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ON THE CAGINALP SYSTEM WITH DYNAMIC BOUNDARY CONDITIONS AND SINGULAR POTENTIALS

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Abstract. This article is devoted to the study of the Caginalp phase field system with dynamic boundary conditions and singular potentials. We first show that, for initial data in H^2, the solutions are strictly separated from the singularities of the potential. This turns out to be our main argument in the proof of the existence and uniqueness of solutions. We then prove the existence of global attractors. In the last part of the article, we adapt well-known results concerning the Łojasiewicz inequality in order to prove the convergence of solutions to steady states.

Keywords: Caginalp phase field system, singular potential, dynamic boundary conditions, global existence, global attractor, Łojasiewicz-Simon inequality, convergence to a steady state

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1. Introduction

We consider in this article the following system of partial differential equations in a bounded smooth domain Ω of $\mathbb{R}^3$:

$$\begin{cases} \varepsilon\,\partial_t w - \Delta w = -\partial_t u, \\ \partial_t u - \Delta u + f(u) = w, \end{cases}$$

$0 < \varepsilon < 1$. This system of equations was proposed by G. Caginalp in [7] in order to model melting-solidification phenomena in certain classes of materials. Here, w corresponds to the relative temperature and u is the order parameter, or phase field, which describes the proportion of either of the phases; the values $u = \pm 1$ correspond to the pure states.

This system, with various types of boundary conditions and for a regular potential f, has been extensively studied, see, e.g., [2], [3], [4], [5], [6], [7], [9], [14], [15],

[24], [33] and the references therein. In particular, one has satisfactory results on the existence and uniqueness of solutions, the existence of finite dimensional attractors and the convergence of solutions to steady states. However, we note that for regular potentials it is not known whether the order parameter remains in the physically relevant interval $[-1, 1]$ in general (see however [2] and [3]).

Now, singular potentials f are also important from the physical point of view; in particular, we have in mind the following thermodynamically relevant logarithmic potential:

$$f(s) = -\kappa_0 s + \kappa_1 \ln \frac{1+s}{1-s}, \quad s \in (-1, 1),\ 0 < \kappa_0 < \kappa_1.$$

Such potentials, in the case of Dirichlet boundary conditions for both w and u, were considered in [16]; in particular, the existence and uniqueness of solutions and the existence of exponential attractors were proved in [16]. The convergence of solutions to steady states was proved in [17] for mixed Dirichlet (for the temperature) and Neumann (for the order parameter) boundary conditions. The case of Neumann boundary conditions, for both w and u, was treated in [8]. We can note that, contrary to regular potentials, such singular potentials allow to prove that the order parameter remains strictly between -1 and 1, as is expected from the physical point of view.

In this article, we supplement the equations with the so-called dynamic boundary conditions (in the sense that the kinetics, i.e., the time derivative of the order parameter, appears explicitly in the boundary conditions). Such boundary conditions have been proposed by physicists (see [10], [11] and [21]; see also [12]) in order to account for the wall effects in confined systems. In particular, the Cahn-Hilliard equation, endowed with these boundary conditions, has been studied in [9], [12], [25], [26], [27], [28] and [32]. The Caginalp system, endowed with dynamic boundary conditions and with regular potentials, was considered in [9], [13] and [14].

Here we are interested in the Caginalp system endowed with dynamic boundary conditions and with a singular potential f (and, in particular, with the above logarithmic potential). We prove the existence and uniqueness of solutions, as well as their regularity. The main ingredient in this study consists in proving that the order parameter u is strictly separated from the singular values of the potential.

We then prove the existence of global attractors. Recall that the global attractor $\mathcal{A}$ associated with the semigroup $S(t)$ on the phase space Φ is the smallest (with respect to inclusion) compact and invariant set which attracts all bounded sets of initial data as time goes to infinity; it thus appears as a suitable object in view of the study of the asymptotic behavior of the system, see, e.g., [31] for a review on this subject.

Another important issue is whether any trajectory converges to some steady state as time goes to infinity. It is important to note that such a question is not a trivial one, as there may be a continuum of steady states. In particular, following [9], we are able to prove the convergence of trajectories to steady states by using an approach based on the so-called Łojasiewicz-Simon inequality and the analyticity of the nonlinear terms. Such an approach, first considered in [30] (based on deep results from the theory of analytic functions of several variables due to S. Łojasiewicz, see [22]) and then simplified and further developed in [20], has been applied with success to many equations and, in particular, to models in phase separation and transition, see, e.g., [1], [8], [17], [18], [19], [23], [27], [29] and [33].

2. Setting of the problem

In this article, we are interested in the study of the following phase-field system:

$$
\begin{cases}
\varepsilon\partial_t w - \Delta w = -\partial_t u, & t>0,\ x\in\Omega,\\
\partial_t u - \Delta u + f(u) = w, & t>0,\ x\in\Omega,\\
\partial_t u - \Delta_\Gamma u + \lambda u + \dfrac{\partial u}{\partial n} + g(u) = 0, & t>0,\ x\in\Gamma=\partial\Omega,\\
\dfrac{\partial w}{\partial n}\Big|_\Gamma = 0, & \\
w|_{t=0} = w_0,\ u|_{t=0} = u_0, &
\end{cases}
\tag{1}
$$

where Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary Γ, Δ_Γ is the Laplace-Beltrami operator and $\partial/\partial n$ is the outward normal derivative. We further assume that $0<\varepsilon<1$ and $\lambda>0$ (actually, the condition $\lambda>0$ is necessary only in order to prove the existence of global attractors; for all the other results, we can also take $\lambda=0$).

The existence and uniqueness of solutions to problem (1) have already been proved in [14] for regular potentials. We are concerned here with singular potentials, namely, we assume that the function f satisfies the following conditions:

$$
f\in\mathcal{C}^3(-1,1),\quad \lim_{s\to\pm1} f(s)=\pm\infty,\quad \lim_{s\to\pm1} f'(s)=+\infty,
\tag{H$_1$}
$$

whereas the function g satisfies

$$
g\in\mathcal{C}^3(\mathbb{R}),\quad \liminf_{s\to\pm\infty} g'(s)\geqslant 0,\quad g(s)s\geqslant \mu|s|^2-\mu'\ \ \forall\, s\in\mathbb{R},
\tag{H$_{2,\mathrm{a}}$}
$$

for some $\mu>0$ and $\mu'\geqslant 0$, and there exists $0<\gamma<1$ such that

$$
\begin{cases}
g(s)\geqslant 0 & \text{on } [\gamma,1],\\
g(s)\leqslant 0 & \text{on } [-1,-\gamma].
\end{cases}
\tag{H$_{2,\mathrm{b}}$}
$$

In view of (H_1), the function f has the following properties (see [16]):

$$f'(s) \geqslant -K_1 \quad \text{and} \quad -\tilde{c} \leqslant F(s) \leqslant f(s)s + \tilde{C} \quad \forall\, s \in (-1,1), \tag{2}$$

where $F(s) = \int_0^s f(r)\,\mathrm{d}r$ and K_1, $\tilde{c}$, $\tilde{C}$ are strictly positive constants. Moreover, according to $(\mathrm{H}_{2,\mathrm{a}})$, the following inequalities hold for g (see [14]):

$$g'(s) \geqslant -K_2 \ \forall\, s \in \mathbb{R}, \quad (G(v) - g(v)v, 1)_\Gamma \leqslant K_2 \|v\|_\Gamma^2 \ \forall\, v \in L^2(\Gamma), \quad K_2 \geqslant 0, \tag{3}$$

where $G(s) = \int_0^s g(r)\,\mathrm{d}r$.

In this article we denote by $\|\cdot\|$ and $(\cdot,\cdot)$ (or $\|\cdot\|_\Gamma$ and $(\cdot,\cdot)_\Gamma$) the norm and the scalar product in $L^2(\Omega)$ (in $L^2(\Gamma)$). Furthermore, the singularities of the potential f lead us to define the quantity $D[u(t)] = (1 - \|u(t)\|_{L^\infty})^{-1}$ for $u \in L^\infty(\Omega)$ and we set $\langle u \rangle = |\Omega|^{-1} \int_\Omega u(x)\,\mathrm{d}x$ for $u \in L^1(\Omega)$.

Throughout the article, c, C_ε will denote positive constants which may vary from line to line and Q, Q_ε will denote increasing functions, C_ε, Q_ε depending on ε.

3. A priori estimates

Following [14], [25], we introduce a further variable $\psi = u|_\Gamma$ and view the dynamic boundary condition as a parabolic equation for ψ on the boundary, namely,

$$\begin{cases} \varepsilon \partial_t w - \Delta w = -\partial_t u, & t > 0,\ x \in \Omega, \\ \partial_t u - \Delta u + f(u) = w, & t > 0,\ x \in \Omega, \\ \partial_t \psi - \Delta_\Gamma \psi + \lambda \psi + \dfrac{\partial u}{\partial n} + g(\psi) = 0, & t > 0,\ x \in \partial\Omega, \\ \dfrac{\partial w}{\partial n}\Big|_\Gamma = 0, \quad u|_\Gamma = \psi, & \\ w|_{t=0} = w_0, \quad u|_{t=0} = u_0, \quad \psi|_{t=0} = \psi_0. & \end{cases} \tag{4}$$

We start with the following theorem. Note that all estimates already depend on ε.

