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Existence results for non-linear singular integral equations with Hilbert kernel in Banach spaces

EXISTENCE RESULTS FOR NON-LINEAR SINGULAR INTEGRAL EQUATIONS WITH HILBERT KERNEL IN BANACH SPACES

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Abstract. A class of non-linear singular integral equations with Hilbert kernel and a related class of quasi-linear singular integro-differential equations are investigated by applying Schauder's fixed point theorem in Banach spaces.

Keywords: non-linear singular integral equation, Schauder's fixed point theorem, Banach space

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1. Introduction

During the last years the theory of non-linear singular integral equations has successfully developed. Many problems of mathematical physics like elasticity, plasticity, viscoelasticity, aerodynamics and fluid mechanics reduced to the solution of non-linear singular integral equations. Non-linear singular integral and integro-differential equations and related Riemann-Hilbert problems have been treated by many authors, see Pogorzelski [16], Guseinov and Mukhtarov [5], Wolfersdorf [17], [18], Junghanns and others [7], [8], Ladopoulos [9]–[12] and many others. The theory of non-linear singular integral and integro-differential equations seems to be a particularly complicated form of the non-linear integral equations. Schauder's fixed point method is one of the basic tools for investigating the existence results of many classes of singular integral and integro-differential equations [1], [2], [4], [6], [7], [17], [18]. We refer to Ladopoulos [9]–[12] for many applications of singular integral and integro-differential equations in engineering and science. In the present paper a class of integral equations with Hilbert kernel and a related class of quasi-linear integro-differential equations are investigated by means of Schauder's fixed point theorem in the Sobolev space W_p^1.

We shall consider the non-linear singular integral equation

$$(1.1) \qquad \varphi(s) + \frac{1}{2\pi} \int_{-\pi}^{\pi} m(s, \sigma, \varphi(\sigma)) \cot \frac{\sigma - s}{2} \, d\sigma = f(s)$$

with an additional condition

$$(1.2) \qquad \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(s) \, ds = \varepsilon,$$

where ε is a positive constant and the functions $\varphi(s)$, $f(s)$, $m(s, \sigma, \varphi(\sigma))$ are continuous positive functions defined on the regions

$$D = \{s \colon s \in [-\pi, \pi]\} \quad \text{and} \quad \tilde{D} = \{(s, \sigma, \varphi(\sigma)) \colon s, \sigma \in [-\pi, \pi], \ -\infty < \varphi < \infty\},$$

respectively.

Throughout the paper $L_p[-\pi, \pi]$ ($1 \leqslant p < \infty$) means the Banach space of all measurable functions u defined on $[-\pi, \pi]$ with the norm

$$\|u\|_{L_p} = \left[\int_{-\pi}^{\pi} |u(t)|^p \, dt \right]^{p^{-1}},$$

and $W_p^1[-\pi, \pi]$, $1 \leqslant p < \infty$, means the Sobolev space of all functions $u \in L_p[-\pi, \pi]$ with $u' \in L_p[-\pi, \pi]$.

We will seek the solution of equation (1.1) in the Sobolev space W_p^1.

2. Reduction to fixed point equation

In this section we seek the solution of equation (1.1) and an estimation of the kernel of a fixed point equation. Differentiating both sides of equation (1.1), we have [14]

$$(2.1) \qquad \varphi'(s) + \frac{1}{2\pi} \int_{-\pi}^{\pi} m_\varphi(s, \sigma, \varphi(\sigma)) \varphi'(\sigma) \cot \frac{\sigma - s}{2} \, d\sigma = F(s),$$

where

$$F(s) = f'(s) - \frac{1}{2\pi} \int_{-\pi}^{\pi} (m_s(s, \sigma, \varphi(\sigma)) + m_\sigma(s, \sigma, \varphi(\sigma))) \cot \frac{\sigma - s}{2} \, d\sigma.$$

