delta(A_y)$; $\partial_r\hat{\psi}_2 = (\partial_j\hat{\psi}_2)_{1\leqslant j\leqslant N}$ with $\partial_j\hat{\psi}_2 = \partial_j \circ \hat{\psi}_2$, ∂_j being taken on $\Delta(A)$ with respect to $\Delta(A_z)$; and the functions $\hat{\psi}_1$ and $\hat{\psi}_2$ are viewed as defined on $\overline{\Omega}$ with values in $\mathcal{D}(\Delta(A))$.

Furthermore, if $(v_\varepsilon)_{\varepsilon\in E}$ is a sequence in $L^p(\Omega)$ such that $v_\varepsilon \to v_0$ reit. in $L^p(\Omega)$-weak Σ as $E \ni \varepsilon \to 0$, then, as $E \ni \varepsilon \to 0$,

$$\int_\Omega a_i^\varepsilon(\cdot, D\Phi_\varepsilon) v_\varepsilon \, \mathrm{d}x \to \iint_{\Omega\times\Delta(A)} b_i(D\psi_0 + \partial_s\hat{\psi}_1 + \partial_r\hat{\psi}_2) v_0 \, \mathrm{d}x \, \mathrm{d}\beta.$$

The basic notation being as above, let $1 < p < \infty$, and let

$$\mathbb{F}_0^p = W_0^{1,p}(\Omega;\mathbb{R}) \times L^p(\Omega; W_\#^{1,p}(\Delta(A_y);\mathbb{R})) \times L^p(\Omega; L^p(\Delta(A_y); W_\#^{1,p}(\Delta(A_z);\mathbb{R}))),$$

where for $\zeta \in \{y,z\}$,

$$W_\#^{1,p}(\Delta(A_\zeta);\mathbb{R}) = \{u \in W_\#^{1,p}(\Delta(A_\zeta))\colon\ \partial_j u \in L^p(\Delta(A_\zeta);\mathbb{R}) \quad (1 \leqslant j \leqslant N)\}.$$

Endowed with the norm

$$\begin{aligned}\|\mathbf{u}\|_{\mathbb{F}_0^p} = \sum_{i=1}^N [\|D_{x_i} u_0\|_{L^p(\Omega)} + \|\partial_i u_1\|_{L^p(\Omega\times\Delta(A_y))} + \|\partial_i u_2\|_{L^p(\Omega\times\Delta(A))}],\\ \mathbf{u} = (u_0,u_1,u_2) \in \mathbb{F}_0^p,\end{aligned}$$

$\mathbb{F}_0^p$ is a Banach space. Furthermore, assuming that the H-structure $\Sigma = \Sigma_y \times \Sigma_z$ is $W^{1,p}(\Omega)$-proper, the space $F_0^\infty = \mathcal{D}_\mathbb{R}(\Omega) \times [\mathcal{D}_\mathbb{R}(\Omega) \otimes J_y(\mathcal{D}(\Delta(A_y);\mathbb{R})/\mathbb{C})] \times [\mathcal{D}_\mathbb{R}(\Omega) \otimes \mathcal{D}(\Delta(A_y);\mathbb{R}) \otimes J_z(\mathcal{D}(\Delta(A_z);\mathbb{R})/\mathbb{C})]$ is dense in $\mathbb{F}_0^p$, where, for $\zeta \in \{y,z\}$, $\mathcal{D}(\Delta(A_\zeta);\mathbb{R})/\mathbb{C} = \{\varphi \in \mathcal{D}(\Delta(A_\zeta);\mathbb{R})\colon\ \int_{\Delta(A_\zeta)} \varphi \,\mathrm{d}\beta_\zeta = 0\}$ and J_ζ denotes the canonical mapping of $W^{1,p}(\Delta(A_\zeta))/\mathbb{C}$ into its separated completion $W_\#^{1,p}(\Delta(A_\zeta))$.

We end this subsection with an existence result.

Lemma 2.11. *Assume the hypotheses* (2.4)–(2.5) *hold true. Assume also that there exists a triplet* $\mathbf{u} = (u_0,u_1,u_2) \in \mathbb{F}_0^p$ *solving the variational problem*

$$\iiint_{\Omega\times\Delta(A_y)\times\hat{G}} b(\mathbb{D}\mathbf{u})\cdot\mathbb{D}\mathbf{v}\,\mathrm{d}x\,\mathrm{d}\beta = M(\chi_G)\int_\Omega f v_0\,\mathrm{d}x \quad \text{for all } \mathbf{v} = (v_0,v_1,v_2) \in \mathbb{F}_0^p, \tag{2.10}$$

where, for each $\mathbf{v} = (v_0,v_1,v_2) \in \mathbb{F}_0^p$, *we denote* $\mathbb{D}\mathbf{v} = Dv_0 + \partial_s v_1 + \partial_r v_2$ *with* $\partial_s v_1 = (\partial_j v_1)_{1\leqslant j\leqslant N}$ *and* $\partial_r v_2 = (\partial_j v_2)_{1\leqslant j\leqslant N}$. *Then,* u_0 *and* u_1 *are unique, and* u_2 *is unique up to an additive function* $g \in L^p(\Omega\times\Delta(A_y); W_\#^{1,p}(\Delta(A_z);\mathbb{R}))$ *such that* $\partial_j g(x,s,r) = 0$ *a.e. in* $\Omega\times\Delta(A_y)\times\hat{G}$ $(1 \leqslant j \leqslant N)$.