Theorem 3.1. *We assume that the functions f and g satisfy assumptions* (H_1), (H_2) *and that the initial data (u_0, ψ_0, w_0) satisfies*

$$D[u_0] + \|u_0\|_{H^2}^2 + \|\psi_0\|_{H^2(\Gamma)}^2 + \|w_0\|_{H^2}^2 < +\infty, \quad D[u_0] > 0, \quad u_0|_\Gamma = \psi_0. \tag{5}$$

Then, for every solution $(u(t), \psi(t), w(t))$ of (4) and every $t \geqslant 0$, we have

$$\begin{aligned} &\|w(t)\|_{H^2}^2 + \|u(t)\|_{H^1}^2 + \|\psi(t)\|_{H^1(\Gamma)}^2 \\ &\quad + \int_0^t \mathrm{e}^{-\alpha(t-s)} (\|\partial_t u(s)\|_{H^1}^2 + \|\partial_t \psi(s)\|_{H^1(\Gamma)}^2 + \varepsilon \|\partial_t w(s)\|_{H^1}^2)\,\mathrm{d}s \\ &\quad + \|\partial_t u(t)\|^2 + \|\partial_t \psi(t)\|_\Gamma^2 \\ &\leqslant Q_\varepsilon(D[u_0] + \|w_0\|_{H^2}^2 + \|u_0\|_{H^2}^2 + \|\psi_0\|_{H^2(\Gamma)}^2)\mathrm{e}^{-\alpha t} + C_{\varepsilon, I_0}, \end{aligned} \tag{6}$$

where $I_0 = \langle \varepsilon w_0 + u_0\rangle$ and the positive constant α and the increasing function Q_ε are independent of (u_0, ψ_0, w_0).

In order to prove Theorem 3.1, we need the following lemma.

Lemma 3.1. *Under the assumptions of Theorem 3.1, any solution $(u(t), \psi(t), w(t))$ to (4) satisfies, for every $t \geqslant 0$:*

$$\begin{aligned}(7)\quad \int_0^t (\|\partial_t u(s)\|^2 + \|\partial_t \psi(s)\|_\Gamma^2 + \|\nabla w(s)\|^2)\mathrm{e}^{-\alpha(t-s)}\,\mathrm{d}s \\ + \varepsilon\|w(t)\|^2 + \|u(t)\|_{H^1}^2 + \|\psi(t)\|_{H^1(\Gamma)}^2 \\ \leqslant Q(D[u_0] + \|u_0\|_{H^1}^2 + \|\psi_0\|_{H^1(\Gamma)}^2 + \|w_0\|_{L^2}^2)\mathrm{e}^{-\alpha t} + C_{\varepsilon, I_0}, \quad \alpha > 0.\end{aligned}$$

Proof. According to (H_1), we assume that, a priori,

$$(8)\qquad \|u\|_{L^\infty(\bar{\Omega}\times\mathbb{R}^+)} < 1.$$

Integrating the first equation of (4) over Ω, we obtain the conservation law

$$\langle \varepsilon w(t) + u(t)\rangle = \langle \varepsilon w_0 + u_0\rangle =: I_0 \quad \forall\, t \geqslant 0.$$

We multiply the first equation of (4) by w, the second by $u + \partial_t u$, sum and integrate over Ω. Using, e.g., the straightforward simplifications (in view of (4), third equation)

$$\begin{aligned}-(\Delta u(t), u(t)) &= \|\nabla u(t)\|^2 - \int_\Gamma \frac{\partial u}{\partial n}(t) u(t)\,\mathrm{d}\sigma \\ &= \|\nabla u(t)\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\psi(t)\|_\Gamma^2 + \|\nabla_\Gamma \psi(t)\|_\Gamma^2 \\ &\quad + \lambda\|\psi(t)\|_\Gamma^2 + (g(\psi(t)), \psi(t))_\Gamma\end{aligned}$$

and

$$\begin{aligned}-(\Delta u(t), \partial_t u(t)) &= (\nabla u(t), \nabla\partial_t u(t)) - \int_\Gamma \frac{\partial u}{\partial n}(t)\partial_t u(t)\,\mathrm{d}\sigma \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|^2 + \|\partial_t\psi(t)\|_\Gamma^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_\Gamma\psi(t)\|_\Gamma^2 \\ &\quad + \frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\psi(t)\|_\Gamma^2 + \frac{\mathrm{d}}{\mathrm{d}t}(G(\psi(t)), 1)_\Gamma,\end{aligned}$$

we obtain

$$\begin{aligned}(9)\quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E(t) + \|\nabla w(t)\|^2 + \|\nabla u(t)\|^2 + \|\nabla_\Gamma\psi(t)\|_\Gamma^2 + \lambda\|\psi(t)\|_\Gamma^2 + (f(u(t)), u(t)) \\ + (g(\psi(t)), \psi(t))_\Gamma + \|\partial_t u(t)\|^2 + \|\partial_t\psi(t)\|_\Gamma^2 = (w(t), u(t))\end{aligned}$$

with

$$E(t) = \varepsilon\|w(t)\|^2 + \|u(t)\|^2 + \|\nabla u(t)\|^2 + 2(F(u(t)),1) + (\lambda+1)\|\psi(t)\|^2_\Gamma \\ + \|\nabla_\Gamma \psi(t)\|^2_\Gamma + 2(G(\psi(t)),1)_\Gamma.$$

Furthermore, employing Friedrich's inequality

$$\frac{1}{2}\|\nabla w(t)\|^2 \geqslant C_\Omega(\|w(t)\|^2 - |\Omega|\langle w(t)\rangle^2), \quad C_\Omega > 0,$$

together with (2), (3) and the following consequence of the conservation law:

$$\langle u(t)\rangle \leqslant 1, \quad |\langle w(t)\rangle| \leqslant \frac{|I_0|+1}{\varepsilon},$$

we deduce that, for $\alpha > 0$ small enough,

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) + \alpha E(t) + \|\nabla w(t)\|^2 + \|\partial_t u(t)\|^2 + \|\partial_t \psi(t)\|^2_\Gamma \leqslant C_{\varepsilon,I_0}.$$

Thus, applying Gronwall's lemma, we finally obtain (7) and Lemma 3.1 is proved. □

P r o o f of Theorem 3.1. We differentiate the second and third equations of (4) with respect to t to find

$$\partial^2_{tt}u - \Delta\partial_t u + f'(u)\partial_t u = \partial_t w, \quad t>0, \ x\in\Omega, \tag{10}$$

$$\partial^2_{tt}\psi - \Delta_\Gamma\partial_t\psi + \lambda\partial_t\psi + \frac{\partial(\partial_t u)}{\partial n} + g'(\psi)\partial_t\psi = 0, \quad t>0, \ x\in\Gamma. \tag{11}$$

Next, we multiply (10) by $\partial_t u$ and the first equation of (4) by $\partial_t w$, sum and integrate over Ω. After straightforward transformations involving (11) we obtain, for $\alpha > 0$ small enough (smaller than the same constant appearing in Lemma 3.1),

$$\frac{\mathrm{d}}{\mathrm{d}t}\{\|\partial_t u(t)\|^2 + \|\partial_t\psi(t)\|^2_\Gamma + \|\nabla w(t)\|^2\} + \alpha(\|\partial_t u(t)\|^2 + \|\partial_t\psi(t)\|^2_\Gamma + \|\nabla w(t)\|^2) \\ + \|\nabla\partial_t u(t)\|^2 + \|\nabla_\Gamma\partial_t\psi(t)\|_\Gamma + \varepsilon\|\partial_t w(t)\|^2 \\ \leqslant C(\|\partial_t u(t)\|^2 + \|\partial_t\psi(t)\|^2_\Gamma + \|\nabla w(t)\|^2).$$

Then we apply Gronwall's lemma together with (7) and the inequalities

$$\|\partial_t\psi(0)\|^2 \leqslant Q(\|\psi_0\|^2_{H^2(\Gamma)} + \|u_0\|^2_{H^2}), \\ \|\partial_t u(0)\|^2 \leqslant Q(D[u_0] + \|u_0\|^2_{H^2} + \|w_0\|^2).$$

This yields

$$\begin{aligned}(12)\qquad &\|\partial_t u(t)\|^2+\|\partial_t\psi(t)\|_\Gamma^2+\|\nabla w(t)\|^2\\&\qquad+\int_0^t \mathrm{e}^{-\alpha(t-s)}\{\|\nabla\partial_t u(s)\|^2+\|\nabla_\Gamma\partial_t\psi(s)\|_\Gamma^2+\varepsilon\|\partial_t w(s)\|^2\}\,\mathrm{d}s\\&\qquad\leqslant Q(D[u_0]+\|\psi_0\|^2_{H^2(\Gamma)}+\|u_0\|^2_{H^2}+\|w_0\|^2_{H^1})\mathrm{e}^{-\alpha t}+C_{\varepsilon,I_0}.\end{aligned}$$

Finally, we multiply the first equation of (4) by $-\Delta\partial_t w-\Delta w$ and integrate over Ω. This implies, for $\alpha>0$ small enough,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\{\|\Delta w(t)\|^2+\varepsilon\|\nabla w(t)\|^2\}+\alpha(\|\Delta w(t)\|^2+\varepsilon\|\nabla w(t)\|^2)+\varepsilon\|\nabla\partial_t w(t)\|^2\\ \leqslant C_\varepsilon(\|\partial_t u(t)\|^2+\|\nabla\partial_t u(t)\|^2+\|\nabla w(t)\|^2).\end{aligned}$$

Hence we complete the proof of Theorem 3.1 by employing (7), (12) and Gronwall's lemma. □

It remains to prove proper H^2-estimates for u and ψ.