Equation (2.1) can be written as the quasi-linear integro-differential equation

$$(2.2) \qquad \varphi'(s) + \frac{1}{2\pi} \int_{-\pi}^{\pi} M(s, \sigma, \varphi(\sigma)) \varphi'(\sigma) \cot \frac{\sigma - s}{2} \, d\sigma = F(s)$$

with the additional condition

$$(2.3) \qquad \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{\varphi}'(s)\, \mathrm{d}s = 0, \quad \varphi(0) = \Lambda,$$

which is equivalent to equation (1.2), where Λ is a positive constant, $M(s,\sigma,\varphi(\sigma)) = m_\varphi(s,\sigma,\varphi(\sigma))$ and $M(s,\sigma,\varphi(\sigma))$ satisfy the Hölder-Lipschitz condition as follows:

$$(2.4) \quad |M(s_2,\sigma_2,\varphi_2(\sigma_2)) - M(s_1,\sigma_1,\varphi_1(\sigma_1))| \leqslant \alpha[|s_2 - s_1|^\delta + |\sigma_2 - \sigma_1|^\delta + |\varphi_2 - \varphi_1|],$$

where α is a positive constant and $0 < \delta < 1$.

From the theory of singular integral equations [3], the solution of equation (2.2) is the solution of the corresponding boundary value problem

$$(2.5) \qquad u(s) + v(s) = c(s)$$

with the additional condition

$$(2.6) \qquad \frac{1}{2\pi} \int_{-\pi}^{\pi} v(s)\, \mathrm{d}s = 0, \quad u(s) = \varphi'(s).$$

Since the boundary value problem (2.5) has index equal to zero [3], its solution has the form

$$(2.7) \qquad F(z) = \mathrm{e}^{\mathrm{i}\nu(z)} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} M(s,\sigma,\varphi(\sigma)) c(\sigma) \mathrm{e}^{w_1(\sigma)} \frac{\mathrm{e}^{\mathrm{i}\sigma} + z}{\mathrm{e}^{\mathrm{i}\sigma} - z}\, \mathrm{d}\sigma + \mathrm{i}\beta_0 \right],$$

where $\nu(z) = w(x,y) + \mathrm{i}w_1(x,y)$ and β_0 is an arbitrary constant.

Setting

$$(2.8) \qquad \mathrm{e}^{\mathrm{i}\nu(z)} = \xi + \mathrm{i}\eta, \quad \psi(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} M(s,\sigma,\varphi(\sigma)) c(\sigma) \mathrm{e}^{w_1(\sigma)} \frac{\mathrm{e}^{\mathrm{i}\sigma} + z}{\mathrm{e}^{\mathrm{i}\sigma} - z}\, \mathrm{d}\sigma$$

and taking into account the properties of Schwarz kernel, we have

$$(2.9) \qquad \operatorname{Re} \psi(t) = M(s,s,\varphi(s)) c(s) \mathrm{e}^{w_1(s)},$$

$$\operatorname{Im} \psi(t) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} M(s,\sigma,\varphi(\sigma)) c(\sigma) \mathrm{e}^{w_1(\sigma)} \cot \frac{\sigma - s}{2}\, \mathrm{d}\sigma.$$

Substituting (2.8) and (2.9) into equation (2.7), we obtain

$$
\begin{aligned}
F(z) &= (-\mathrm{i})(\xi + \mathrm{i}\eta)[\psi(z) + \mathrm{i}\beta_0] \\
&= (-\mathrm{i}\xi + \eta)\bigg[M(s, s, \varphi(s))c(s)\mathrm{e}^{w_1(s)} \\
&\qquad - \frac{\mathrm{i}}{2\pi} \int_{-\pi}^{\pi} M(s, \sigma, \varphi(\sigma))c(\sigma)\mathrm{e}^{w_1(\sigma)} \cot \frac{\sigma - s}{2}\, \mathrm{d}\sigma + \mathrm{i}\beta_0 \bigg] \\
&= \bigg[\xi(s)\beta_0 + \eta(s)M(s, s, \varphi(s))c(s)\mathrm{e}^{w_1(s)} \\
&\qquad - \frac{\xi(s)}{2\pi} \int_{-\pi}^{\pi} M(s, \sigma, \varphi(\sigma))c(\sigma)\mathrm{e}^{w_1(\sigma)} \cot \frac{\sigma - s}{2}\, \mathrm{d}\sigma \bigg] \\
&\quad + \mathrm{i}\bigg[\eta(s)\beta_0 - \xi(s)M(s, s, \varphi(s))c(s)\mathrm{e}^{w_1(s)} \\
&\qquad - \frac{\eta(s)}{2\pi} \int_{-\pi}^{\pi} M(s, \sigma, \varphi(\sigma))c(\sigma)\mathrm{e}^{w_1(\sigma)} \cot \frac{\sigma - s}{2}\, \mathrm{d}\sigma \bigg].
\end{aligned}
$$

