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Abstract. Mikheev, starting from a Moufang loop, constructed a groupoid and reported that this groupoid is in fact a group which, in an appropriate sense, is universal with respect to enveloping the Moufang loop. Later Grishkov and Zavarnitsine gave a complete proof of Mikheev's results. Here we give a direct and self-contained proof that Mikheev's groupoid is a group, in the process extending the result from Moufang loops to Bol loops.

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1. Introduction

A *groupoid* $(Q, \circ)$ is a set Q endowed with a binary product $\circ: Q \times Q \longrightarrow Q$. The groupoid is a *quasigroup* if, for each $x \in Q$, the right translation map $\mathrm{R}_x: Q \longrightarrow Q$ and left translation map $\mathrm{L}_x: Q \longrightarrow Q$ given by

$$a\mathrm{R}_x = a \circ x \quad \text{and} \quad a\mathrm{L}_x = x \circ a$$

are both permutations of Q.

The groupoid $(Q, \circ)$ is a *groupoid with identity* if it has a two-sided identity element:

$$1 \circ x = x = x \circ 1, \qquad \text{for all } \; x \in Q.$$

That is, R_1 and L_1 are Id_Q, the identity permutation of Q. A quasigroup with identity is a *loop*.

The loop $(Q, \circ)$ is a (*right*) *Bol* loop if it identically has the right Bol property:

$$\text{for all } \; a, b, x \in Q, \quad a((xb)x) = ((ax)b)x.$$

(We often abuse notation by writing pq in place of $p \circ q$.) The loop is a *Moufang* loop if it has the Moufang property:

$$\text{for all } \; a, b, x \in Q, \quad a(x(bx)) = ((ax)b)x.$$

Finally the loop is a *group* if it has the associative property:

$$\text{for all } \; a, b, x \in Q, \quad a(xb) = (ax)b.$$

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The Moufang property is clearly a weakened form of the associative property. Furthermore, with $a = 1$ the Moufang property gives $x(bx) = (xb)x$ identically; so the Bol property is a consequence of the Moufang property. Thus every group is a Moufang loop and every Moufang loop is a Bol loop. The reverse implications do not hold in general; see [6, Examples IV.1.1 and IV.6.2].

A (*right*) *pseudo-automorphism* of the groupoid with identity $(Q, \circ)$ is a permutation A of Q equipped with an element $a \in Q$, a *companion* of A, for which R_a is a permutation (always true for $(Q, \circ)$ a loop) and such that

$$xA \circ (yA \circ a) = (xy)A \circ a$$

for all $x, y \in Q$. We shall abuse this terminology by referring to the pair (A, a) as a pseudo-automorphism. The set of all pseudo-automorphisms (A, a) is then denoted $\mathrm{PsAut}(Q, \circ)$ and admits the group operation

$$(A, a)(D, d) = (AD, aD \circ d),$$

as we shall verify in Proposition 2.1 below.

In the research report [5] Mikheev, starting from a Moufang loop $(Q, \circ)$, constructed a groupoid on the set $\mathrm{PsAut}(Q, \circ) \times Q$. The main results reported by Mikheev are that this groupoid is in fact a group and that, in an appropriate sense, it is universal with respect to "enveloping" the Moufang loop $(Q, \circ)$.

In [3] Grishkov and Zavarnitsine gave a complete proof of Mikheev's results (and a great deal more). Concerning Mikheev's construction they proved:

Theorem 1.1. *Let $(Q, \circ)$ be a Moufang loop.*

(a) *The groupoid $(\mathrm{PsAut}(Q, \circ) \times Q, \star)$ given by*

$$\text{(Mk)}\qquad \begin{gathered} \{(A, a), x\} \star \{(B, b), y\} = \{(A, a)(B, b)(C, c), (xB)y\} \quad \textit{with} \\ (C, c) = \Big(\mathrm{R}^{-1}_{xB,b}, (((xB)b)^{-1}b)xB\Big)\Big(\mathrm{R}_{xB,y}, ((xB)^{-1}y^{-1})((xB)y)\Big) \end{gathered}$$

is a group $\mathcal{W}(Q, \circ)$.

(b) *The group $\mathcal{W}(Q, \circ)$ admits a group of triality automorphisms and is universal (in an appropriate sense) among all the groups admitting triality that envelope the Moufang loop $(Q, \circ)$.*

Here for each p, q in an arbitrary loop $(Q, \circ)$ we have set $\mathrm{R}_{p,q} = \mathrm{R}_p\mathrm{R}_q\mathrm{R}^{-1}_{pq}$.