Theorem 3.2. *Under the assumptions of Theorem* 3.1, *the solutions* $u(t)$ *and* $\psi(t)$ *of problem* (4) *are strictly separated from the singularities* ±1 *of the function* f, *i.e., there exists a constant* $0<\delta<1$ *depending on* $D[u_0]$, $\|u_0\|_{H^2}$, $\|\psi_0\|_{H^2(\Gamma)}$, $\|w_0\|_{H^2}$ *and* ε *such that*

$$(13)\qquad \|\psi(t)\|_{L^\infty(\Gamma)}\leqslant\delta\quad\text{and}\quad\|u(t)\|_{L^\infty}\leqslant\delta\quad\forall\,t\geqslant0.$$

P r o o f. From Theorem 3.1 we infer the existence of a constant $\beta>0$ such that

$$\|w(t)\|_{L^\infty}\leqslant c\|w(t)\|_{H^2}\leqslant\beta\quad\forall\,t\geqslant0.$$

Hence we denote by δ a strictly positive constant depending on $D[u_0]$, $\|u_0\|_{H^2}$, $\|\psi_0\|_{H^2(\Gamma)}$, $\|w_0\|_{H^2}$ and ε, which satisfies (we know that such a constant exists, owing to (H_1) and $(\mathrm{H}_{2,\mathrm{b}})$)

$$(14)\qquad \|u_0\|_{L^\infty(\bar\Omega)}<\delta<1,\quad g(\delta)\geqslant0\quad\text{and}\quad f(\delta)>\beta\geqslant\|w(t)\|_{L^\infty}\quad\forall\,t\geqslant0.$$

We then set $v=u-\delta$, $\varphi=\psi-\delta$ and rewrite the second and third equations of (4) as

$$\begin{aligned}(15)\qquad \partial_t v-\Delta v+f(u)-f(\delta)&=w-f(\delta),\\ \partial_t\varphi-\Delta_\Gamma\varphi+\lambda\varphi+\frac{\partial v}{\partial n}+g(\psi)-g(\delta)&=-\lambda\delta-g(\delta).\end{aligned}$$

We multiply the first equation of (15) by $v_+ = \max(v, 0)$ and integrate over Ω. Applying (2) and (14) (i.e., $w(t,x) - f(\delta) \leqslant 0\ \forall\, t \geqslant 0,\ \forall\, x \in \Omega$), we arrive at

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_+(t)\|^2 + \|\nabla v_+(t)\|^2 - \int_\Gamma \frac{\partial v(t)}{\partial n} v_+(t)\,\mathrm{d}\sigma \leqslant K_1\|v_+(t)\|^2.$$

We find, noting that $v|_\Gamma = \varphi$ and owing to (3) and (H_2) (thus, $\lambda\delta + g(\delta) \geqslant 0$),

$$-\int_\Gamma \frac{\partial v(t)}{\partial n}\varphi_+(t)\,\mathrm{d}\sigma \geqslant \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_+(t)\|_\Gamma^2 + \|\nabla_\Gamma\varphi_+(t)\|_\Gamma^2 + \lambda\|\varphi_+(t)|_\Gamma^2 - K_2\|\varphi_+(t)|_\Gamma^2.$$

Then we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{\|v_+(t)\|^2 + \|\varphi_+(t)\|_\Gamma^2\} + \|\nabla v_+(t)\|^2 + \|\nabla_\Gamma\varphi_+(t)\|_\Gamma^2 + \lambda\|\varphi_+(t)\|_\Gamma^2 \\ \leqslant K_1\|v_+(t)\|^2 + K_2\|\varphi_+(t)\|_\Gamma^2.\end{aligned}$$

Thus Gronwall's lemma leads to

$$\|v_+(t)\|^2 + \|\varphi_+(t)\|_\Gamma^2 \leqslant (\|v_+(0)\|^2 + \|\varphi_+(0)\|_\Gamma^2)\mathrm{e}^{Kt}$$

with $K = 2\max(K_1, K_2)$. According to (14) we have

$$v_+(0) = 0 = \varphi_+(0).$$

Hence we obtain

$$v_+(t) = 0 = \varphi_+(t) \quad \forall\, t \geqslant 0,$$

i.e.,

$$v(t,x) \leqslant 0\ \ \forall\, t \geqslant 0,\ \text{ for a.e. } x \in \Omega, \quad \varphi(t,x) \leqslant 0\ \ \forall\, t \geqslant 0,\ \text{ for a.e. } x \in \Gamma.$$

We then conclude that

$$u(t,x) \leqslant \delta\ \ \forall\, t \geqslant 0,\ \text{ for a.e. } x \in \Omega, \quad \psi(t,x) \leqslant \delta\ \ \forall\, t \geqslant 0,\ \text{ for a.e. } x \in \Gamma.$$

It remains to prove that $u(t,x) \geqslant -\delta$ for all $t > 0$, for a.e. $x \in \Omega$. In order to do so, we can assume, owing to (H_1), that the constant $0 < \delta < 1$ introduced in (14) also satisfies

$$f(-\delta) < -\beta \leqslant -\|w(t)\|_{L^\infty} \quad \forall\, t \geqslant 0.$$

Then we set $v = u - \delta'$ with $\delta' = -\delta$. We again consider equation (15), with δ replaced by δ', and multiply this equation by $v_- = \min(0, v)$. We omit the rest of the proof of Theorem 3.2, the arguments being exactly the same as above. □

Corollary 3.1. *Under the assumptions of Theorem* 3.1 *there exists a constant* $M_\delta > 0$ *depending on the constant* $\delta = \delta(D[u_0], \|u_0\|_{H^2}, \|\psi_0\|_{H^2(\Gamma)}, \|w_0\|_{H^2}, \varepsilon)$ *appearing in Theorem* 3.2 *such that the following estimate holds:*

$$\|u(t)\|_{H^2} + \|\psi(t)\|_{H^2(\Gamma)} \leqslant M_\delta \quad \forall\, t \geqslant 0.$$

Proof. According to Theorems 3.1 and 3.2, we can rewrite the second and third equations of (4) as

$$\begin{cases} -\Delta u = h_1, & h_1 = w - \partial_t u - f(u), \\ -\Delta_\Gamma \psi + \lambda\psi + \dfrac{\partial u}{\partial n} = h_2, & h_2 = -\partial_t \psi - g(\psi), \end{cases}$$

with $\|h_1\| \leqslant C_1$ and $\|h_2\|_{L^2(\Gamma)} \leqslant C_2$, the constants C_1 and C_2 depending on δ. Arguing then as in [25, Lemma A.1], we obtain the estimate of Corollary 3.1. $\square$

Theorem 3.3.

(i) *Under the assumptions of Theorem* 3.1 *there exists a constant* M_1 *depending on* t_1, δ, $D[u_0]$, $\|u_0\|_{H^2}$, $\|\psi_0\|_{H^2(\Gamma)}$, $\|w_0\|_{H^2}$, ε *such that the following estimate holds for some* $t_1 > 0$*:*

$$\|u(t)\|_{H^3} + \|\psi(t)\|_{H^3(\Gamma)} + \|w(t)\|_{H^3} \leqslant M_1 \qquad \forall\, t \geqslant t_1. \tag{16}$$

(ii) *Furthermore, if we assume that*

$$D[u_0] + \|u_0\|_{H^3} + \|\psi_0\|_{H^3(\Gamma)} + \|w_0\|_{H^3} < +\infty,$$

then there exists a constant M_2 *depending on* $D[u_0]$, $\|u_0\|_{H^3}$, $\|\psi_0\|_{H^3(\Gamma)}$, $\|w_0\|_{H^3}$, δ, ε *such that*

$$\|u(t)\|_{H^3} + \|\psi(t)\|_{H^3(\Gamma)} + \|w(t)\|_{H^3} \leqslant M_2 \quad \forall\, t \geqslant 0. \tag{17}$$

Proof. We multiply equation (10) by $\partial^2_{tt} u$, integrate over Ω and use (11) to obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\{\|\nabla \partial_t u(t)\|^2 + \|\nabla_\Gamma \partial_t \psi(t)\|^2_\Gamma + \lambda \|\partial_t \psi(t)\|^2_\Gamma\} + \|\partial^2_{tt} u(t)\|^2 + \|\partial^2_{tt}\psi(t)\|^2_\Gamma \\ \leqslant C(\|\partial_t u(t)\|^2 + \|\partial_t w(t)\|^2 + \|\partial_t \psi(t)\|^2_\Gamma). \end{aligned} \tag{18}$$

Then we multiply this inequality by s and integrate over $[0,t]$. Standard integrations by parts and Theorem 3.1 lead to

$$
\begin{aligned}
(19)\quad & t\|\nabla\partial_t u(t)\|^2 + t\|\nabla_\Gamma\partial_t\psi(t)\|_\Gamma^2 + \lambda t\|\partial_t\psi(t)\|_\Gamma^2 \\
&\quad + \int_0^t (s\|\partial_{tt}^2 u(s)\|^2 + s\|\partial_{tt}^2\psi(s)\|_\Gamma^2)\,\mathrm{d}s \\
&\leqslant \int_0^t (\|\nabla\partial_t u(s)\|^2 + \|\nabla_\Gamma\partial_t\psi(s)\|_\Gamma^2 + \lambda\|\partial_t\psi(s)\|_\Gamma^2)\,\mathrm{d}s \\
&\quad + Ct\int_0^t (\|\partial_t u(s)\|^2 + \|\partial_t w(s)\|^2 + \|\partial_t\psi(s)\|_\Gamma^2)\,\mathrm{d}s \leqslant C_\varepsilon t + C'_\varepsilon
\end{aligned}
$$

where the above constants depend on $D[u_0]$, $\|u_0\|_{H^2}$, $\|\psi_0\|_{H^2(\Gamma)}$, $\|w_0\|_{H^2}$, δ and ε. In particular, we infer that

$$
\begin{aligned}
(20)\quad & \|\nabla\partial_t u(t)\|^2 + \|\nabla_\Gamma\partial_t\psi(t)\|_\Gamma^2 + \lambda\|\partial_t\psi(t)\|_\Gamma^2 \\
&\leqslant C_\varepsilon + \frac{C'_\varepsilon}{t} \leqslant C_\varepsilon + \frac{C'_\varepsilon}{t_1} =: C_1 \quad \forall\, t \geqslant t_1 > 0.
\end{aligned}
$$

Next, we differentiate the first equation of (4) with respect to t, multiply the resulting equation by $t\,\partial_{tt}^2 w$ and integrate over Ω. We find

$$
\varepsilon t\|\partial_{tt}^2 w(t)\|^2 + t\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\partial_t w(t)\|^2 \leqslant \frac{t}{\varepsilon}\|\partial_{tt}^2 u(t)\|^2.
$$