Proof. If $\mathbf{u} = (u_0,u_1,u_2)$ and $\mathbf{v} = (v_0,v_1,v_2)$ are two solutions of (2.10), then we have

$$\iiint_{\Omega\times\Delta(A_y)\times\hat{G}} (b(\mathbb{D}\mathbf{u}) - b(\mathbb{D}\mathbf{v}))\cdot\mathbb{D}\mathbf{w}\,\mathrm{d}x\,\mathrm{d}\beta = 0 \quad \forall\,\mathbf{w} \in \mathbb{F}_0^p.$$

Taking in particular $\mathbf{w} = \mathbf{u} - \mathbf{v}$, and by a classical argument using the reverse Hölder inequality and (2.8), we obtain

$$\begin{aligned}\iiint_{\Omega\times\Delta(A_y)\times\hat{G}} &(b(\mathbb{D}\mathbf{u}) - b(\mathbb{D}\mathbf{v}))\cdot\mathbb{D}(\mathbf{u}-\mathbf{v})\,\mathrm{d}x\,\mathrm{d}\beta \\ &\geqslant c_1(\|\mathbb{D}\mathbf{u}\|^p_{L^p(\Omega\times\Delta(A_y)\times\hat{G})} + \|\mathbb{D}\mathbf{v}\|^p_{L^p(\Omega\times\Delta(A_y)\times\hat{G})})^{(p-\alpha_2)/p} \\ &\quad\times\|\mathbb{D}(\mathbf{u}-\mathbf{v})\|^{\alpha_2}_{L^p(\Omega\times\Delta(A_y)\times\hat{G})},\end{aligned}$$

and hence $\|\mathbb{D}(\mathbf{u}-\mathbf{v})\|_{L^p(\Omega\times\Delta(A_y)\times\hat{G})} = 0$, which amounts to saying that $u_0 = v_0$ a.e. in Ω, $u_1 = v_1$ a.e. in $\Omega\times\Delta(A_y)$ and $\partial_j(v_2 - u_2) = 0$ a.e. in $\Omega\times\Delta(A_y)\times\hat{G}$ $(1\leqslant j\leqslant N)$, from which the lemma follows. $\square$

2.3. The abstract homogenization result

We can now state and prove the main result of the paper. The basic notation is as in Subsection 2.2.

Theorem 2.12. *Let $1 < p < \infty$. Suppose* (2.4)–(2.6) *hold true and further the H-structure Σ is $W^{1,p}(\Omega)$-proper. For each real $\varepsilon > 0$, let $u_\varepsilon \in W^{1,p}(\Omega^\varepsilon;\mathbb{R})$ be the solution of* (1.4)*, and P_ε the extension operator of Proposition* 2.8*. Then, as $\varepsilon\to 0$,*

$$P_\varepsilon u_\varepsilon \to u_0 \quad \text{in } W_0^{1,p}(\Omega)\text{-weak,} \tag{2.11}$$

where u_0 is the unique function in $W_0^{1,p}(\Omega)$ with the following property:

(P) *There exists a unique $u_1 \in L^p(\Omega; W^{1,p}_\#(\Delta(A_y);\mathbb{R}))$ and there is some $u_2 \in L^p(\Omega\times\Delta(A_y); W^{1,p}_\#(\Delta(A_z);\mathbb{R}))$ such that the triplet $\mathbf{u} = (u_0, u_1, u_2)$ is a solution of* (2.10).

P r o o f. For fixed $\varepsilon > 0$, we have $u_\varepsilon \in V_\varepsilon$ and

$$\int_\Omega a^\varepsilon(\cdot, Du_\varepsilon)\cdot Dv\,\mathrm{d}x = \int_\Omega fv\,\mathrm{d}x \quad \forall\, v\in V_\varepsilon. \tag{2.12}$$

Taking in particular $v = u_\varepsilon$, we arrive at $\sup\limits_{\varepsilon>0}\|u_\varepsilon\|_{V_\varepsilon} < \infty$. Using Proposition 2.8, it follows that $(P_\varepsilon u_\varepsilon)_{\varepsilon>0}$ is bounded in $W_0^{1,p}(\Omega)$. Let E be a fundamental sequence. Σ being $W^{1,p}(\Omega)$-proper, there exist a subsequence E' from E and a triplet $\mathbf{u} = (u_0, u_1, u_2) \in \mathbb{F}_0^p$ such that, as $E'\ni\varepsilon\to 0$, we have (2.11) and

$$\frac{\partial P_\varepsilon u_\varepsilon}{\partial x_j} \to \mathbb{D}_j\mathbf{u} = \frac{\partial u_0}{\partial x_j} + \partial_j u_1 + \partial_j u_2 \text{ reit. in } L^p(\Omega)\text{-weak } \Sigma \quad (1\leqslant j\leqslant N). \tag{2.13}$$

It remains to show that $\mathbf{u} = (u_0, u_1, u_2)$ satisfies (2.10). Here u_0 being the sole function satisfying (2.10), (2.11) holds for all E with $E \ni \varepsilon \to 0$, hence for $0 < \varepsilon \to 0$. Let us now verify that $\mathbf{u}$ satisfies (2.10). To that end, let $\Phi = (\psi_0, J_y \circ \hat{\psi}_1, J_z \circ \hat{\psi}_2) \in F_0^\infty$ with $\psi_0 \in \mathcal{D}_{\mathbb{R}}(\Omega)$, $\psi_1 \in \mathcal{D}_{\mathbb{R}}(\Omega) \otimes ({}_{\mathbb{R}}A_y^\infty/\mathbb{C})$ and $\psi_2 \in \mathcal{D}_{\mathbb{R}}(\Omega) \otimes ({}_{\mathbb{R}}A_y^\infty) \otimes ({}_{\mathbb{R}}A_z^\infty/\mathbb{C})$, where ${}_{\mathbb{R}}A_y^\infty/\mathbb{C} = \{\psi \in {}_{\mathbb{R}}A_y^\infty \colon\ M(\psi) = 0\}$ (M being the mean value on $\mathbb{R}_y^N$ for $\mathcal{H}$), ${}_{\mathbb{R}}A_z^\infty/\mathbb{C} = \{\psi \in {}_{\mathbb{R}}A_z^\infty \colon\ M(\psi) = 0\}$ (M being the mean value on $\mathbb{R}_z^N$ for $\mathcal{H}'$), ${}_{\mathbb{R}}A_y^\infty = A_y^\infty \cap \mathcal{C}(\mathbb{R}_y^N; \mathbb{R})$ and similar definition for ${}_{\mathbb{R}}A_z^\infty$, $\hat{\psi}_1 = \mathcal{G} \circ \psi_1$ is viewed as a mapping of Ω into $\mathcal{D}(\Delta(A_y))$, and $\hat{\psi}_2 = \mathcal{G} \circ \psi_2$ as a mapping of $\Omega \times \Delta(A_y)$ into $\mathcal{D}(\Delta(A_z))$. Define Φ_ε ($\varepsilon > 0$) as in (2.9). Then $\Phi_\varepsilon \in \mathcal{D}_{\mathbb{R}}(\Omega)$ and further all the functions Φ_ε ($\varepsilon > 0$) have their supports contained in a fixed compact set $K \subset \Omega$. In view of Lemma 2.7, there is some $\varepsilon_0 > 0$ such that

$$\Phi_\varepsilon = 0 \quad \text{in } \Omega^\varepsilon \setminus Q^\varepsilon \quad (0 < \varepsilon \leqslant \varepsilon_0).$$