The expression (Mk) can be simplified somewhat. By Moufang's Theorem ([1, p. 117] and [6, Cor. IV.2.9]) Moufang loops generated by two elements are groups, so within $(Q, \circ)$ commutators

$$[p, q] = p^{-1}q^{-1}pq = (qp)^{-1}pq$$

are well-defined, as seen in Mikheev's original formulation [5]. Also in Moufang loops we have $\mathrm{R}_{p,q}^{-1} = \mathrm{R}_{q,p}$ by [1, Lemma VII.5.4]. Therefore Grishkov and Zavarnitsine could give (Mk) in the pleasing form

$$(C, c) = (\mathrm{R}_{b,xB}, [b, xB])\,(\mathrm{R}_{xB,y}, [xB, y]).$$

Grishkov and Zavarnitsine [3, Corollary 1] verify Mikheev's construction by first constructing from $(Q, \circ)$ a particular group admitting triality and then showing that Mikheev's groupoid is a quotient of that group and especially is itself a group. Their construction displays universal properties for the two groups admitting triality and so also for Mikheev's enveloping group. (Grishkov and Zavarnitsine also correct several small misprints from [5].)

In this short note we take a different approach. In particular we give a direct and self-contained proof that Mikheev's groupoid is a group. In the process we extend the result from Moufang loops to Bol loops, and we see that the groupoid has a natural life as a group.

An *autotopism* (A, B, C) of the groupoid $(Q, \circ)$ is a triple of permutations of Q such that

$$xA \circ yB = (x \circ y)C$$

for all $x, y \in Q$. Clearly the set $\mathrm{Atop}(Q, \circ)$ of all autotopisms of $(Q, \circ)$ forms a group under composition.

We then have

Theorem 1.2. *Let* $(Q, \circ)$ *be a Bol loop. The groupoid* $(\mathrm{PsAut}(Q, \circ) \times Q, \star)$ *with product given by* (Mk) *is isomorphic to the autotopism group* $\mathrm{Atop}(Q, \circ)$*. In particular* $(\mathrm{PsAut}(Q, \circ) \times Q, \star)$ *is a group.*

Theorem 1.2 gives Theorem 1.1(a) immediately, and 1.1(b) directly follows. Indeed following Doro [2], the group G *admits triality* if G admits the symmetric group of degree three, $\mathrm{Sym}(3) = S$, as a group of automorphism such that, for σ of order 2 and τ of order 3 in S, the identity $[g, \sigma][g, \sigma]^{\tau}[g, \sigma]^{\tau^2} = 1$ holds for all $g \in G$. Doro proved that the set $\{\, [g, \sigma] \mid g \in G \,\}$ naturally carries the structure of a Moufang loop $(Q, \circ)$; we say that G *envelopes* $(Q, \circ)$. Many nonisomorphic groups admitting triality envelope Moufang loops isomorphic to $(Q, \circ)$. Among these the autotopism group $A = \mathrm{Atop}\,(Q, \circ)$ is the largest that is additionally *faithful*, which is to say that the centralizer of S^A within $A \rtimes S$ is the identity. That is, for every group G admitting triality that is faithful and envelopes $(Q, \circ)$ there is an S-injection of G into A. This is the universal property examined by Mikheev, Grishkov, and Zavarnitsine. See [4, §10.3] for further details.

The general references for this note are the excellent books [1] and [6]. Several of the results given here are related to ones from [6] — both as exact versions ("see") and as variants or extensions ("compare").

2. Autotopisms of groupoids

Proposition 2.1. *Let $(Q,\circ)$ be a groupoid with identity 1. The map*

$$\psi\colon (A,a)\mapsto (A,A\mathrm{R}_a,A\mathrm{R}_a)$$

gives a bijection of $\mathrm{PsAut}(Q,\circ)$ *with the subgroup of* $\mathrm{Atop}(Q,\circ)$ *consisting of all autotopisms (A,B,C) for which $1A=1$. For such an autotopism we have*

$$\psi^{-1}(A,B,C)=(A,1C).$$

In particular $\mathrm{PsAut}(Q,\circ)$ *is a group under the composition*

$$(A,a)(D,d)=(AD,aD\circ d).$$

Proof: (Compare [6, III.4.14].) If (A,a) is a pseudo-automorphism then $(A,A\mathrm{R}_a,A\mathrm{R}_a)$ is an autotopism by definition. In particular for every $x\in Q$ we have $1A\circ xA\mathrm{R}_a=(1\circ x)A\mathrm{R}_a=xA\mathrm{R}_a$. As $A\mathrm{R}_a$ is a permutation of Q, there is an x with $1=xA\mathrm{R}_a$. Thus $1A=1A\circ 1=1$.