Then integration over $(0,t)$, combined with integration by parts, (19) and Theorem 3.1, implies

$$
\begin{aligned}
t\|\nabla\partial_t w(t)\|^2 + \varepsilon\int_0^t s\|\partial_{tt}^2 w(s)\|^2\,\mathrm{d}s &\leqslant \int_0^t \frac{s}{\varepsilon}\|\partial_{tt}^2 u(s)\|^2\,\mathrm{d}s + \int_0^t \|\nabla\partial_t w(s)\|^2\,\mathrm{d}s \\
&\leqslant C''_\varepsilon t + C'''_\varepsilon.
\end{aligned}
$$

From the second equation of (4), we infer, applying (20), that

$$
\|\nabla(\Delta u(t))\| \leqslant \|\nabla w(t)\| + \|\nabla f(u(t))\| + \|\nabla\partial_t u(t)\| \leqslant C_2 \quad \forall\, t \geqslant t_1.
$$

Hence we conclude, in view of Corollary 3.1, that

$$
(21)\qquad \|u(t)\|_{H^3} \leqslant C_3 \quad \forall\, t \geqslant t_1,
$$

with C_3 depending on t_1, $D[u_0]$, $\|u_0\|_{H^2}$, $\|\psi_0\|_{H^2(\Gamma)}$, $\|w_0\|_{H^2}$, δ, ε. In a similar way we can write

$$
\|\nabla_\Gamma(\Delta_\Gamma\psi(t))\|_\Gamma \leqslant \|\nabla_\Gamma\partial_t\psi(t)\|_\Gamma + \lambda\|\nabla_\Gamma\psi(t)\|_\Gamma + \Big\|\nabla_\Gamma\Big(\frac{\partial u}{\partial n}(t)\Big)\Big\|_\Gamma + \|\nabla_\Gamma g(\psi(t))\|_\Gamma
$$

with

$$\Big\|\nabla_\Gamma\Big(\frac{\partial u}{\partial n}(t)\Big)\Big\|_\Gamma \leqslant C\|u(t)\|_{H^3}.$$

Consequently, estimates (20), (21) and Corollary 3.1 imply

$$\|\psi(t)\|_{H^3(\Gamma)} \leqslant C_4 \quad \forall t \geqslant t_1.$$

The same arguments hold for the H^3-estimate of w and (16) is proved.

In order to prove (17), we now assume that u_0, w_0 belong to $H^3(\Omega)$ and that ψ_0 belongs to $H^3(\Gamma)$. We again integrate (18) over $(0,t)$ and use Theorem 3.1 together with

$$\begin{aligned}\|\nabla\partial_t u(0)\| &\leqslant \|\nabla(\Delta u_0)\| + \|f'(u_0)\nabla u_0\| + \|\nabla w_0\|\\ &\leqslant C(\|u_0\|_{H^3} + \|w_0\|_{H^1}),\end{aligned}$$

$$\begin{aligned}\|\nabla_\Gamma\partial_t\psi(0)\|_\Gamma &\leqslant C'(\|\psi_0\|_{H^3(\Gamma)} + \|u_0\|_{H^2(\Gamma)})\\ &\leqslant C''(\|\psi_0\|_{H^3(\Gamma)} + \|u_0\|_{H^3}).\end{aligned}$$

Then it follows that

$$\begin{aligned}\|\nabla\partial_t u(t)\|^2 &+ \|\nabla_\Gamma\partial_t\psi(t)\|_\Gamma^2 + \lambda\|\partial_t\psi(t)\|_\Gamma^2\\ &+ \int_0^t (\|\partial_{tt}^2 u(s)\|^2 + \|\partial_{tt}^2\psi(s)\|_\Gamma^2)\,\mathrm{d}s \leqslant C'''\end{aligned} \tag{22}$$

with C''' depending on $D[u_0]$, $\|u_0\|_{H^3}$, $\|\psi_0\|_{H^3(\Gamma)}$, $\|w_0\|_{H^2}$ and ε.

Next, we differentiate the first equation of (4) with respect to t and multiply the resulting equation by $\partial_{tt}^2 w$ and integrate over Ω. Then we find

$$\varepsilon\|\partial_{tt}^2 w(t)\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\partial_t w(t)\|^2 \leqslant \frac{1}{\varepsilon}\|\partial_{tt}^2 u(t)\|^2.$$

Integrating this estimate over $(0,t)$ and using

$$\begin{aligned}\|\nabla\partial_t w(0)\| &\leqslant \frac{1}{\varepsilon}(\|\nabla\Delta w_0\| + \|\nabla\partial_t u(0)\|)\\ &\leqslant C_\varepsilon(\|w_0\|_{H^3} + \|u_0\|_{H^3}),\end{aligned}$$

we obtain, in view of (22),

$$\|\nabla\partial_t w(t)\|^2 + \varepsilon\int_0^t \|\partial_{tt}^2 w(s)\|^2\,\mathrm{d}s \leqslant \tilde{C},$$

with $\tilde{C}$ depending on $\|\psi_0\|_{H^3(\Gamma)}$, $\|u_0\|_{H^3}$, $\|w_0\|_{H^3}$, $D[u_0]$, ε. Writing

$$\|\nabla(\Delta u(t))\| \leqslant \|\nabla w(t)\| + \|f'(u(t))\nabla u(t)\| + \|\nabla \partial_t u(t)\|$$

and applying (22), we conclude that

$$\|u(t)\|_{H^3} \leqslant M \quad \forall\, t \geqslant 0,$$

with M depending on $D[u_0]$, $\|u_0\|_{H^3}$, $\|\psi_0\|_{H^3(\Gamma)}$ and $\|w_0\|_{H^3}$. Similar arguments apply to the H^3-estimates of ψ and w, which completes the proof of Theorem 3.3. □

We conclude this section by giving three lemmas concerned with the difference of two solutions to problem (4). They furnish the Lipschitz continuous dependence of the solutions on the initial data at any fixed time. In particular, we infer from Lemma 3.2 below the uniqueness of solutions to problem (4) (and (1)).

Lemma 3.2. *Let functions f and g satisfy assumptions* (H_1) *and* (H_2), *respectively. Let (u_1, ψ_1, w_1), (u_2, ψ_2, w_2) be two solutions to problem* (4) *with initial data satisfying* (5). *Then the following estimate holds for $t \geqslant 0$:*

$$\begin{aligned}
&\|w_1(t) - w_2(t)\|^2 + \|u_1(t) - u_2(t)\|^2 + \|\psi_1(t) - \psi_2(t)\|_\Gamma^2 \\
&\quad \leqslant C_1 \mathrm{e}^{C_2 t}(\|w_1(0) - w_2(0)\|^2 + \|u_1(0) - u_2(0)\|^2 + \|\psi_1(0) - \psi_2(0)\|_\Gamma^2),
\end{aligned}$$

where C_1, C_2 depend on ε, but are independent of the initial data.

The proof of this lemma is the same as the one of [8, Lemma 3.1] (see also [16] and [25]) and we thus omit the details here.

Lemma 3.3. *Let functions f and g satisfy assumptions* (H_1) *and* (H_2), *respectively. Let (u_1, ψ_1, w_1), (u_2, ψ_2, w_2) be two solutions to problem* (4) *with initial data satisfying* (5). *Then the following estimate holds for $t \geqslant 0$:*

$$\begin{aligned}
&\varepsilon\|w_1(t) - w_2(t)\|^2 + \|u_1(t) - u_2(t)\|_{H^1}^2 \\
&\qquad + \|\psi_1(t) - \psi_2(t)\|_{H^1(\Gamma)}^2 + \langle \varepsilon(w_1(t) - w_2(t)) + u_1(t) - u_2(t)\rangle^2 \\
&\quad \leqslant C_3 \mathrm{e}^{C_4 t}(\varepsilon\|w_1(0) - w_2(0)\|^2 + \|u_1(0) - u_2(0)\|_{H^1}^2 + \|\psi_1(0) - \psi_2(0)\|_{H^1(\Gamma)}^2),
\end{aligned}$$

with C_3, C_4 depending on $\|w_{i0}\|_{H^2}$, $\|u_{i0}\|_{H^2}$, $\|\psi_{i0}\|_{H^2(\Gamma)}$, $D[u_{i0}]$, $i = 1, 2$, and ε.

P r o o f. We set $w = w_1 - w_2$, $u = u_1 - u_2$ and $\psi = \psi_1 - \psi_2$. Thus (u, ψ, w) is a solution to

$$
\begin{cases}
\varepsilon \partial_t w - \Delta w = -\partial_t u, \\
\partial_t u - \Delta u + l(t) u = w, \\
\partial_t \psi - \Delta_\Gamma \psi + \lambda \psi + \dfrac{\partial u}{\partial n} + h(t)\psi = 0, \\
\dfrac{\partial w}{\partial n}\Big|_\Gamma = 0, \quad u|_\Gamma = \psi, \\
w|_{t=0} = w_0, \quad u|_{t=0} = u_0, \quad \psi|_{t=0} = \psi_0,
\end{cases}
\tag{23}
$$

where $l(t) = \int_0^1 f'(s u_1(t) + (1-s) u_2(t))\,\mathrm{d}s$ and $h(t) = \int_0^1 g'(s\psi_1(t) + (1-s)\psi_2(t))\,\mathrm{d}s$.

We have the conservation law

$$
\langle \varepsilon w(t) + u(t) \rangle = \langle \varepsilon w_{10} + u_{10} \rangle - \langle \varepsilon w_{20} + u_{20} \rangle =: \tilde{I}_0 \quad \forall\, t \geqslant 0.
$$

Since f and g are of class $\mathcal{C}^1$ and u_1, u_2, ψ_1, ψ_2 are strictly separated from ± 1, we infer that

$$
\|l(t)\|_{L^\infty} + \|h(t)\|_{L^\infty(\Gamma)} \leqslant C \quad \forall\, t \geqslant 0,
\tag{24}
$$

with C depending on $\|w_{i0}\|_{H^2}$, $\|u_{i0}\|_{H^2}$, $\|\psi_{i0}\|_{H^2(\Gamma)}$, $D[u_{i0}]$, $i = 1, 2$, and ε.