This being so, we choose in (2.12) $v = \Phi_\varepsilon|_{Q^\varepsilon}$ (the restriction of Φ_ε to Q^ε) with $0 < \varepsilon \leqslant \varepsilon_0$. We use the decomposition $\Omega^\varepsilon = Q^\varepsilon \cup (\Omega^\varepsilon \setminus Q^\varepsilon)$ and the equality $Q^\varepsilon = \Omega \cap \varepsilon^2 G$ to obtain

$$\int_\Omega a^\varepsilon(\cdot, D(P_\varepsilon u_\varepsilon)) \cdot D\Phi_\varepsilon \chi_G^\varepsilon \, \mathrm{d}x = \int_\Omega f \Phi_\varepsilon \chi_G^\varepsilon \, \mathrm{d}x \quad (0 < \varepsilon \leqslant \varepsilon_0),$$

since

$$\begin{aligned}
&\int_\Omega a^\varepsilon(\cdot, D(P_\varepsilon u_\varepsilon)) \cdot D\Phi_\varepsilon \chi_G^\varepsilon \, \mathrm{d}x \\
&\quad = \int_\Omega a^\varepsilon(\cdot, D(P_\varepsilon u_\varepsilon)) \cdot D\Phi_\varepsilon \chi_{Q^\varepsilon} \, \mathrm{d}x = \int_{Q^\varepsilon} a^\varepsilon(\cdot, D(P_\varepsilon u_\varepsilon)) \cdot D\Phi_\varepsilon \, \mathrm{d}x \\
&\quad = \int_{Q^\varepsilon} a^\varepsilon(\cdot, Du_\varepsilon) \cdot D\Phi_\varepsilon \, \mathrm{d}x = \int_{\Omega^\varepsilon} a^\varepsilon(\cdot, Du_\varepsilon) \cdot D\Phi_\varepsilon \, \mathrm{d}x.
\end{aligned}$$

We make use of (1.5) to obtain

$$\int_\Omega (a^\varepsilon(\cdot, D(P_\varepsilon u_\varepsilon)) - a^\varepsilon(\cdot, D\Phi_\varepsilon)) \cdot D(P_\varepsilon u_\varepsilon - \Phi_\varepsilon) \chi_G^\varepsilon \, \mathrm{d}x \geqslant 0. \tag{2.14}$$

But

$$\int_\Omega a^\varepsilon(\cdot, D(P_\varepsilon u_\varepsilon)) \cdot D(P_\varepsilon u_\varepsilon - \Phi_\varepsilon) \chi_G^\varepsilon \, \mathrm{d}x = \int_\Omega f(P_\varepsilon u_\varepsilon - \Phi_\varepsilon) \chi_G^\varepsilon \, \mathrm{d}x$$

and

$$\int_\Omega f(P_\varepsilon u_\varepsilon - \Phi_\varepsilon) \chi_G^\varepsilon \, \mathrm{d}x \to M(\chi_G) \int_\Omega f(u_0 - \psi_0) \, \mathrm{d}x \quad \text{when } E' \ni \varepsilon \to 0 \text{ with } 0 < \varepsilon \leqslant \varepsilon_0$$

(observe that $\chi_G^\varepsilon f \to M(\chi_G) f$ in $L^{p'}(\Omega)$-weak and $(P_\varepsilon u_\varepsilon - \Phi_\varepsilon) \to (u_0 - \psi_0)$ in $L^p(\Omega)$ because of the compactness of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$).

Let us evaluate $\lim_{E' \ni \varepsilon \to 0,\, 0 < \varepsilon \leqslant \varepsilon_0} \int_\Omega a^\varepsilon(\cdot, D\Phi_\varepsilon) \cdot D(P_\varepsilon u_\varepsilon - \Phi_\varepsilon)\chi_G^\varepsilon \,\mathrm{d}x$. First of all, let us show that as $E' \ni \varepsilon \to 0$,

$$D_{x_j}(P_\varepsilon u_\varepsilon - \Phi_\varepsilon)\chi_G^\varepsilon \to \mathbb{D}_j(\mathbf{u} - \Phi)\chi_{\hat{G}} \text{ reit. in } L^p(\Omega)\text{-weak } \Sigma \quad (1 \leqslant j \leqslant N).$$

For this purpose, let $g \in \mathcal{C}(\overline{\Omega}; A)$; then $\chi_G g \in \mathcal{C}(\overline{\Omega}; \mathfrak{X}_\Sigma^{p'} \cap L^\infty(\mathbb{R}_y^N; \mathcal{B}(\mathbb{R}_z^N)))$, since $r \geqslant p'$ (see (2.4)), and so, as $E' \ni \varepsilon \to 0$, $0 < \varepsilon \leqslant \varepsilon_0$ [18, Proposition 3.3],

$$\int_\Omega D_{x_j}(P_\varepsilon u_\varepsilon - \Phi_\varepsilon)\chi_G^\varepsilon g^\varepsilon \,\mathrm{d}x \to \iint_{\Omega \times \Delta(A)} \mathbb{D}_j(\mathbf{u} - \Phi)\chi_{\hat{G}} \hat{g} \,\mathrm{d}x \,\mathrm{d}\beta. \tag{2.15}$$