Now we have

$$
(2.10) \quad
\begin{cases}
u(s) = & \xi(s)\beta_0 + \eta(s)M(s, s, \varphi(s))c(s)\mathrm{e}^{w_1(s)} \\
& - \dfrac{\xi(s)}{2\pi} \displaystyle\int_{-\pi}^{\pi} M(s, \sigma, \varphi(\sigma))c(\sigma)\mathrm{e}^{w_1(\sigma)} \cot \dfrac{\sigma - s}{2}\, \mathrm{d}\sigma, \\[2ex]
v(s) = & \eta(s)\beta_0 - \xi(s)M(s, s, \varphi(s))c(s)\mathrm{e}^{w_1(s)} \\
& - \dfrac{\eta(s)}{2\pi} \displaystyle\int_{-\pi}^{\pi} M(s, \sigma, \varphi(\sigma))c(\sigma)\mathrm{e}^{w_1(\sigma)} \cot \dfrac{\sigma - s}{2}\, \mathrm{d}\sigma.
\end{cases}
$$

In view of condition (2.6), we have

$$
\int_{-\pi}^{\pi} \bigg[\eta(s)\beta_0 - \xi(s)M(s, s, \varphi(s))c(s)\mathrm{e}^{w_1(s)} \\
- \frac{\eta(s)}{2\pi} \int_{-\pi}^{\pi} M(s, \sigma, \varphi(\sigma))c(\sigma)\mathrm{e}^{w_1(\sigma)} \cot \frac{\sigma - s}{2}\, \mathrm{d}\sigma \bigg] = 0.
$$

Hence

$$
(2.11) \quad \int_{-\pi}^{\pi} \eta(s)\beta_0\, \mathrm{d}s - \int_{-\pi}^{\pi} \xi(s)M(s, s, \varphi(s))c(s)\mathrm{e}^{w_1(s)}\, \mathrm{d}s \\
- \int_{-\pi}^{\pi} \eta(s)\, \mathrm{d}s \int_{-\pi}^{\pi} \frac{1}{2\pi} M(s, \sigma, \varphi(\sigma))c(\sigma)\mathrm{e}^{w_1(\sigma)} \cot \frac{\sigma - s}{2}\, \mathrm{d}\sigma = 0,
$$

and by changing the order of integration in the last term of equation (2.11), we obtain

$$(2.12) \quad \beta_0 \int_{-\pi}^{\pi} \eta(\sigma) \, d\sigma - \int_{-\pi}^{\pi} \xi(\sigma) M(s, \sigma, \varphi(\sigma)) c(\sigma) e^{w_1(\sigma)} \, d\sigma$$

$$+ \int_{-\pi}^{\pi} M(s, \sigma, \varphi(\sigma)) c(\sigma) e^{w_1(\sigma)} \, d\sigma \frac{1}{2\pi} \int_{-\pi}^{\pi} \eta(s) \cot \frac{s - \sigma}{2} \, ds = 0.$$

Now by using the Hilbert formula

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} v(\sigma) \cot \frac{\sigma - s}{2} \, d\sigma = u(s) - u_0 \qquad \text{where } u_0 = u(0, 0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\sigma) \, d\sigma$$

we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \eta(s) \cot \frac{s - \sigma}{2} \, ds = \xi(\sigma) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \xi(\sigma) \, d\sigma.$$