If (A,a) and (B,b) are pseudo-automorphisms with $(A,A\mathrm{R}_a,A\mathrm{R}_a)$ equal to $(B,B\mathrm{R}_b,B\mathrm{R}_b)$, then $A=B$ and $a=1A\mathrm{R}_a=1B\mathrm{R}_b=b$; so ψ is an injection of $\mathrm{PsAut}(Q,\circ)$ into the described subgroup of $\mathrm{Atop}(Q,\circ)$.

Now suppose that (A,B,C) is an autotopism with $1A=1$. Always $1\circ x=x$, so

$$xB=1A\circ xB=(1\circ x)C=xC,$$

giving $B=C$.

Again $x\circ 1=x$ and

$$xA\circ 1C=(x\circ 1)C=xC.$$

That is $B=C=A\mathrm{R}_{1C}$, and in particular R_{1C} is a permutation. Therefore $(A,B,C)=(A,A\mathrm{R}_a,A\mathrm{R}_a)$, the image of the pseudo-automorphism (A,a) for $a=1C$. The map ψ is indeed a bijection.

Those autotopisms with $1A$=1 clearly form a subgroup, so ψ^{-1} gives $\mathrm{PsAut}(Q,\circ)$ a natural group structure. We find

$$\begin{aligned}\psi(A,a)\psi(D,d)&=(A,A\mathrm{R}_a,A\mathrm{R}_a)(D,D\mathrm{R}_d,D\mathrm{R}_d)\\&=(AD,A\mathrm{R}_aD\mathrm{R}_d,A\mathrm{R}_aD\mathrm{R}_d)\\&=\psi(AD,e)\end{aligned}$$

for some e with $A\mathrm{R}_aD\mathrm{R}_d=AD\mathrm{R}_e$. Indeed $e=1AD\mathrm{R}_e=1A\mathrm{R}_aD\mathrm{R}_d=aD\circ d$. Therefore multiplication in $\mathrm{PsAut}(Q,\circ)$ is given by

$$(A,a)(D,d)=(AD,aD\circ d),$$

as stated here and above. $\square$

From now on we identify $\mathrm{PsAut}(Q,\circ)$ with its isomorphic image under ψ in $\mathrm{Atop}(Q,\circ)$.

Corollary 2.2. (a) *Let (A,B,C) and (D,E,F) be autotopisms of the groupoid with identity $(Q,\circ)$. Then we have $(A,B,C)=(D,E,F)$ if and only if $A=D$ and $1C=1F$.*

(b) *Let (A,B,C) and (D,E,F) be autotopisms of the loop $(Q,\circ)$. Then we have $(A,B,C)=(D,E,F)$ if and only if $A=D$ and there is an $x\in Q$ with $xC=xF$.*

PROOF: (Compare [6, III.3.1].) One direction is clear.

Now suppose that $A=D$.

$$\begin{aligned}(X,Y,Z)&=(A,B,C)(D,E,F)^{-1}\\&=(AD^{-1},BE^{-1},CF^{-1})\\&=(\mathrm{Id}_Q,BE^{-1},CF^{-1})\\&=(\mathrm{Id}_Q,\mathrm{Id}_Q\,\mathrm{R}_e,\mathrm{Id}_Q\,\mathrm{R}_e)\\&=(\mathrm{Id}_Q,\mathrm{R}_e,\mathrm{R}_e)\end{aligned}$$

for $e=1CF^{-1}$ by the proposition.

For any x with $xC=xF$ we then have

$$x\circ 1=x=xCF^{-1}=xZ=x\circ e,$$

so in both parts of the corollary we find $e=1$. Therefore (X,Y,Z) is equal to $(\mathrm{Id}_Q,\mathrm{Id}_Q,\mathrm{Id}_Q)$, the identity of $\mathrm{Atop}(Q,\circ)$. □

A particular consequence of the corollary is that we may (if we wish) denote the autotopism (A,B,C) by $(A,*,C)$, since A and C determine B uniquely.