The sequence $(D_{x_j}(P_\varepsilon u_\varepsilon - \Phi_\varepsilon))_{\varepsilon \in E'}$ is bounded in $L^p(\Omega)$, and the function χ_G^ε is the characteristic function of Q^ε in Ω, so, belongs to $L^\infty(\Omega)$. We therefore deduce that the sequence $(D_{x_j}(P_\varepsilon u_\varepsilon - \Phi_\varepsilon)\chi_G^\varepsilon)_{\varepsilon \in E'}$ is bounded in $L^p(\Omega)$, hence the existence of a subsequence from E', still denoted by E', and a function $v_j \in L^p(\Omega \times \Delta(A))$ such that for $E' \ni \varepsilon \to 0$, we have $D_{x_j}(P_\varepsilon u_\varepsilon - \Phi_\varepsilon)\chi_G^\varepsilon \to v_j$ reit. in $L^p(\Omega)$-weak Σ. Thus, for the above g, we have, as $E' \ni \varepsilon \to 0$,

$$\int_\Omega D_{x_j}(P_\varepsilon u_\varepsilon - \Phi_\varepsilon)\chi_G^\varepsilon g^\varepsilon \,\mathrm{d}x \to \iint_{\Omega \times \Delta(A)} v_j \hat{g} \,\mathrm{d}x \,\mathrm{d}\beta. \tag{2.16}$$

Taking the particular g in $\mathcal{K}(\Omega; A)$ and comparing (2.15) with (2.16), we are immediately led to $v_j = \mathbb{D}_j(\mathbf{u} - \Phi)\chi_{\hat{G}}$. Consequently, Corollary 2.10 yields

$$\int_\Omega a^\varepsilon(\cdot, D\Phi_\varepsilon) \cdot D(P_\varepsilon u_\varepsilon - \Phi_\varepsilon)\chi_G^\varepsilon \,\mathrm{d}x \to \iiint_{\Omega \times \Delta(A_y) \times \hat{G}} b(\mathbb{D}\Phi) \cdot \mathbb{D}_j(\mathbf{u} - \Phi) \,\mathrm{d}x \,\mathrm{d}\beta$$

for $E' \ni \varepsilon \to 0$, $0 < \varepsilon \leqslant \varepsilon_0$. Finally, by passing to the limit as $E' \ni \varepsilon \to 0$ with $0 < \varepsilon \leqslant \varepsilon_0$ in (2.14), we obtain

$$M(\chi_G)\int_\Omega f(u_0 - \psi_0) \,\mathrm{d}x - \iiint_{\Omega \times \Delta(A_y) \times \hat{G}} b(\mathbb{D}\Phi) \cdot \mathbb{D}_j(\mathbf{u} - \Phi) \,\mathrm{d}x \,\mathrm{d}\beta \geqslant 0 \tag{2.17}$$

for all $\Phi \in F_0^\infty$. The relation (2.17) still holds for $\Phi \in \mathbb{F}_0^p$ (this follows by the density of F_0^∞ in $\mathbb{F}_0^p$). Then choosing in (2.17) the particular functions $\Phi = \mathbf{u} - t\mathbf{v}$ with $\mathbf{v} \in \mathbb{F}_0^p$ and $t > 0$, and then dividing by t, and finally changing $\mathbf{v}$ into $-\mathbf{v}$ leads to (2.10). This completes the proof. $\square$

3. Some concrete homogenization problems for (1.4)

This section deals with the study of a few concrete homogenization problems for (1.4). Before we proceed any further, however, we need some preliminary results.

3.1. Preliminaries

The basic notation and hypotheses are as in Section 2. The holes $(k+T)$ $(k \in S)$ being pairwise disjoint, the characteristic function χ_Θ of the set Θ in $\mathbb{R}^N_z$ is given by $\chi_\Theta = \sum_{k\in S} \chi_{k+T}$ (a locally finite sum) or more suitably

$$\chi_\Theta = \sum_{k\in\mathbb{Z}^N} \theta(k)\chi_{k+T}, \tag{3.1}$$

where χ_{k+T} is the characteristic function of the set $k+T$ in $\mathbb{R}^N_z$ and θ is that of S in $\mathbb{Z}^N$. We shall refer to θ as the *distribution function of the holes* [23].

We end this subsection with two very useful results (see [23] for the proofs).

Proposition 3.1. *Let Σ_z be an H-structure on $\mathbb{R}^N_z$ (for $\mathcal{H}'$) with image A_z. Assume that the distribution function of the holes belongs to the space of essential functions on $\mathbb{Z}^N$, $\mathrm{ES}(\mathbb{Z}^N)$ (see [20]). On the other hand, assume that for every φ in $\mathcal{K}(Z)$ (the space of all continuous complex functions on $\mathbb{R}^N_z$ with compact supports contained in $Z = (-\frac{1}{2}, \frac{1}{2})^N$), the function $\sum_{k\in\mathbb{Z}^N} \theta(k)\tau_k\varphi$ (where $\tau_k\varphi(z) = \varphi(z-k)$ for $z \in \mathbb{R}^N$) lies in A_z. Then $\chi_\Theta \in \mathfrak{X}^p_{\Sigma_z}(\mathbb{R}^N_z)$ $(1 \leqslant p < \infty)$ and further*

$$M(\chi_\Theta) = \mathfrak{M}(\theta)\lambda(T), \tag{3.2}$$

where λ is the Lebesgue measure on $\mathbb{R}^N$ and $\mathfrak{M}(\theta)$ the essential mean of θ [21].

Corollary 3.2. *Let the hypotheses be those of Proposition 3.1. Then (2.4) and (2.5) hold true.*

We are now in a position to state and solve some concrete homogenization problems.

3.2. The holes are equidistributed

The holes are equidistributed means that each cell $k+Y$ contains a hole $k+T$. In this case the distribution function of the holes is given by $\theta(k) = 1$ for all $k \in \mathbb{Z}^N$, which concretely means that $S = \mathbb{Z}^N$. With that hypothesis, it is established in [23, Section 3.2] that

$$\chi_G \in \mathfrak{X}^r_{\Sigma_{\mathbb{Z}^N}}(\mathbb{R}^N_z) \;\; (1 \leqslant r < \infty) \quad \text{and} \quad M(\chi_G) > 0, \tag{3.3}$$

where $\Sigma_{\mathbb{Z}^N}$ is the periodic H-structure represented by $\mathbb{Z}^N$.