Hence equation (2.11) takes on the form

$$\beta_0 \int_{-\pi}^{\pi} \eta(\sigma) \, d\sigma - \int_{-\pi}^{\pi} \xi(\sigma) M(s, \sigma, \varphi(\sigma)) c(\sigma) e^{w_1(\sigma)} \, d\sigma$$

$$+ \int_{-\pi}^{\pi} M(s, \sigma, \varphi(\sigma)) c(\sigma) e^{w_1(\sigma)} \, d\sigma \left[\xi(\sigma) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \xi(\sigma) \, d\sigma \right] = 0.$$

Therefore

$$\beta_0 \int_{-\pi}^{\pi} \eta(\sigma) \, d\sigma - \frac{1}{2\pi} \int_{-\pi}^{\pi} \xi(\sigma) \, d\sigma \int_{-\pi}^{\pi} M(s, \sigma, \varphi(\sigma)) c(\sigma) e^{w_1(\sigma)} \, d\sigma = 0.$$

In the case of $\beta_0 \int_{-\pi}^{\pi} \eta(\sigma) \, d\sigma \neq 0$ the corresponding homogeneous equation has no solution. In this case we determine the constant β_0 and consequently the nonhomogeneous equation (2.2) has the unique solution (2.10).

Hence

$$u(s) = \xi(s) \beta_0 + \eta(s) M(s, s, \varphi(s)) c(s) e^{w_1(s)}$$

$$- \frac{\xi(s)}{2\pi} \int_{-\pi}^{\pi} M(s, \sigma, \varphi(\sigma)) c(\sigma) e^{w_1(\sigma)} \cot \frac{\sigma - s}{2} \, d\sigma.$$

From (2.6) we have

$$(2.13) \qquad \varphi'(s) = \xi(s) \beta_0 + \eta(s) M(s, s, \varphi(s)) c(s) e^{w_1(s)}$$

$$- \frac{\xi(s)}{2\pi} \int_{-\pi}^{\pi} M(s, \sigma, \varphi(\sigma)) c(\sigma) e^{w_1(\sigma)} \cot \frac{\sigma - s}{2} \, d\sigma.$$

Taking into account the additional condition (2.3), we obtain an equivalent fixed point equation

$$(2.14) \qquad\qquad S\varphi = \varphi,$$

where the operator S is defined for any 2π-periodic continuous function $\varphi(s)$ by

$$(2.15) \qquad\qquad (S\varphi)(s) = \Lambda + \int_0^s k(s, \sigma, \varphi(\sigma))\, \mathrm{d}\sigma,$$

with the kernel function

$$(2.16) \qquad k(s, \sigma, \varphi(\sigma)) = \xi(s)\beta_0 + \eta(s)M(s, s, \varphi(s))c(s)\mathrm{e}^{w_1(s)}$$
$$- \frac{\xi(s)}{2\pi} \int_{-\pi}^{\pi} M(s, \sigma, \varphi(\sigma))c(\sigma)\mathrm{e}^{w_1(\sigma)} \cot \frac{\sigma - s}{2}\, \mathrm{d}\sigma.$$

Lemma 2.1. *The kernel function $k(s, \sigma, \varphi(\sigma))$ given by (2.16) is bounded in the space $L_p[-\pi, \pi]$, $p > 1$, under the conditions*

$$(2.17) \qquad \begin{cases} |\xi(s_2) - \xi(s_1)| \leqslant \lambda_1 |s_2 - s_1|^\delta, \\ |\eta(s_2) - \eta(s_1)| \leqslant \lambda_2 |s_2 - s_1|^\delta, \\ |c(s_2) - c(s_1)| \leqslant \lambda_3 |s_2 - s_1|^\delta, \text{ and} \\ |w_1(s_2) - w_1(s_1)| \leqslant \lambda_4 |s_2 - s_1|^\delta, \end{cases}$$

where λ_i $(i = 1, 2, 3, 4)$ are positive constants.