We now multiply the first equation of (23) by w, the second by $\partial_t u$, sum and integrate over Ω to find

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{\varepsilon\|w(t)\|^2 + \|\nabla u(t)\|^2 + \|\nabla_\Gamma \psi(t)\|_\Gamma^2 + \lambda\|\psi(t)\|_\Gamma^2\} + \|\nabla w(t)\|^2 \\
+ \frac{1}{2}\|\partial_t u(t)\|^2 + \frac{1}{2}\|\partial_t \psi(t)\|_\Gamma^2 \\
\leqslant c'\|u(t)\|^2 + c\|\psi(t)\|_\Gamma^2.
\end{aligned}
$$

Using again Friedrich's inequality and the fact that

$$
\frac{\mathrm{d}}{\mathrm{d}t}(\langle \varepsilon w(t) + u(t) \rangle^2) = 0 \quad \forall\, t \geqslant 0
$$

we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{\varepsilon\|w(t)\|^2 + \|\nabla u(t)\|^2 + \|\nabla_\Gamma \psi(t)\|_\Gamma^2 + \lambda\|\psi(t)\|_\Gamma^2 + \langle \varepsilon w(t) + u(t) \rangle^2\} \\
+ \|\nabla w(t)\|^2 + \frac{1}{2}\|\partial_t u(t)\|^2 + \frac{1}{2}\|\partial_t \psi(t)\|_\Gamma^2 \\
\leqslant c\|\psi(t)\|_\Gamma^2 + c''\|\nabla u(t)\|^2 + c'''\langle u(t) \rangle^2.
\end{aligned}
$$

Next, we write
$$\langle u\rangle = \langle \varepsilon w + u\rangle - \langle \varepsilon w\rangle.$$
Thus we have
$$\langle u\rangle^2 \leqslant 2\langle \varepsilon w + u\rangle^2 + 2\varepsilon^2\langle w\rangle^2 \leqslant 2\langle \varepsilon w + u\rangle^2 + \frac{2\varepsilon}{|\Omega|}\|w\|^2$$
and, finally, we find
$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\{\varepsilon\|w(t)\|^2 + \|\nabla u(t)\|^2 + \|\nabla_\Gamma\psi(t)\|_\Gamma^2 + \lambda\|\psi(t)\|_\Gamma^2 + \langle \varepsilon w(t) + u(t)\rangle^2\}\\ &\leqslant C(\|\psi(t)\|_\Gamma^2 + \|\nabla u(t)\|^2 + \langle \varepsilon w(t) + u(t)\rangle^2 + \varepsilon\|w(t)\|^2)\\ &\leqslant \tilde{C}(\varepsilon\|w(t)\|^2 + \|\nabla u(t)\|^2 + \|\nabla_\Gamma\psi(t)\|_\Gamma^2 + \lambda\|\psi(t)\|_\Gamma^2\\ &\quad + \langle \varepsilon w(t) + u(t)\rangle^2).\end{aligned}$$
Then we deduce that
$$\begin{aligned}\varepsilon\|w(t)\|^2 + \|\nabla u(t)\|^2 + \|\psi(t)\|_{H^1(\Gamma)}^2 + \langle \varepsilon w(t) + u(t)\rangle^2&\\ \leqslant C(\varepsilon\|w(0)\|^2 + \|u(0)\|_{H^1}^2 + \|\psi(0)\|_{H^1(\Gamma)}^2)\mathrm{e}^{\tilde{C}t}&.\end{aligned}$$
Again, this estimate, together with the inequality $\langle u\rangle^2 \leqslant 2\langle \varepsilon w + u\rangle^2 + 2\varepsilon|\Omega|^{-1}\|w\|^2$, obviously leads to
$$\langle u(t)\rangle^2 \leqslant 2C(\varepsilon\|w(0)\|^2 + \|u(0)\|_{H^1}^2 + \|\psi(0)\|_{H^1(\Gamma)}^2)\mathrm{e}^{\tilde{C}t}.$$
Hence we obtain the H^1-estimate of u and Lemma 3.3 is proved. □

We also have the following result.

Lemma 3.4. *Let functions f and g satisfy assumptions* (H_1) *and* (H_2), *respectively. Let* (u_1, ψ_1, w_1), (u_2, ψ_2, w_2) *be two solutions to problem* (4), *with initial data satisfying* (5). *Then we have, for $t \geqslant 0$,*
$$\begin{aligned}\|w_1(t) - w_2(t)\|_{H^1}^2 + \|u_1(t) - u_2(t)\|_{H^1}^2 + \|\psi_1(t) - \psi_2(t)\|_{H^1(\Gamma)}^2&\\ + \|\partial_t u_1(t) - \partial_t u_2(t)\|^2 + \|\partial_t\psi_1(t) - \partial_t\psi_2(t)\|_\Gamma^2&\\ \leqslant C_5\mathrm{e}^{C_6t}(\|w_{10} - w_{20}\|_{H^1}^2 + \|u_{10} - u_{20}\|_{H^2}^2 + \|\psi_{10} - \psi_{20}\|_{H^2(\Gamma)}^2)&\end{aligned}$$
with C_5, C_6 depending on $\|w_{i0}\|_{H^2}$, $\|u_{i0}\|_{H^2}$, $\|\psi_{i0}\|_{H^2(\Gamma)}$, $D[u_{i0}]$, $i = 1, 2$, and ε.

P r o o f. We differentiate the second and third equations of (23) with respect to t to obtain

$$\tag{25} \begin{cases} \partial_{tt}^2 u - \Delta \partial_t u + l_t(t)u + l(t)\partial_t u = \partial_t w, \\ \partial_{tt}^2 \psi - \Delta_\Gamma \partial_t \psi + \lambda \partial_t \psi + \dfrac{\partial(\partial_t u)}{\partial n} + h_t(t)\psi + h(t)\partial_t \psi = 0, \\ u_t|_\Gamma = \psi_t. \end{cases}$$

Obviously, it follows from Theorems 3.1 and 3.2 that, in addition to (24), we also have

$$\|l_t(t)\| + \|h_t(t)\|_\Gamma \leqslant C \quad \forall\, t \geqslant 0.$$

We multiply the first equation of (25) by $\partial_t u$ and the first equation of (23) by $\partial_t w$, sum and integrate over Ω. Then, using (H_1) and (H_2), it follows from the conditions

$$l(t) \geqslant -K_1 \quad \text{and} \quad h(t) \geqslant -K_2 \quad \forall\, t \geqslant 0$$

and straightforward simplifications that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{\|\nabla w(t)\|^2 + \|\partial_t u(t)\|^2 + \|\partial_t \psi(t)\|_\Gamma^2\} + \|\nabla \partial_t u(t)\|^2 \\ + \|\nabla_\Gamma \partial_t \psi(t)\|_\Gamma^2 + \lambda \|\partial_t \psi(t)\|_\Gamma^2 + \varepsilon \|\partial_t w(t)\|^2 \\ \leqslant K_1 \|\partial_t u(t)\|^2 + K_2 \|\partial_t \psi(t)\|_\Gamma^2 \\ - (l_t(t)u(t), \partial_t u(t)) - (h_t(t)\psi(t), \partial_t \psi(t))_\Gamma. \end{aligned}$$

Since we have (cf. (24) and recall that $H^1(\Omega) \hookrightarrow L^6(\Omega)$)

$$\begin{aligned} |(l_t(t)u(t), \partial_t u(t))| &\leqslant \|l_t(t)\| \|u(t)\partial_t u(t)\| \\ &\leqslant c\|u(t)\|_{L^4} \|\partial_t u(t)\|_{L^4} \\ &\leqslant c'\|u(t)\|_{H^1} \|\partial_t u(t)\|_{H^1} \\ &\leqslant \frac{1}{2}\|\partial_t u(t)\|_{H^1}^2 + C\|u(t)\|_{H^1}^2, \end{aligned}$$

and, similarly,

$$|(h_t(t)\psi(t), \partial_t \psi(t))| \leqslant \frac{1}{2}\|\partial_t \psi(t)\|_{H^1(\Gamma)}^2 + C'\|\psi(t)\|_{H^1(\Gamma)}^2,$$

we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{\|\nabla w(t)\|^2 + \|\partial_t u(t)\|^2 + \|\partial_t \psi(t)\|_\Gamma^2\} + \frac{1}{2}\|\partial_t u(t)\|_{H^1}^2 \\ + \frac{1}{2}\|\partial_t \psi(t)\|_{H^1(\Gamma)}^2 + \varepsilon\|\partial_t w(t)\|^2 \\ \leqslant (K_1 + 1)\|\partial_t u(t)\|^2 + (K_2 + 1)\|\partial_t \psi(t)\|_\Gamma^2 + C\|u(t)\|_{H^1}^2 \\ + C'\|\psi(t)\|_{H^1(\Gamma)}^2. \end{aligned}$$

Thus

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\{\|\nabla w(t)\|^2 + \|\partial_t u(t)\|^2 + \|\partial_t \psi(t)\|^2_\Gamma\} &+ \|\partial_t u(t)\|^2_{H^1} \\ &+ \|\partial_t \psi(t)\|^2_{H^1(\Gamma)} + 2\varepsilon\|\partial_t w(t)\|^2 \\ &\leqslant C''(\|\nabla w(t)\|^2 + \|\partial_t u(t)\|^2 + \|\partial_t \psi(t)\|^2_\Gamma) \\ &+ C'''(\|u(t)\|^2_{H^1} + \|\psi(t)\|^2_{H^1(\Gamma)}).\end{aligned}$$