Under the preceding perforation hypothesis, we are going to solve the following problems.

3.2.1. Problem I (Periodic setting). We intend to solve the homogenization problem for (1.4) under the periodicity hypothesis

$$(3.4)\qquad a(y+k,\lambda)=a(y,\lambda)\quad \text{a.e. in } y\in\mathbb{R}^N,\ \text{ for all } k\in\mathbb{Z}^N \text{ and all } \lambda\in\mathbb{R}^N.$$

The suitable H-structures are $\Sigma_y=\Sigma_z=\Sigma_{\mathbb{Z}^N}$, and so, $\Sigma=\Sigma_{\mathbb{Z}^N}\times\Sigma_{\mathbb{Z}^N}$, a $W^{1,p}(\Omega)$-proper H-structure with image $A=\mathcal{C}_{\text{per}}(Y\times Z)$ (the space of $Y\times Z$-periodic continuous complex functions on $\mathbb{R}^N_y\times\mathbb{R}^N_z$), where $Y=Z=(-\frac{1}{2},\frac{1}{2})^N$. The homogenization process of (1.4) will be ended as soon as (2.6) will be proved. To that end, fix $1\leqslant i\leqslant N$ and $\Psi\in(A_{\mathbb{R}})^N$. By repeating the argument used in [18, Subsection 4.2] we see that:

(i) For fixed z in $\mathbb{R}^N$, the function $y\to a_i(y,\Psi(y,z))$ is measurable from $\mathbb{R}^N$ to $\mathbb{R}$.

(ii) For almost all y in $\mathbb{R}^N$, the function $z\to a_i(y,\Psi(y,z))$ (denoted by $a_i(y,\Psi(y,\cdot))$) is continuous on $\mathbb{R}^N$ and lies in $\mathcal{C}_{\text{per}}(Z)$ (the Z-periodic continuous complex functions on $\mathbb{R}^N_z$).

(iii) Taking account of (3.4), the function $y\to a_i(y,\Psi(y,\cdot))$ (denoted by $a_i(\cdot,\Psi)$) is measurable from $\mathbb{R}^N$ into $\mathcal{C}_{\text{per}}(Z)$.

We deduce from all this, that $a_i(\cdot,\Psi)\in L^\infty_{\text{per}}(Y;\mathcal{C}_{\text{per}}(Z))$ (also use (1.2) and (1.3)). But $L^\infty_{\text{per}}(Y;\mathcal{C}_{\text{per}}(Z))\subset L^{p'}_{\text{per}}(Y;\mathcal{C}_{\text{per}}(Z))$, where $L^\infty_{\text{per}}(Y;\mathcal{C}_{\text{per}}(Z))$ (resp. $L^{p'}_{\text{per}}(Y;\mathcal{C}_{\text{per}}(Z))$) denotes the space of Y-periodic functions in $L^\infty(\mathbb{R}^N_y;\mathcal{C}_{\text{per}}(Z))$ (resp. $L^{p'}_{\text{loc}}(\mathbb{R}^N_y;\mathcal{C}_{\text{per}}(Z))$). Therefore, (2.6) follows by the fact that $A=\mathcal{C}_{\text{per}}(Y\times Z)$ is dense in $L^{p'}_{\text{per}}(Y;\mathcal{C}_{\text{per}}(Z))$ and the latter space is continuously embedded in $\Xi^{p'}(\mathbb{R}^N_y;\mathcal{B}(\mathbb{R}^N_z))$.

3.2.2. Problem II (Almost periodic setting). We denote by $L^p_{\text{AP}}(\mathbb{R}^N_y)$ the space of functions in $L^p_{\text{loc}}(\mathbb{R}^N_y)$ that are almost periodic in the sense of Stepanov [3], [13], [22]. $L^p_{\text{AP}}(\mathbb{R}^N_y)$ is a Banach space with the norm

$$\|u\|_{p,\infty}=\sup_{k\in\mathbb{Z}^N}\left(\int_{k+Y}|u(y)|\,\mathrm{d}y\right)^{1/p}\quad (u\in L^p_{\text{AP}}(\mathbb{R}^N_y)).$$

It is worth noting that the space $\text{AP}(\mathbb{R}^N_y)$ is a dense vector subspace of $L^p_{\text{AP}}(\mathbb{R}^N_y)$, $\text{AP}(\mathbb{R}^N_y)$ being the space of all continuous complex almost periodic functions on $\mathbb{R}^N_y$ [16], [22].

After these preliminaries, our aim here is to homogenize the problem (1.4) under the hypotheses

$$(3.5)\qquad a_i(\cdot,\lambda)\in L^2_{\text{AP}}(\mathbb{R}^N_y)\ \text{ for all } \lambda\in\mathbb{R}^N\quad (1\leqslant i\leqslant N)$$

and

(3.6) for $\Psi \in \mathrm{AP}(\mathbb{R}_y^N \times \mathbb{R}_z^N; \mathbb{R})^N$ we have

$$\sup_{k \in \mathbb{Z}^{2N}} \int_{k+Y\times Z} |a_i(y-t, \Psi(y,z)) - a_i(y, \Psi(y,z))|^2 \, \mathrm{d}y \, \mathrm{d}z \to 0 \text{ as } |t| \to 0.$$