P r o o f. We shall estimate the kernel $k(s, \sigma, \varphi(\sigma))$ for $p > 1$ as follows:

$$(2.18) \qquad \|k(s, \sigma, \varphi(\sigma))\|_p \leqslant \|\xi(s)\beta_0\|_p + \|N_1(s)\|_p + \|N_2(s)\|_p$$

where

$$(2.19) \qquad\qquad N_1(s) = \eta(s)M(s, s, \varphi(s))c(s)\mathrm{e}^{w_1(s)}$$

and

$$(2.20) \qquad N_2(s) = \frac{\xi(s)}{2\pi} \int_{-\pi}^{\pi} M(s, \sigma, \varphi(\sigma))c(\sigma)\mathrm{e}^{w_1(\sigma)} \cot \frac{\sigma - s}{2}\, \mathrm{d}\sigma.$$

Now we estimate $\|\xi(s)\beta_0\|_p$.

By using conditions (2.17), we have

$$(2.21) \qquad\qquad |\xi(s)| \leqslant n_1 \quad \text{with} \quad n_1 = 2\pi\lambda_1 + |\xi(0)|.$$

Therefore

$$(2.22) \qquad \|\xi(s)\beta_0\|_p \leqslant Q_1 \equiv \text{const},$$

where

$$Q_1 = |\beta_0| n_1 (2\pi)^{p^{-1}}.$$

By using conditions (2.4), (2.17) and equality (2.19), from [7], [15] we can estimate $\|N_1(s)\|_p$ as follows:

$$(2.23) \qquad \|N_1(s)\|_p \leqslant \|\eta(s)\|_{p_1} \|c(s)\|_{p_2} \|e^{w_1(s)}\|_{p_3} \|M(s,s,\varphi(s))\|_{p_4},$$

where $p_1^{-1} + p_2^{-1} + p_3^{-1} + p_4^{-1} = p^{-1}$.

To estimate the norm given in (2.23), we proceed in several steps:

(a) Estimation of $\|\eta(s)\|_{p_1}$.

As in (2.21) it is easy to see that

$$(2.24) \qquad |\eta(s)| \leqslant n_2 \quad \text{with} \quad n_2 = 2\pi\lambda_2 + |\eta(0)|.$$

Therefore

$$(2.25) \qquad \|\eta(s)\|_{p_1} < \gamma_1,$$

where $\gamma_1 = (2\pi)^{p_1^{-1}} n_2$.

Similarly, from (2.24) it is easy to obtain an estimate of $\|c(s)\|_{p_2}$ as follows:

$$(2.26) \qquad |c(s)| \leqslant n_3 \quad \text{with} \quad n_3 = 2\pi\lambda_3 + |c(0)|,$$

therefore

$$(2.27) \qquad \|c(s)\|_{p_2} < \gamma_2,$$

where $\gamma_2 = (2\pi)^{p_2^{-1}} n_3$.

Also, an estimate of $\|M(s,s,\varphi(s))\|_{p_4}$ follows:

$$(2.28) \quad |M(s,s,\varphi(s))| \leqslant n_4 \quad \text{with} \quad n_4 = \alpha R + B, \quad B = \max_{s,\sigma \in [-\pi,\pi]} |M(s,s,\varphi(0))|,$$

therefore

$$(2.29) \qquad \|M(s,s,\varphi(s))\|_{p_4} < \gamma_3,$$

where $\gamma_3 = (2\pi)^{p_4^{-1}} n_4$.

(b) Estimation of $\|e^{w_1(s)}\|_{p_3}$.

It follows from [5] that

$$\|e^{w_1(s)}\|_{p_3} \leqslant \left[1 + \left(\int_{-\pi}^{\pi} (n_5)^{p_3}\, ds\right)^{p_3^{-1}}\right] \exp\left(\int_{-\pi}^{\pi} (n_5)^{p_3}\, ds\right)^{1/p_3} = (1 + \gamma_4)e^{\gamma_4},$$

where

$$(2.30) \qquad |w_1(s)| \leqslant n_5, \quad n_5 = (2\pi)\lambda_5 + |w_1(0)|, \quad \gamma_4 = (2\pi)^{p_3^{-1}} n_5.$$

Therefore

$$(2.31) \qquad \qquad \|e^{w_1(s)}\|_{p_3} \leqslant (1 + \gamma_4)e^{\gamma_4}.$$

Substituting (2.25), (2.27), (2.29), and (2.31) into (2.23), we obtain

$$(2.32) \qquad \qquad \|N_1(s)\|_p \leqslant Q_2,$$

where $Q_2 = \gamma_1 \gamma_2 \gamma_3 (1 + \gamma_4)e^{\gamma_4}$.