3. Autotopisms of Bol loops

Recall that a Bol loop $(Q,\circ)$ is a loop with

$$\text{for all }\ a,b,x\in Q,\quad a((xb)x)=((ax)b)x.$$

Lemma 3.1. (a) *The loop $(Q,\circ)$ is a Bol loop if and only if $(\mathrm{R}_x^{-1},\mathrm{L}_x\mathrm{R}_x,\mathrm{R}_x)$ is an autotopism for all $x\in Q$.*

(b) *The Bol loop $(Q,\circ)$ is a right inverse property loop. That is, for x^{-1} defined by $xx^{-1}=1$ we have $(x^{-1})^{-1}=x$ and $(ax)x^{-1}=a$, for all $a,x\in Q$. In particular $\mathrm{R}_x^{-1}=\mathrm{R}_{x^{-1}}$ for all x.*

PROOF: (a) (See [6, Theorem IV.6.7].) For a fixed $x\in Q$ we have $a((xb)x)=((ax)b)x$ for all $a,b\in Q$ if and only if $(c\mathrm{R}_x^{-1})(b\mathrm{L}_x\mathrm{R}_x)=(cb)\mathrm{R}_x$ for all c $(=ax)$, $b\in Q$ if and only if $(\mathrm{R}_x^{-1},\mathrm{L}_x\mathrm{R}_x,\mathrm{R}_x)$ is an autotopism.

(b) (See [6, Theorem IV.6.3].) In the identity $a((xb)x)=((ax)b)x$ set $b=x^{-1}$ to find $ax=a((xx^{-1})x)=((ax)x^{-1})x$. That is, $a\mathrm{R}_x=(ax)x^{-1}\mathrm{R}_x$ and so

$a = (ax)x^{-1}$. Further set $a = x^{-1}$ in this identity, giving

$$1\mathrm{R}_{x^{-1}} = x^{-1} = (x^{-1}x)x^{-1} = (x^{-1}x)\mathrm{R}_{x^{-1}},$$

whence $1 = x^{-1}x$ and $(x^{-1})^{-1} = x$. $\square$

Throughout the balance of this section let $(Q, \circ)$ be a Bol loop.
For all p in the Bol loop $(Q, \circ)$ set

$$\mathrm{r}_p = (\mathrm{R}_{p^{-1}}^{-1}, \mathrm{L}_{p^{-1}}\mathrm{R}_{p^{-1}}, \mathrm{R}_{p^{-1}}) = (\mathrm{R}_p, \mathrm{L}_{p^{-1}}\mathrm{R}_{p^{-1}}, \mathrm{R}_{p^{-1}}),$$

and set $\mathrm{r}_{p,q} = \mathrm{r}_p\mathrm{r}_q\mathrm{r}_{pq}^{-1}$ for all p, q. By the lemma, each r_p and $\mathrm{r}_{p,q}$ is an autotopism of $(Q, \circ)$.

Lemma 3.2. (a) $\mathrm{r}_p^{-1} = \mathrm{r}_{p^{-1}}$.
(b) $\mathrm{r}_{p,q} = \big(\mathrm{R}_{p,q}, (p^{-1}q^{-1})(pq)\big)$.
(c) $\mathrm{r}_{p,q}^{-1} = \big(\mathrm{R}_{p,q}^{-1}, ((pq)^{-1}q)p\big)$.

Proof: By Lemma 3.1

$$\mathrm{r}_p^{-1} = (\mathrm{R}_p^{-1}, *, \mathrm{R}_{p^{-1}}^{-1}) = (\mathrm{R}_p^{-1}, *, \mathrm{R}_p) = (\mathrm{R}_{p^{-1}}, *, \mathrm{R}_p) = \mathrm{r}_{p^{-1}},$$

as in (a). Therefore

$$\mathrm{r}_{p,q} = \mathrm{r}_p\mathrm{r}_q\mathrm{r}_{pq}^{-1} = (\mathrm{R}_p\mathrm{R}_q\mathrm{R}_{pq}^{-1}, *, \mathrm{R}_{p^{-1}}\mathrm{R}_{q^{-1}}\mathrm{R}_{pq}) = (\mathrm{R}_{p,q}, *, \mathrm{R}_{p^{-1}}\mathrm{R}_{q^{-1}}\mathrm{R}_{pq})$$

and

$$\mathrm{r}_{p,q}^{-1} = (\mathrm{R}_{p,q}, *, \mathrm{R}_{p^{-1}}\mathrm{R}_{q^{-1}}\mathrm{R}_{pq})^{-1} = (\mathrm{R}_{p,q}^{-1}, *, \mathrm{R}_{(pq)^{-1}}\mathrm{R}_q\mathrm{R}_p).$$

Here

$$\begin{aligned}
1\,\mathrm{R}_p\mathrm{R}_q\mathrm{R}_{pq}^{-1} &= (pq)(pq)^{-1} = 1\,;\\
1\,\mathrm{R}_{p^{-1}}\mathrm{R}_{q^{-1}}\mathrm{R}_{pq} &= (p^{-1}q^{-1})(pq)\,;\\
1\,\mathrm{R}_{(pq)^{-1}}\mathrm{R}_q\mathrm{R}_p &= ((pq)^{-1}q)p.
\end{aligned}$$