By a simple argument (see [25]) we may consider a countable subgroup R of $\mathbb{R}_y^N$ such that $a_i(\cdot, \lambda) \in L^2_{\mathrm{AP},R}(\mathbb{R}_y^N)$ for all $\lambda \in \mathbb{R}^N$ $(1 < i \leqslant N)$, where $L^2_{\mathrm{AP},R}(\mathbb{R}_y^N)$ is the space of those $u \in L^2_{\mathrm{AP}}(\mathbb{R}_y^N)$ with spectrum $\mathrm{Sp}(u) = \{k \in \mathbb{R}^N \colon M(\overline{\gamma}_k u) \neq 0\}$ (where γ_k is defined on $\mathbb{R}^N$ by $\gamma_k(y) = \exp(2\mathrm{i}\pi k \cdot y)$) contained in R. This suggests to put $\Sigma_y = \Sigma_R$ (the almost periodic H-structure on $\mathbb{R}_y^N$ represented by R) and then, $\Sigma = \Sigma_R \times \Sigma_{\mathbb{Z}^N}$. It can be easily proven that (see [19]) $\Sigma = \Sigma_{\mathcal{R}}$, where $\mathcal{R} = R \times \mathbb{Z}^N$ and $\Sigma_{\mathcal{R}}$ is the almost periodic H-structure on $\mathbb{R}_y^N \times \mathbb{R}_z^N$ represented by the countable subgroup $\mathcal{R}$ of $\mathbb{R}^N \times \mathbb{R}^N$. Moreover, Σ is $W^{1,2}(\Omega)$-proper (see Example 2.2). With all this in mind, we see that to solve the homogenization problem for (1.4) under the preceding hypotheses, it suffices to check (2.6). For that purpose, let $(\theta_n)_{n\geqslant 1}$ be a mollifier on $\mathbb{R}_y^N$, i.e., $(\theta_n)_{n\geqslant1} \subset \mathcal{C}_0^\infty(\mathbb{R}_y^N)$ with $\theta_n \geqslant 0$, $\int \theta_n(y)\,\mathrm{d}y = 1$, θ_n having support contained in $\frac{1}{n}\overline{B}_N$, where $\overline{B}_N$ is the closed unit ball in $\mathbb{R}_y^N$. Let n be freely fixed in $\mathbb{N}^*$. For $1 \leqslant i \leqslant N$ set

$$q_n^i(y,\lambda) = \int \theta_n(t) a_i(y-t,\lambda)\,\mathrm{d}t \quad (y, \lambda \in \mathbb{R}^N).$$

It can easily be proven that $q_n^i(\cdot,\lambda)$ lies in $\mathrm{AP}_R(\mathbb{R}_y^N) = \{u \in \mathrm{AP}(\mathbb{R}_y^N) \colon \mathrm{Sp}(u) \subset R\}$ (the image of Σ_R) for all $\lambda \in \mathbb{R}^N$. Let $A = \mathrm{AP}_{\mathcal{R}}(\mathbb{R}_y^N \times \mathbb{R}_z^N)$ (the image of Σ); we will first prove the following:

$$q_n^i(\cdot,\Psi) \in A \quad \text{for all } \Psi \in (A_{\mathbb{R}})^N. \tag{3.7}$$

To do this, fix Ψ in $(A_{\mathbb{R}})^N$. Let K be a compact set in $\mathbb{R}^N$ such that $\Psi(y,z) \in K$ for all $(y,z) \in \mathbb{R}^N \times \mathbb{R}^N$. According to the fact that $q_n^i(\cdot,\lambda)$ lies in ${}_{\mathbb{R}}A_y = \mathrm{AP}_R(\mathbb{R}_y^N; \mathbb{R})$, we may view q_n^i as a function $\lambda \to q_n^i(\cdot,\lambda)$ of $\mathbb{R}^N$ into ${}_{\mathbb{R}}A_y$, which function lies in $\mathcal{C}(\mathbb{R}^N; {}_{\mathbb{R}}A_y)$ (it is a fact that q_n^i has properties identical to (1.2)–(1.3)). Still calling q_n^i the restriction of the latter function to K, we have $q_n^i \in \mathcal{C}(K; {}_{\mathbb{R}}A_y)$. $\mathcal{C}(K;\mathbb{R}) \otimes {}_{\mathbb{R}}A_y$ being dense in $\mathcal{C}(K; {}_{\mathbb{R}}A_y)$, there exists a sequence $(g_m)_{m\geqslant1}$ in $\mathcal{C}(K;\mathbb{R}) \otimes {}_{\mathbb{R}}A_y$ such that

$$\sup_{(y,\lambda)\in\mathbb{R}^N\times K} |g_m(y,\lambda) - q_n^i(y,\lambda)| \to 0 \quad \text{as } m \to \infty.$$

Hence, $g_m(\cdot,\Psi) \to q_n^i(\cdot,\Psi)$ in $\mathcal{B}(\mathbb{R}_y^N \times \mathbb{R}_z^N)$ as $m \to \infty$. Thus, (3.7) is proved if we can check that each $g_m(\cdot,\Psi)$ lies in A. However, it is enough to verify that we have

$g(\cdot,\Psi)\in A$ for any $g\colon \mathbb{R}^N_y\times\mathbb{R}^N_\lambda\to\mathbb{R}$ of the form

$$g(y,\lambda)=\chi(\lambda)\phi(y)\ \ (y,\lambda\in\mathbb{R}^N)\ \text{ with }\ \chi\in\mathcal{C}(K;\mathbb{R})\ \text{ and }\ \phi\in{}_{\mathbb{R}}A_y.$$

For such g, the Stone-Weierstrass theorem reveals that there is a sequence (f_m) of polynomials in $\lambda=(\lambda_1,\ldots,\lambda_N)\in K$ such that $f_m\to\chi$ in $\mathcal{C}(K)$ as $m\to\infty$, hence $f_m(\Psi)\to\chi(\Psi)$ in $\mathcal{B}(\mathbb{R}^N_y\times\mathbb{R}^N_z)$ as $m\to\infty$, where $f_m(\Psi)$ stands for $f_m\circ\Psi$ (usual composition) and $\chi(\Psi)$ stands for $\chi\circ\Psi$. We deduce that $\chi(\Psi)$ lies in $A_{\mathbb{R}}$, since the same is true of each $f_m(\Psi)$ (recall that $A_{\mathbb{R}}$ is an algebra). Thus, we have $g(\cdot,\Psi)\in A_{\mathbb{R}}$, since ${}_{\mathbb{R}}A_y\subset A_{\mathbb{R}}$. This proves (3.7).

Finally by mere computations (using the definition of q^i_n and the hypothesis (3.6)) we are led to (2.6).