Similarly, we can estimate $\|N_2(s)\|_p$ as follows:

$$(2.33) \quad \|N_2(s)\|_p = \left\|\frac{\xi(s)}{2\pi} \int_{-\pi}^{\pi} M(s, \sigma, \varphi(\sigma)) c(\sigma) e^{w_1(\sigma)} \cot \frac{\sigma - s}{2}\, d\sigma\right\|$$

$$\leqslant \frac{1}{2\pi} \|\xi(s)\|_{q_1} \left\|\int_{-\pi}^{\pi} M(s, \sigma, \varphi(\sigma)) c(\sigma) e^{w_1(\sigma)} \cot \frac{\sigma - s}{2}\, d\sigma\right\|_{q_2}.$$

Using the well-known inequality, [5],

$$(2.34) \qquad \qquad \left\|\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\varphi(s)}{s - x}\right\| \leqslant \varrho \|\varphi(s)\|,$$

where ϱ is a positive constant, we conclude that

$$(2.35) \qquad \|N_2(s)\|_p \leqslant \varrho \|\xi(s)\|_{q_1} \|M(s, \sigma, \varphi(\sigma)) c(\sigma) e^{w_1(\sigma)}\|_{q_2},$$

where $p^{-1} = q_1^{-1} + q_2^{-1}$.

Hence,

$$(2.36) \qquad \qquad \|\xi(s)\|_{q_1} \leqslant n_1 (2\pi)^{q_1^{-1}}.$$

Also,

$$(2.37) \qquad \|c(\sigma) e^{w_1(\sigma)} M(s, \sigma, \varphi(\sigma))\|_{q_2} \leqslant \|c(\sigma)\|_{k_1} \|e^{w_1(\sigma)}\|_{k_2} \|M(s, \sigma, \varphi(\sigma))\|_{k_3},$$

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where $q_2^{-1} = k_1^{-1} + k_2^{-1} + k_3^{-1}$, and so

(2.38) $$\|c(\sigma)\|_{k_1} \leqslant (2\pi)^{k_1^{-1}} n_3,$$

(2.39) $$\|\mathrm{e}^{w_1(\sigma)}\|_{k_2} \leqslant (1 + n_5(2\pi)^{k_2^{-1}})\mathrm{e}^{(2\pi)^{k_2^{-1}} n_5}$$

and

(2.40) $$\|M(s, \sigma, \varphi(\sigma))\|_{k_3} \leqslant (2\pi)^{k_3^{-1}} n_4.$$

Substituting (2.38), (2.39), and (2.40) into (2.37), we obtain

(2.41) $$\|c(\sigma)\mathrm{e}^{w_1(\sigma)} M(s, \sigma, \varphi(\sigma))\|_{q_2} \leqslant n_3 n_4 (2\pi)^{k_1^{-1}+k_3^{-1}} (1 + n_5(2\pi)^{k_2^{-1}})\mathrm{e}^{n_5(2\pi)^{k_2^{-1}}};$$

using (2.36) and (2.41) in (2.35), we have

(2.42) $$\|N_2(s)\|_p \leqslant Q_3,$$

where

$$Q_3 = \varrho n_1 n_3 n_4 (2\pi)^{q_1^{-1}+k_1^{-1}+k_3^{-1}} (1 + n_5(2\pi)^{k_2^{-1}})\mathrm{e}^{n_5(2\pi)^{k_2^{-1}}}.$$

Substituting (2.22), (2.32), and (2.42) into (2.18), we conclude that

(2.43) $$\|k(s, \sigma, \varphi(\sigma))\| \leqslant Q,$$

where

$$Q = Q_1 + Q_2 + Q_3 \equiv \mathrm{const}.$$

$\square$

3. Existence theorem

For non-negative constants R and μ we define the following convex and compact set:

(3.1) $$A_{R,\mu}^{\delta} = \{\varphi \in W_p^1[-\pi, \pi], \ |\varphi| \leqslant R, \ |\varphi(s_2) - \varphi(s_1)| \leqslant \mu|s_2 - s_1|^{\delta},$$
$$s_1, s_2 \in [-\pi, \pi]\}.$$

We are going to prove some assertions about the set $A_{R,\mu}^{\delta}$ and its image $S(A_{R,\mu}^{\delta})$. It is easy to see that the set $A_{R,\mu}^{\delta}$ is a convex set.