Since

$$\begin{aligned}\|\partial_t u(0)\|^2 &\leqslant C(\|u_0\|^2_{H^2} + \|w_0\|^2), \\ \|\partial_t \psi(0)\|^2_\Gamma &\leqslant C'(\|\psi_0\|^2_{H^2(\Gamma)} + \|u_0\|^2_{H^2}),\end{aligned}$$

we conclude by Gronwall's lemma and Lemma 3.3 that

$$\|\nabla w(t)\|^2 + \|\partial_t u(t)\|^2 + \|\partial_t \psi(t)\|^2_\Gamma \leqslant C(\|w_0\|^2_{H^1} + \|u_0\|^2_{H^2} + \|\psi_0\|^2_{H^2(\Gamma)})\mathrm{e}^{C''t}$$

and Lemma 3.4 is proved. □

4. Existence and uniqueness of solutions

Theorem 4.1. *Let the nonlinearities f and g satisfy assumptions* (H_1) *and* (H_2). *Then, for any initial data $(u_0, \psi_0, w_0) \in H^2(\Omega) \times H^2(\Gamma) \times H^2_N(\Omega)$ satisfying*

$$D[u_0] + \|u_0\|^2_{H^2} + \|\psi_0\|^2_{H^2(\Gamma)} + \|w_0\|^2_{H^2} < +\infty, \quad D[u_0] > 0, \quad u_0|_\Gamma = \psi_0,$$

problem (4) *(or* (1)*) possesses a unique solution (u, ψ, w) which satisfies all the estimates of the previous section. Here $H^2_N(\Omega) = \{w \in H^2(\Omega),\ \partial w/\partial n|_\Gamma = 0\}$.*

Proof. The above a priori estimates lead us to introduce the approximate function

$$f_\delta(s) = \begin{cases} s + \delta + f(-\delta), & s \in (-\infty, -\delta], \\ f(s), & s \in [-\delta, \delta], \\ s + f(\delta) - \delta, & s \in [\delta, +\infty), \end{cases}$$

where the constant $\delta > 0$ is the one appearing in Theorem 3.2 (δ depends on $D[u_0]$, $\|u_0\|_{H^2}$, $\|\psi_0\|_{H^2(\Gamma)}$, $\|w_0\|_{H^2}$ and ε). Increasing δ ($0 < \delta < 1$) if necessary, we can assume that

$$f(\delta) \geqslant \delta \quad \text{and} \quad f(-\delta) \leqslant -\delta.$$

Then we consider the following regularized problem:

$$(26)\qquad \begin{cases} \varepsilon\partial_t w - \Delta w = -\partial_t u, & t>0,\ x\in\Omega,\\ \partial_t u - \Delta u + f_\delta(u) = w, & t>0,\ x\in\Omega,\\ \partial_t\psi - \Delta_\Gamma\psi + \lambda\psi + \dfrac{\partial u}{\partial n} + g(\psi) = 0, & t>0,\ x\in\Gamma,\\ \dfrac{\partial w}{\partial n}\Big|_\Gamma = 0,\ u|_\Gamma = \psi, & \\ w|_{t=0} = w_0,\ u|_{t=0} = u_0,\ \psi|_{t=0} = \psi_0. & \end{cases}$$

We can check, without any difficulty, that the function f_δ satisfies the following assumption, required in [14]: there exist constants $\eta_1 > 0$ and $\eta_2 \geqslant 0$ such that

$$f_\delta(s)s \geqslant \eta_1 s^2 - \eta_2 \quad \forall\, s\in\mathbb{R}.$$

Thus, arguing as in [14], we infer the existence of a solution $(u_\delta, \psi_\delta, w_\delta)$ to problem (26) belonging to

$$\mathcal{C}_{\mathrm{w}}([0,T], H^2(\Omega)\times H^2(\Gamma)\times H^2_N(\Omega)) \cap (W^{1,2}_p(\Omega_T)\times W^{1,2}_p(\Gamma_T)\times W^{1,2}_p(\Omega_T))$$

with $p\in(3,10/3)$. (Here we have set $\Omega_T = [0,T]\times\Omega$ and $\Gamma_T = [0,T]\times\Gamma$, and $W^{1,2}_p(\Omega_T)$ denotes the set of functions which, together with their first time derivative and first and second space derivatives, belong to $L^p(\Omega_T)$.)

In order to make sure that this solution is suitable for problem (4), we show in the next lemma that f_δ satisfies (2).

Lemma 4.1. *We set $F_\delta(r) = \int_0^r f_\delta(s)\,\mathrm{d}s$. The functions f_δ and F_δ possess the following properties:*

$$f'_\delta(r) \geqslant -K_1, \quad r\neq\pm\delta, \quad \text{and} \quad -\tilde{c} \leqslant F_\delta(r) \leqslant f_\delta(r)r + \tilde{C} \quad \forall\, r\in\mathbb{R},$$

where K_1, $\tilde{c}$, $\tilde{C}$ are the strictly positive constants appearing in (2).

Proof. We only detail the case $r\in\,]\delta,+\infty)$. The other ones are very similar and are omitted.

It follows from the definition of f_δ that $f'_\delta(r) = 1 \geqslant -K_1\ \forall\, r\in\,]\delta,+\infty)$. Moreover, since f satisfies (2) and $f(\delta)\geqslant\delta$, we obtain

$$F_\delta(r) = \int_0^\delta f_\delta(s)\,\mathrm{d}s + \int_\delta^r f_\delta(s)\,\mathrm{d}s = F(\delta) + \frac{r^2-\delta^2}{2} + (f(\delta)-\delta)(r-\delta) \geqslant -\tilde{c}.$$

Then, since $f(\delta) = f_\delta(r) - r + \delta$ for $r \geqslant \delta$, (2) also leads to

$$\begin{aligned}F_\delta(r) &= F(\delta) + (r-\delta)\Big(\frac{r-\delta}{2} + f(\delta)\Big)\\ &\leqslant f(\delta)\delta + \tilde{C} + (r-\delta)\Big(\frac{r-\delta}{2} + f(\delta)\Big)\\ &\leqslant f(\delta)r + \tilde{C} + \frac{(r-\delta)^2}{2}\\ &\leqslant f_\delta(r)r + \tilde{C} + (\delta - r)\Big(\frac{r+\delta}{2}\Big) \leqslant f_\delta(r)r + \tilde{C}.\end{aligned}$$

As a consequence of Lemma (4.1), the a priori estimates established in Section 2 for the solutions to problem (4) still hold for the solutions to $(\mathcal{P}_\delta)$. In particular, we deduce from Theorem 3.2 that

$$\|u_\delta(t)\|_{L^\infty} \leqslant \delta \quad \forall\, t \geqslant 0.$$

Hence we have $f_\delta(u_\delta) = f(u_\delta)$ and we conclude that $(u_\delta, \psi_\delta, w_\delta)$ is also a solution to problem (4). Since the uniqueness of the solution to problem (4) is a direct consequence of Lemma 3.3, we finally conclude that $(u_\delta, \psi_\delta, w_\delta)$ is the unique solution to problem (4) and Theorem 4.1 is proved. $\square$

R e m a r k 4.1. Actually, in order to apply the results of [14], the function f_δ needs to be of class $\mathcal{C}^1$. However, the existence of a solution for this less regular function f_δ follows from standard regularization arguments (in particular, we can consider a regularized potential f_δ^ξ (which approximates f_δ) of class $\mathcal{C}^1$.

R e m a r k 4.2. Lemma 3.2 allows to prove (by continuity) the existence (and also the uniqueness) of a solution for initial data belonging to the closure L of

$$\Phi = \Big\{(u,w,\psi) \in H^2(\Omega) \times H^2(\Omega) \times H^2(\Gamma),\ u|_\Gamma = \psi,\ \frac{\partial w}{\partial n}\Big|_\Gamma = 0,\ \|u\|_{L^\infty} < 1\Big\}$$

in $L^2(\Omega) \times L^2(\Omega) \times L^2(\Gamma)$, namely,

$$L = \{(u,w,\psi) \in L^\infty(\Omega) \times L^2(\Omega) \times L^2(\Gamma),\ \|u\|_{L^\infty} \leqslant 1\}$$

(see also [8], [16] and [25] where similar situations are encountered). This allows in particular to consider initial data containing also the pure states (i.e., u_0 can take the values ± 1). Now, contrary to the case of Dirichlet or Neumann boundary conditions (see [8] and [16]), we have not been able to prove that the system mixes instantaneously, i.e, that the solutions are separated from the singular values of the potential as soon as $t > 0$ (or in finite time, i.e., for $t \geqslant t_0 > 0$). The difficulties here come from the dynamic boundary condition; essentially, we would need to prove that, for a solution starting in L, $\|u(t_0)\|_{L^\infty} < 1$, where $t_0 > 0$ is arbitrarily small.

5. Existence of global attractors

Owing to the results of the previous section, we can define the semigroup

$$S(t)\colon\ \Phi_M \to \Phi_M, \quad S(t)(u_0,w_0,\psi_0)=(u(t),w(t),\psi(t)),$$

where (u,w,ψ) is the unique solution to (1) with initial data (u_0,w_0,ψ_0) and

$$\Phi_M=\Big\{(u,w,\psi)\in H^2(\Omega)\times H^2(\Omega)\times H^2(\Gamma),\ u|_\Gamma=\psi,\ \frac{\partial w}{\partial n}\Big|_\Gamma=0,\ \|u\|_{L^\infty}<1,\\ |I_0|\leqslant M\Big\}.$$

Now, the estimate of Corollary 3.1 does not allow to prove the existence of a bounded absorbing set (i.e., that the system is dissipative), since the constant δ is chosen such that $\|u_0\|_{L^\infty(\bar\Omega)}<\delta<1$, which implies that the constant M_δ depends on $\|u_0\|_{L^\infty}$ and is not bounded as $\|u_0\|_{L^\infty}\to 1$. Thus, in order to have a dissipative estimate on $\|u\|_{L^\infty}$, we need to proceed in a more accurate way. To do so, we set $y_+(t)=\max(\tilde\delta,1-\alpha t)$, where $0<\tilde\delta<1$ and $\alpha>0$ are to be fixed below. We thus have, setting $t_0=(1-\tilde\delta)/\alpha$,

$$y_+(t)=\begin{cases}1-\alpha t & \text{if } \ 0\leqslant t\leqslant t_0,\\ \tilde\delta & \text{if } \ t\geqslant t_0.\end{cases}$$