3.2.3. Problem III. Our goal here is to investigate, under the equidistributed perforation, the asymptotic behavior as $\varepsilon\to 0$ of u_ε, under the structure hypothesis

$$a_i(\cdot,\lambda)\in\mathcal{B}_{\infty,\mathbb{Z}^N}(\mathbb{R}^N_y)\quad(1\leqslant i\leqslant N)\quad\text{for any }\ \lambda\in\mathbb{R}^N,\tag{3.8}$$

where $\mathcal{B}_{\infty,\mathbb{Z}^N}(\mathbb{R}^N_y)$ denotes the closure in $\mathcal{B}(\mathbb{R}^N_y)$ of the space of finite sums

$$\sum_{\text{finite}}\varphi_i u_i\quad(\varphi_i\in\mathcal{B}_\infty(\mathbb{R}^N_y),\ \ u_i\in\mathcal{C}_{\text{per}}(Y)),$$

$\mathcal{B}_\infty(\mathbb{R}^N_y)$ being the space of continuous complex functions on $\mathbb{R}^N_y$ that converge finitely at infinity. $\mathcal{B}_{\infty,\mathbb{Z}^N}(\mathbb{R}^N_y)$ is an H-algebra and the associated H-structure is denoted by $\Sigma_{\infty,\mathbb{Z}^N}$. Set $\Sigma=\Sigma_{\infty,\mathbb{Z}^N}\times\Sigma_{\mathbb{Z}^N}$; Σ is $W^{1,p}(\Omega)$-proper for any real $p>1$ (Example 2.4), and its image is $A=\mathcal{C}_{\text{per}}(Z;\mathcal{B}_{\infty,\mathbb{Z}^N}(\mathbb{R}^N_y))$ (the space of complex continuous periodic functions of $\mathbb{R}^N_z$ into $\mathcal{B}_{\infty,\mathbb{Z}^N}(\mathbb{R}^N_y)$), since A coincides with the closure of $\mathcal{B}_{\infty,\mathbb{Z}^N}(\mathbb{R}^N_y)\otimes\mathcal{C}_{\text{per}}(Y)$ in $\mathcal{B}(\mathbb{R}^N_y\times\mathbb{R}^N_z)$. Repeating the proof of (3.7), we obtain that $a_i(\cdot,\Psi)$ lies in A for all $\Psi\in(A_{\mathbb{R}})^N$, from which we get the homogenization of (1.4) under the hypothesis (3.8) and for any real $p>1$.

Remark 3.1. The hypothesis

$$a_i(\cdot,\lambda)\in\mathcal{B}_\infty(\mathbb{R}^N_y)\quad(1\leqslant i\leqslant N)\quad\text{for all }\ \lambda\in\mathbb{R}^N$$

is a particular case of (3.8).

3.3. The holes are periodically distributed

We assume that the function θ is periodic, that is, there exists a network $\mathcal{R}_1$ in $\mathbb{R}^N_z$ with $\mathcal{R}_1\subset\mathbb{Z}^N$ such that

$$\theta(k+r)=\theta(k)\quad\text{for all }\ k\in\mathbb{Z}^N\ \text{ and all }\ r\in\mathcal{R}_1.$$

Let $\Sigma_{\mathcal{R}_1}$ be the periodic H-structure on $\mathbb{R}_z^N$ represented by $\mathcal{R}_1$. Then χ_G lies in $\mathfrak{X}_{\Sigma_{\mathcal{R}_1}}^r(\mathbb{R}_z^N)$ $(1 \leqslant r < \infty)$ (see [23, Subsection 3.3]), and hence (3.3) holds. Therefore, Problems I–III carry over without slightest change to the present situation.

3.4. The holes are distributed in an almost periodic fashion

We assume that the function θ is almost periodic, i.e., the translates $\tau_h\theta$ $(h \in \mathbb{Z}^N)$ form a relatively compact set in $\ell^\infty(\mathbb{Z}^N)$. Then we have (see [23, Subsection 3.4]) the existence of a countable subgroup R_0 of $\mathbb{R}^N$ such that

$$\chi_G \in \mathfrak{X}_{\Sigma_{R_0}}^r(\mathbb{R}_z^N) \quad (1 \leqslant r < \infty) \quad \text{with } M(\chi_G) > 0.$$

The conclusion of Problem II still holds when $\mathcal{R}$ is replaced by $\mathcal{R}' = R \times R_0$. In particular, under the periodicity hypothesis on $a(\cdot, \lambda)$, we are also led to a result of the same type as that of Problem II. It is also possible to work out the homogenization of (1.4) under other structure hypotheses.

3.5. The holes are concentrated in a neighborhood of the origin in $\mathbb{R}^N$

We assume that Ω contains the origin of $\mathbb{R}^N$. Let $\mathcal{B}_\infty(\mathbb{Z}^N)$ be the space of all mappings $u\colon \mathbb{Z}^N \to \mathbb{C}$ that converge finitely at infinity. We assume that $\theta \in \mathcal{B}_\infty(\mathbb{Z}^N)$. Then, proceeding as in [20] we can prove that

$$\chi_G \in \mathfrak{X}_{\Sigma_\infty^0}^r(\mathbb{R}_z^N) \quad (1 \leqslant r < \infty) \quad \text{with } M(\chi_G) > 0,$$

where the H-structure Σ_∞^0 is defined in [20]. Before we go any further, however, let us briefly define the H-structure Σ_∞^0: let F be the set of all continuous complex functions f on $\mathbb{R}_z^N$ of the form $f = \sum_{k \in \mathbb{Z}^N} d(k)\tau_k\varphi$ with $d \in \mathcal{B}_\infty(\mathbb{Z}^N)$ and $\varphi \in \mathcal{K}(Z)$ (Z and $\mathcal{K}(Z)$ being as in Proposition 3.1), and let $\mathcal{B}_\infty^0(\mathbb{R}_z^N)$ be the closure in $\mathcal{B}(\mathbb{R}_z^N)$ of the space of all functions of the form $\psi = c + \sum_{\text{finite}} f_i$ with $c \in \mathbb{C}$ and $f_i \in F$. The space $\mathcal{B}_\infty^0(\mathbb{R}_z^N)$ is an H-algebra on $\mathbb{R}_z^N$ [20, Proposition 3.3]. We set $\mathcal{B}_\infty^0(\mathbb{R}_z^N) = \mathcal{J}(\Sigma_\infty^0)$.