Now we shall prove that the operator S defined by (2.15) transforms $A_{R,\mu}^{\delta}$ into itself.

For each $\varphi \in A_{R,\mu}^{\delta}$, we have

$$(S\varphi)(s) = \psi(s), \quad \psi(s) \in W_p^1[-\pi, \pi], \tag{3.2}$$

where $(S\varphi)(s) = \Lambda + \int_0^s k(s, \sigma, \varphi(\sigma))\, d\sigma$.

Then for any $s \in [-\pi, \pi]$ we have from (2.43) and (3.2) that

$$|\psi(s)| \leqslant \Lambda + \int_{-\pi}^{\pi} |k(s, \sigma, \varphi(\sigma))|\, d\sigma \leqslant \Lambda + \|k(s, \sigma, \varphi(\sigma))\|_p (2\pi)^{k^{-1}} \leqslant R, \tag{3.3}$$

where

$$R \equiv Q(2\pi)^{k^{-1}} + \Lambda \quad \text{and} \quad p^{-1} + k^{-1} = 1.$$

Now, we evaluate $|\psi(s_2) - \psi(s_1)|$ for $p > 1$, $p^{-1} + k^{-1} = 1$, $k^{-1} = \delta$, $s_1, s_2 \in [-\pi, \pi]$. We have

$$\begin{aligned} |\psi(s_2) - \psi(s_1)| &= \left| \int_0^{s_2} k(s, \sigma, \varphi(\sigma))\, d\sigma - \int_0^{s_1} k(s, \sigma, \varphi(\sigma))\, d\sigma \right| \\ &= \left| \int_{s_1}^{s_2} k(s, \sigma, \varphi(\sigma))\, d\sigma \right| \leqslant \|k(s, \sigma, \varphi(\sigma))\|_p |s_2 - s_1|^{k^{-1}} \\ &\leqslant Q|s_2 - s_1|^{k^{-1}}. \end{aligned} \tag{3.4}$$

If $Q \equiv \mu$, then the operator S maps the space $W_p^1[-\pi, \pi]$ into its convex compact subset $A_{R,\mu}^{\delta}$. In particular, S maps $A_{R,\mu}^{\delta}$ into itself, so that all transformed functions $\psi(s)$ belong to $A_{R,\mu}^{\delta}$.

Hence the following lemma is valid.

Lemma 3.1. *Let functions $\xi(s)$, $\eta(s)$, $c(s)$, $w_1(s)$ and $M(s, \sigma, \varphi(\sigma))$ satisfy the Hilbert-Lipschitz conditions (2.17). Then, for arbitrary $\varphi \in A_{R,\mu}^{\delta}$, the transformed points $(S\varphi)(s) = \psi(s)$ belong to the set $A_{R,\mu}^{\delta}$.*

Lemma 3.2. *The operator S defined in (3.2) which transforms the set $A_{R,\mu}^{\delta}$ into itself is continuous.*

P r o o f.   Let $\{\varphi_n\}_{n=1}^{\infty}$ be a sequence of elements of the set $A_{R,\mu}^{\delta}$ converging uniformly to an element $\varphi \in A_{R,\mu}^{\delta}$.