Furthermore, setting $v=u-y_+$ and $\varphi=\psi-y_+$, we have

$$\partial_t v-\Delta v+f(u)-f(\tilde\delta)=w-f(\tilde\delta),$$
$$\partial_t\varphi-\Delta_\Gamma\varphi+\lambda\varphi+\partial v/\partial n+g(\psi)-g(\tilde\delta)=-\lambda\tilde\delta-g(\tilde\delta)$$

for $t>t_0$, and

$$\partial_t v-\Delta v+f(u)-f(1-\alpha t)=w-f(1-\alpha t)+\alpha,$$
$$\partial_t\varphi-\Delta_\Gamma\varphi+\lambda\varphi+\partial v/\partial n+g(\psi)-g(1-\alpha t)=-\lambda(1-\alpha t)-g(1-\alpha t)+\alpha$$

for $t<t_0$. Proceeding then exactly as in the proof of Theorem 3.2 we now see that, choosing $\tilde\delta$ such that

$$g(\tilde\delta)\geqslant 0, \quad f(\tilde\delta)>\beta\geqslant\|w(t)\|_{L^\infty} \quad \forall\, t\geqslant 0$$

and taking then α small enough such that (owing to (H_1) and $(\mathrm{H}_{2,\mathrm{b}})$ and noting that $\lambda>0$)

$$\lambda(1-\alpha t)+g(1-\alpha t)-\alpha\geqslant 0, \quad \|w(t)\|_{L^\infty}-f(1-\alpha t)+\alpha\leqslant 0$$

$\forall\, t\in[0,t_0]$, we have

$$u(t,x)\leqslant y_+(t), \quad x\in\Omega,\ t\geqslant 0,\ t\neq t_0,$$
$$\psi(t,x)\leqslant y_+(t), \quad x\in\Gamma,\ t\geqslant 0,\ t\neq t_0.$$

Similarly, setting $y_-(t) = \min(-\tilde{\delta}, -1+\alpha t)$, we have (see the proof of Theorem 3.2)

$$u(t,x) \geqslant y_-(t), \quad x \in \Omega, \ t \geqslant 0, \ t \neq t_0,$$
$$\psi(t,x) \geqslant y_-(t), \quad x \in \Gamma, \ t \geqslant 0, \ t \neq t_0.$$

We now deduce from the above estimates that

$$\|u(t)\|_{L^\infty} \leqslant \tilde{\delta} \quad \text{for } t > t_0.$$

Furthermore, it follows from Theorem 3.1 that, if $R_0 = R_0(M)$ is large enough, then for $t \geqslant t_1 = t_1(R_0)$ we have $\|w(t)\|_{H^2} \leqslant R_0$. Therefore, taking $t > t_0 \geqslant t_1$, t_0 large enough, we have, proceeding as above,

$$\|u(t)\|_{L^\infty} \leqslant \tilde{\delta},$$

where $\tilde{\delta} = \tilde{\delta}(R_0)$ is now independent of the initial data.

We finally deduce from the estimates performed in Section 2 that, if $R_1 = R_1(M)$ is large enough, then

$$B^i_{R_1} = \{(u, w, \psi) \in H^i(\Omega) \times H^i(\Omega) \times H^i(\Gamma), \ \|u\|_{H^i} + \|w\|_{H^i} + \|\psi\|_{H^i} \leqslant R_1\} \cap \Phi_M$$

is a bounded absorbing set for $S(t)$ on $H^i(\Omega) \times H^i(\Omega) \times H^i(\Gamma)$, $i = 2, 3$. This yields the following result (note that it is not difficult to prove that $S(t)\colon \Phi_M \to \Phi_M$ is continuous $\forall\, t \geqslant 0$).

Theorem 5.1. *The semigroup $S(t)$ possesses the compact global attractor $\mathcal{A}_M$ on Φ_M which is bounded in $H^3(\Omega) \times H^3(\Omega) \times H^3(\Gamma)$.*

Remark 5.1. It is now not difficult to prove, in view of the strict separation property of u, that $\mathcal{A}_M$ has finite dimension (in the sense of the Hausdorff or the fractal dimension, see, e.g., [31]); to do so, we essentially proceed as in the case of regular potentials (see, e.g., [8] and [14]).

In addition to (H_1), the function f will be assumed to satisfy

$$(\mathrm{H}_3) \qquad f \text{ is real analytic in } (-1,1).$$

The main result of this section is given in the following theorem.

Theorem 6.1. *Let f satisfy assumptions* (H_1), (H_3), $g=0$ *and let* $(u, u|_\Gamma, w)$ *be a solution to* (1) *with initial data* $(u_0, u_0|_\Gamma, w_0)$ *satisfying* (5). *Then* $\lim_{t\to+\infty} u(t) =: \bar{u}$ *and* $\lim_{t\to+\infty} w(t) =: \overline{w}$ *exist in* $H^2(\Omega)\cap H^2(\Gamma)$ *and* $H^2(\Omega)$, *respectively, and the functions* $\bar{u}$, $\overline{w}$ *are solutions to the equilibrium problem*

$$\begin{cases} -\Delta\bar{u} + f(\bar{u}) = \overline{w}, & x\in\Omega, \\ -\Delta_\Gamma\bar{u} + \dfrac{\partial\bar{u}}{\partial n} + \lambda\bar{u} = 0, & x\in\Gamma, \\ \langle\varepsilon\overline{w} + \bar{u}\rangle = I_0 \ (= \langle\varepsilon w_0 + u_0\rangle). \end{cases} \tag{27}$$

We will only outline the proof, which follows the arguments of R. Chill et al. [9] (see also [27]). To this aim, we first make a change of unknowns in problem (1).

R e m a r k 6.1. We slightly change our notation in this section and set, following [9], $H^i(\Omega)\cap H^i(\Gamma) = \{u\in H^i(\Omega),\ u|_\Gamma \in H^i(\Gamma)\}$, endowed with the norm

$$\|u\|^2_{H^i(\Omega)\cap H^i(\Gamma)} = \|u\|^2_{H^i} + \|u\|^2_{H^i(\Gamma)},$$

$i = 0,1,2$, being understood that, for $i=0$, we assume that the trace exists.

6.1. Modified problem and the solving semigroup

We set $v = w - I_0/\varepsilon$ and $\tilde{f}(u) = f(u) - I_0/\varepsilon$. Then we can rewrite problem (1) (for $g=0$) as

$$\begin{cases} \varepsilon\partial_t v - \Delta v = -\partial_t u, & t>0,\ x\in\Omega, \\ \partial_t u - \Delta u + \tilde{f}(u) = v, & t>0,\ x\in\Omega, \\ \partial_t u - \Delta_\Gamma u + \lambda u + \dfrac{\partial u}{\partial n} = 0, & t>0,\ x\in\Gamma, \\ \dfrac{\partial v}{\partial n}\Big|_\Gamma = 0, & \\ v|_{t=0} = w_0 - \dfrac{I_0}{\varepsilon},\ u|_{t=0} = u_0. & \end{cases} \tag{28}$$

Here we consider a modified problem in order to have the homogeneous conservation law

$$\langle \varepsilon v(t) + u(t) \rangle = 0 \quad \forall\, t \geqslant 0$$

(see [9]). It is clear that the function $\tilde{f}$ also satisfies assumptions (H_1) (and, consequently, (2)) and (H_3). Thus all the a priori estimates of the preceding sections still hold for the solutions to problem (28).

We can now define the solving semigroup associated with problem (28), namely,

$$S(t)\colon\ \Phi \to \Phi, \quad S(t)(u_0, v_0) = (u(t), v(t)),$$

where $(u, u|_\Gamma, v)$ is the unique solution to problem (28) with initial data $(u_0, u_0|_\Gamma, v_0)$ and

$$\Phi = \Big\{ (u, v) \in (H^2(\Omega) \cap H^2(\Gamma)) \times H^2(\Omega),\ \frac{\partial v}{\partial n}\Big|_\Gamma = 0,\ \|u\|_{L^\infty} < 1,\ \langle \varepsilon v + u \rangle = 0 \Big\},$$

endowed with the norm

$$\|(u, v)\|_\Phi^2 = \|u\|_{H^2}^2 + \|u\|_{H^2(\Gamma)}^2 + \|v\|_{H^2}^2.$$

Of course, $S(t)(u_0, w_0 - I_0/\varepsilon) = (u(t), v(t))$, where $(u, u|_\Gamma, v + I_0/\varepsilon)$ is the unique solution to problem (4) with initial data $(u_0, u_0|_\Gamma, w_0)$.

We then define a functional $E\colon\ \Phi \to \mathbb{R}$ as

$$E(u(t), v(t)) = \frac{1}{2}\|\nabla u(t)\|^2 + \int_\Omega \tilde{F}(u(t))\,\mathrm{d}x + \frac{1}{2}\|\nabla_\Gamma u(t)\|_\Gamma^2 + \frac{\lambda}{2}\|u(t)\|_\Gamma^2 + \frac{\varepsilon}{2}\|v(t)\|^2,$$

where $\tilde{F}(s) = \int_0^s \tilde{f}(r)\,\mathrm{d}r$. This functional is a Lyapunov function for our problem, since it satisfies

$$\begin{aligned}(29)\qquad \frac{\mathrm{d}}{\mathrm{d}t}E(u(t), v(t)) &= (-\Delta u(t) + \tilde{f}(u(t)), \partial_t u(t)) \\ &\quad + \Big(-\Delta_\Gamma u(t) + \frac{\partial u(t)}{\partial n} + \lambda u(t), \partial_t u(t))_\Gamma + \varepsilon(v(t), \partial_t v(t)\Big) \\ &= -\|\partial_t u(t)\|^2 - \|\partial_t u(t)\|_\Gamma^2 + (v(t), \partial_t u(t) + \varepsilon \partial_t v(t)) \\ &= -\|\partial_t u(t)\|^2 - \|\partial_t u(t)\|_\Gamma^2 - \|\nabla v(t)\|^2 \leqslant 0 \quad \forall\, t \geqslant 0.\end{aligned}$$

Moreover, assume that there exists $\tilde{t} > 0$ such that $E(S(\tilde{t})(u_0, v_0)) = E(u_0, v_0)$. Then it follows that $\partial_t u(t) = 0$, $\partial_t u|_\Gamma(t) = 0$, $\nabla v(t) = 0\ \forall\, t \in (0, \tilde{t})$. Hence $\partial_t v(t) = 0\ \forall\, t \in (0, \tilde{t})$ and $(u_0, u_0|_\Gamma, v_0)$ is a stationary solution. Finally, we infer from (16) that the orbit $\bigcup_{t \geqslant t_1} S(t)(u_0, v_0)$ is relatively compact in Φ. Thus $(\Phi, S(t), E)$ is a gradient system, from which it follows that the ω-limit set (with respect to the topology of Φ) $\omega(u_0, v_0)$ consists of equilibria. Furthermore, the equilibria coincide with the critical points of E. The proofs of these assertions resemble the ones given in [9], owing to the strict separation property on u, and we thus omit the details. We now have to prove that $\omega(u_0, v_0)$ is a singleton.