With this in mind, let us illustrate the preceding setting with two problems.

3.5.1. Problem IV. We intend to study the homogenization of (1.4) under the above perforation hypothesis and the periodicity hypothesis

$$a_i(\cdot, \lambda) \in \mathcal{C}_{\text{per}}(Y) \quad (1 \leqslant i \leqslant N) \quad \text{for each } \lambda \in \mathbb{R}^N. \tag{3.9}$$

Proposition 3.3. *Under the above hypotheses we have* (2.6) *with* $\Sigma = \Sigma_{\mathbb{Z}^N} \times \Sigma_\infty^0$, *where* $1 < p < \infty$ *is arbitrarily fixed.*

Proof. Set $A = \mathcal{C}_{\mathrm{per}}(Y; \mathcal{B}_\infty^0(\mathbb{R}_z^N))$; since $\mathcal{C}_{\mathrm{per}}(Y)$ can be identified with the space $\mathcal{C}(\mathbb{T}^N)$ of continuous complex functions on the N-dimensional torus $\mathbb{T}^N$, A coincides with the closure of $\mathcal{C}_{\mathrm{per}}(Y) \otimes \mathcal{B}_\infty^0(\mathbb{R}_z^N)$ in $\mathcal{B}(\mathbb{R}_y^N \times \mathbb{R}_z^N)$. Hence, it follows that A is the image of the H-structure $\Sigma = \Sigma_{\mathbb{Z}^N} \times \Sigma_\infty^0$. Thus, the proposition is proved if we can check that $a_i(\cdot, \Psi)$ lies in A for all $\Psi \in (A_{\mathbb{R}})^N$ and all $1 \leqslant i \leqslant N$. But then, by repeating the proof of (3.7) we are led to the result. $\square$

Now, put $\Sigma_z^1 = \Sigma_\infty^0 + \Sigma_{\mathbb{Z}^N}$ and $\Sigma_1 = \Sigma_{\mathbb{Z}^N} \times \Sigma_z^1$. It can be shown, using a procedure similar to that followed in [18, Example 2.9], that Σ_1 is a $W^{1,p}(\Omega)$-proper H-structure on $\mathbb{R}_y^N \times \mathbb{R}_z^N$ (for $\mathcal{H}^*$), for any real $p > 1$. Furthermore, we have $\mathfrak{X}_\Sigma^{p'} \subset \mathfrak{X}_{\Sigma_1}^{p'}$, where Σ is the H-structure of Proposition 3.3, and

$$a_i(\cdot, \Psi) \in \mathfrak{X}_{\Sigma_1}^{p'} \quad \text{for all} \quad \Psi \in (A_{\mathbb{R}})^N \quad (1 \leqslant i \leqslant N),$$

where A is as in the proof of Proposition 3.3. Hence, the conclusion of Theorem 2.12 follows from the above proposition.

3.5.2. Problem V. Our aim here is to solve the homogenization problem for (1.4) under the perforation hypothesis of the current subsection and the structure hypothesis

$$a_i(\cdot, \lambda) \in \mathcal{B}_\infty(\mathbb{R}_y^N) \quad \text{for all} \quad \lambda \in \mathbb{R}^N \quad (1 \leqslant i \leqslant N). \tag{3.10}$$

Proposition 3.4. *Under these hypotheses, we have* (2.6) *with* $\Sigma = \Sigma_\infty \times \Sigma_\infty^0$ *and* $p = 2$.

Proof. The hypothesis (3.10) suggests to take $\Sigma_y = \Sigma_\infty$, and so $\Sigma = \Sigma_\infty \times \Sigma_\infty^0$ with image $A = \mathcal{B}_\infty(\mathbb{R}_y^N; \mathcal{B}_\infty^0(\mathbb{R}_z^N))$, the space of all complex continuous functions $u\colon \mathbb{R}_y^N \to \mathcal{B}_\infty^0(\mathbb{R}_z^N)$ such that $u(y)$ has a limit in $\mathcal{B}_\infty^0(\mathbb{R}_z^N)$ as $|y| \to \infty$.

This being so, define the function q on $\mathbb{R}_y^N \times \mathbb{R}_z^N \times \mathbb{R}_\lambda^N$ by

$$q(y, z, \lambda) = a(y, z) \quad (y, z, \lambda \in \mathbb{R}^N).$$

Then q satisfies all the properties (1.2)–(1.4) of [18] so that the proof of the above proposition is quite similar to that of [18, Proposition 4.4]. $\square$

One can also show that the H-structure $\Sigma_1 = \Sigma_\infty \times (\Sigma_\infty^0 + \Sigma_{\mathbb{Z}^N})$ is $W^{1,2}(\Omega)$-proper and satisfies $\mathfrak{X}_\Sigma^2 \subset \mathfrak{X}_{\Sigma_1}^2$ (where Σ is the H-structure of the above proposition), so that the problem (1.4), (3.10) is solvable under the perforation hypothesis of the present subsection.

3.6. Concluding remarks

The problem (1.4) has just been solved under many hypotheses on the perforation (of the open set Ω) and on the structure of the function $a(\cdot,\lambda)$ (for fixed λ). One may also consider other perforation hypotheses (see, e.g., [23, Subsection 3.5]) and other structure hypotheses on $a(\cdot,\lambda)$. Equally, the study carried out in this paper can, to a certain extent, be easily applied to the homogenization in perforated domains (with holes of size ε or ε^2) of the problem studied in [18].

If Ω is a domain in $\mathbb{R}^4$, the equidistributed perforation considered here leads to the holes of critical size [10] up to a multiplicative positive constant. However, we do not have in this case the appearance of the "strange term" [10]. This is perhaps due to the reiteration property.

Acknowledgment. The author would like to thank the anonymous referee for valuable remarks and suggestions.

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