We consider the difference

$$|\psi_n(s) - \psi(s)| \leqslant \int_{-\pi}^{\pi} |k(s, \sigma, \varphi_n(\sigma)) - k(s, \sigma, \varphi(\sigma))|\, d\sigma,$$

where
$$|k(s, \sigma, \varphi_n(\sigma)) - k(s, \sigma, \varphi(\sigma))| \leqslant |(N_n^1)(s)| + |(N_n^2)(s)|$$

with

$$(N_n^1)(s) = \eta(s)c(s)e^{w_1(s)}M(s, s, \varphi_n(s)) - \eta(s)c(s)e^{w_1(s)}M(s, s, \varphi(s))$$

and

$$(N_n^2)(s) = \frac{\xi(s)}{2\pi} \int_{-\pi}^{\pi} c(\sigma)e^{w_1(\sigma)}M(s, \sigma, \varphi_n(\sigma)) \cot \frac{\sigma - s}{2} \, d\sigma$$
$$- \frac{\xi(s)}{2\pi} \int_{-\pi}^{\pi} c(\sigma)e^{w_1(\sigma)}M(s, \sigma, \varphi(\sigma)) \cot \frac{\sigma - s}{2} \, d\sigma.$$

Now, we will show that
$$\lim_{n \to \infty} |\psi_n(s) - \psi(s)| = 0.$$

We have

$$\lim_{n \to \infty} |(N_n^1)(s)| = \lim_{n \to \infty} |\eta(s)c(s)e^{w_1(s)}M(s, s, \varphi_n(s)) - \eta(s)c(s)e^{w_1(s)}M(s, s, \varphi(s))|$$
$$\leqslant |\eta(s)c(s)e^{w_1(s)}| \lim_{n \to \infty} |M(s, s, \varphi_n(s)) - M(s, s, \varphi(s))|$$
$$\leqslant |\eta(s)c(s)e^{w_1(s)}| \lim_{n \to \infty} \alpha |\varphi_n(s) - \varphi(s)|,$$

but

$$\lim_{n \to \infty} |\varphi_n(s) - \varphi(s)| = 0.$$

Hence

(3.5) $$\lim_{n \to \infty} |(N_n^1)(s)| = 0.$$

Also,

$$\lim_{n \to \infty} |(N_n^2)(s)|$$
$$= \lim_{n \to \infty} \left| \frac{\xi(s)}{2\pi} \int_{-\pi}^{\pi} c(\sigma)e^{w_1(\sigma)}M(s, \sigma, \varphi_n(\sigma)) \cot \frac{\sigma - s}{2} \, d\sigma \right.$$
$$\left. - \frac{\xi(s)}{2\pi} \int_{-\pi}^{\pi} c(\sigma)e^{w_1(\sigma)}M(s, \sigma, \varphi(\sigma)) \cot \frac{\sigma - s}{2} \, d\sigma \right|$$
$$\leqslant |\xi(s)| \lim_{n \to \infty} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} c(\sigma)e^{w_1(\sigma)}[M(s, \sigma, \varphi_n(\sigma)) - M(s, \sigma, \varphi(\sigma))] \cot \frac{\sigma - s}{2} \, d\sigma \right|$$

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and using inequality (2.34), we arrive at

$$\lim_{n\to\infty} |(N_n^2)(s)|$$
$$\leqslant |\xi(s)| \lim_{n\to\infty} \varrho \|c(\sigma)\mathrm{e}^{w_1(\sigma)}[M(s,\sigma,\varphi_n(\sigma)) - M(s,\sigma,\varphi(\sigma))]\|$$
$$\leqslant \varrho|\xi(s)||c(\sigma)\mathrm{e}^{w_1(\sigma)}| \lim_{n\to\infty} \alpha|\varphi_n(s) - \varphi(s)| = 0.$$

Hence

$$(3.6) \qquad\qquad\qquad \lim_{n\to\infty} |(N_n^2)(s)| = 0.$$

From (3.5) and (3.6) we obtain

$$\lim_{n\to\infty} |\psi_n(s) - \psi(s)| = 0.$$

Consequently, the operator S is continuous. □

By the preceding lemmas and the Arzela-Ascoli theorem [13], [16] the image of $A_{R,\mu}^{\delta}$ is compact. Therefore, we can use Schauder's fixed point theorem. Hence the operator S has at least one fixed point. Thus, we can state the main theorem as follows.

Theorem 3.1. *If the conditions of Lemmas 3.1 and 3.2 are satisfied, then equation (1.1) has at least one solution in the space $W_p^1[-\pi, \pi]$.*

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