Remark 6.2. It follows from the relative compactness and the existence of a Lyapunov function that (27) possesses at least one solution, i.e., that there is at least one stationary solution. Alternatively, proceeding as in [32], we can prove that $u \in H^3(\Omega)$, $\langle u \rangle = I_0 - \varepsilon\overline{w}$, $\|u\|_{L^\infty} \leqslant \delta < 1$, where the constants $\overline{w}$ and δ satisfy $I_0 - \varepsilon\overline{w} \leqslant \delta$, is a solution to (27) if and only if it is a critical point of

$$J(u(t)) = \frac{1}{2}\|\nabla u(t)\|^2 + \int_\Omega F(u(t))\,\mathrm{d}x + \frac{1}{2}\|\nabla_\Gamma u(t)\|_\Gamma^2 + \frac{\lambda}{2}\|u(t)\|_\Gamma^2 + \frac{\varepsilon}{2}\|\overline{w}\|^2$$

over

$$K = \{u \in H^1(\Omega) \cap H^1(\Gamma),\ \langle u \rangle = I_0 - \varepsilon\overline{w}\}.$$

Then, noting that $K_\delta = \{u \in K,\ \|u\|_{L^\infty} \leqslant \delta\}$ (where $\overline{w}$ and δ are as above) is weakly closed and that J is bounded from below on K_δ (note that f is actually regular on K_δ), we can prove that J possesses a minimizer u in K_δ, hence the existence of a solution to (27). We refer the reader to [32] for more details.

6.2. A Łojasiewicz-Simon type inequality

We set

$$V = \{(u, v) \in (H^1(\Omega) \cap H^1(\Gamma)) \times L^2(\Omega),\ \langle \varepsilon w + u \rangle = 0\}.$$

We obviously have $\Phi \hookrightarrow V$, hence $V' \hookrightarrow \Phi'$.

The proof of Theorem 6.1 requires a Łojasiewicz-Simon type inequality related to E. Although the potential in (28) is singular, we can use a result proved in [9] for regular potentials. Indeed, we saw in Theorem 3.2 that the solutions are regular and strictly separated from the singularities ± 1. Consequently, the nonlinearity $\tilde{f}(u)$ is bounded in $L^\infty(\Omega)$. Thus the arguments of [9] are still valid in our case (see in particular [9, Proposition 6.6]; see also [27]), and we have

Proposition 6.1. *Let $(\bar{u}, \overline{v}) \in \Phi$ be a critical point of the functional E. Then there exist constants $\overline{\sigma} > 0$, $\overline{C} > 0$ and $\bar{\theta} \in (0, \frac{1}{2}]$ depending on $(\bar{u}, \overline{v})$ such that*

$$|E(u, v) - E(\bar{u}, \overline{v})|^{1-\bar{\theta}} \leqslant \overline{C}\|E'(u, v)\|_{V'},$$

whenever $\|(u, v) - (\bar{u}, \overline{v})\|_\Phi \leqslant \overline{\sigma}$ and $\|u\|_{L^\infty} \leqslant \delta$, $\delta < 1$.

Thus the functional E satisfies the Łojasiewicz-Simon inequality near every $(\varphi,\psi)\in\omega(u_0,v_0)$. Since the ω-limit set $\omega(u_0,v_0)$ is compact in Φ and E is constant ($=E_\infty$) on $\omega(u_0,v_0)$, there exist uniform constants $\theta\in(0,\frac{1}{2}]$, $C\geqslant 0$ and a neighborhood U of $\omega(u_0,v_0)$ in Φ such that, for every $(u,v)\in U$,

$$|E(u,v)-E_\infty|^{1-\theta}\leqslant C\|E'(u,v)\|_{V'}$$

(see [9] for more details). Furthermore, since $\lim\limits_{t\to+\infty}\operatorname{dist}((u(t),v(t)),\omega(u_0,v_0))=0$ in Φ, there exists $T_L\geqslant 0$ such that

$$(u(t),v(t))\in U\quad\forall\, t\geqslant T_L.$$

Let (u,v) belong to Φ. In order to estimate $\|E'(u,v)\|_{V'}$, we note that, for every $(h,k)\in V$,

$$\begin{aligned}
&\langle E'(u,v),(h,k)\rangle_{V',V}\\
&\qquad=(\nabla u,\nabla h)+(\tilde f(u),h)+(\nabla_\Gamma u,\nabla_\Gamma h)_\Gamma+\lambda(u,h)_\Gamma+\varepsilon(v,k)\\
&\qquad=(-\Delta u+\tilde f(u)-v,h)+\Big(-\Delta_\Gamma u+\frac{\partial u}{\partial n}+\lambda u,h\Big)_\Gamma+\int_\Omega(vh+\varepsilon vk)\,\mathrm{d}x\\
&\qquad=(-\Delta u+\tilde f(u)-v,h)+\Big(-\Delta_\Gamma u+\frac{\partial u}{\partial n}+\lambda u,h\Big)_\Gamma+(v-\langle v\rangle,h+\varepsilon k).
\end{aligned}$$

Thus (28) implies

$$\begin{aligned}
\|E'(u,v)\|_{V'}&\leqslant\|\Delta u-\tilde f(u)+v\|+\|\Delta_\Gamma u-\frac{\partial u}{\partial n}-\lambda u\|_\Gamma+C\|\nabla v\|\\
&\leqslant C(\|\partial_t u\|+\|\partial_t u\|_\Gamma+\|\nabla v\|).
\end{aligned}$$

6.3. Proof of Theorem 6.1

By definition of an ω-limit set, there exists $(\bar u,\overline{v})\in\omega(u_0,v_0)$ and a sequence $t_n\to+\infty$ such that

$$u(t_n)\to\bar u\quad\text{in }H^2(\Omega)\cap H^2(\Gamma)\quad\text{and}\quad v(t_n)\to\overline{v}\quad\text{in }H^2(\Omega).$$

If $E(u(\tilde t),v(\tilde t))=E(\bar u,\overline{v})$ ($=E_\infty$) for some $\tilde t\geqslant 0$, then $E(u(t),v(t))=E(\bar u,\overline{v})$ $\forall\, t\geqslant\tilde t$ and it follows from (29) that

$$u(t)=\bar u,\quad v(t)=\overline{v}\quad\forall\, t\geqslant\tilde t.$$

Hence Theorem 6.1 is proved in that case. So, we may assume that $E(u(t),v(t))>E_\infty$ $\forall\, t\geqslant 0$.

We note that

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}&(E(u(t),v(t)) - E_\infty)^\theta \\
&= \theta\,(E(u(t),v(t)) - E_\infty)^{\theta-1}(\|\partial_t u(t)\|^2 + \|\partial_t u(t)\|_\Gamma^2 + \|\nabla v(t)\|^2) \\
&\geqslant \frac{\theta}{4}\,(E(u(t),v(t)) - E_\infty)^{\theta-1}\,(\|\partial_t u(t)\| + \|\partial_t u(t)\|_\Gamma + \|\nabla v(t)\|)^2.
\end{aligned}
$$

Thus the above uniform Łojasiewicz-Simon inequality implies, for $t \geqslant T_L$,

$$
-\frac{\mathrm{d}}{\mathrm{d}t}(E(u(t),v(t)) - E_\infty)^\theta \geqslant \frac{\theta}{4C}(\|\partial_t u(t)\| + \|\partial_t u(t)\|_\Gamma + \|\nabla v(t)\|).
$$

By integrating this inequality over $(T_L, +\infty)$, we infer that

$$
\partial_t u \in L^1(T_L, +\infty, L^2(\Omega) \cap L^2(\Gamma)), \quad \nabla v \in L^1(T_L, +\infty, L^2(\Omega)).
$$

Since

$$
\begin{aligned}
\varepsilon\|\partial_t v(t)\|_{H^{-1}} &\leqslant \|\Delta v(t)\|_{H^{-1}} + \|\partial_t u(t)\|_{H^{-1}} \\
&\leqslant C(\|\nabla v(t)\| + \|\partial_t u(t)\|),
\end{aligned}
$$

we also deduce that $\partial_t v \in L^1(T_L, +\infty, H^{-1}(\Omega))$ and conclude that

$$
\lim_{t\to\infty}(u(t), v(t)) = (\bar{u}, \overline{v})
$$

exists in $(L^2(\Omega) \cap L^2(\Gamma)) \times H^{-1}(\Omega)$ and that $(\bar{u}, \overline{v})$ is a solution to the stationary problem associated with (28). By the relative compactness of the orbit, this limit also exists in the space $(H^2(\Omega) \cap H^2(\Gamma)) \times H^2(\Omega)$. We finally conclude that

$$
\lim_{t\to\infty}(u(t), w(t)) = (\bar{u}, \overline{w})
$$

strongly in $(H^2(\Omega) \cap H^2(\Gamma)) \times H^2(\Omega)$, where $(\bar{u}, \overline{w})$ is a solution to (27).

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