The first calculation tells us that $\mathrm{r}_{p,q}$ (and $\mathrm{r}_{p,q}^{-1}$) are in $\mathrm{PsAut}(Q, \circ)$. The second, together with Proposition 2.1, then gives (b) and the third (c). $\square$

Proposition 3.3. *Let (X, Y, Z) be in $\mathrm{Atop}(Q, \circ)$. Set $x = 1X$, $A = X\mathrm{R}_x^{-1}$, and $a = 1Z \circ x$. Then*

$$(X, Y, Z) = (A, a)\,\mathrm{r}_x.$$

In particular $\{\, \mathrm{r}_x \mid x \in Q \,\}$ is a set of right coset representatives for the subgroup $\mathrm{PsAut}(Q, \circ)$ in $\mathrm{Atop}(Q, \circ)$.

Proof: (Compare [6, III.4.16, IV.6.8].)

$$\begin{aligned}(X,Y,Z) &= (X,Y,Z)\mathrm{r}_x^{-1}\mathrm{r}_x\\ &= (X\mathrm{R}_x^{-1}, Y\mathrm{R}_{x^{-1}}^{-1}\mathrm{L}_{x^{-1}}^{-1}, Z\mathrm{R}_{x^{-1}}^{-1})\mathrm{r}_x\\ &= (X\mathrm{R}_{x^{-1}},\ *\ , Z\mathrm{R}_x)\mathrm{r}_x.\end{aligned}$$

For $A = X\mathrm{R}_x^{-1} = X\mathrm{R}_{x^{-1}}$ we have $1A = 1X\mathrm{R}_{x^{-1}} = x \circ x^{-1} = 1$. Furthermore $1Z\mathrm{R}_x = 1Z \circ x = a$, so by Proposition 2.1 we have $(X,Y,Z) = (A,a)\mathrm{r}_x$. ☐

Proposition 3.4. $\mathrm{r}_x(B,b) = (B,b)\mathrm{r}_{(xB)b}\mathrm{r}_b^{-1}$

Proof: We have

$$\begin{aligned}\mathrm{r}_x(B,b) &= (\mathrm{R}_x, \mathrm{L}_{x^{-1}}\mathrm{R}_{x^{-1}}, \mathrm{R}_{x^{-1}})(B, B\mathrm{R}_b, B\mathrm{R}_b)\\ &= (\mathrm{R}_x B,\ *\ , \mathrm{R}_{x^{-1}}B\mathrm{R}_b)\end{aligned}$$

and

$$\begin{aligned}(B,b)\mathrm{r}_{(xB)b}\mathrm{r}_b^{-1} &= (B, B\mathrm{R}_b, B\mathrm{R}_b)(\mathrm{R}_{(xB)b},\ *\ , \mathrm{R}_{((xB)b)^{-1}})(\mathrm{R}_b^{-1},\ *\ , \mathrm{R}_b)\\ &= (B\mathrm{R}_{(xB)b}\mathrm{R}_b^{-1},\ *\ , B\mathrm{R}_b\mathrm{R}_{((xB)b)^{-1}}\mathrm{R}_b).\end{aligned}$$

First we observe that $x\,\mathrm{R}_{x^{-1}}B\mathrm{R}_b = 1B\mathrm{R}_b = b$ and

$$x\,B\mathrm{R}_b\mathrm{R}_{((xB)b)^{-1}}\mathrm{R}_b = ((xB)b)\mathrm{R}_{((xB)b)^{-1}}\mathrm{R}_b = 1\mathrm{R}_b = b.$$

Therefore by Corollary 2.2 we need only verify $\mathrm{R}_x B = B\mathrm{R}_{(xB)b}\mathrm{R}_b^{-1}$ to prove the proposition.

As (B,b) is a pseudo-automorphism

$$\begin{aligned}p\mathrm{R}_x B\mathrm{R}_b &= (px)B\mathrm{R}_b\\ &= (px)B \circ b\\ &= pB \circ (xB \circ b)\\ &= pB\mathrm{R}_{(xB)b}\end{aligned}$$

for every $p \in Q$. Therefore $\mathrm{R}_x B\mathrm{R}_b = B\mathrm{R}_{(xB)b}$ and $\mathrm{R}_x B = B\mathrm{R}_{(xB)b}\mathrm{R}_b^{-1}$ as desired. ☐

Corollary 3.5. $(A,a)\mathrm{r}_x(B,b)\mathrm{r}_y = (A,a)(B,b)\mathrm{r}_{xB,b}^{-1}\mathrm{r}_{xB,y}\mathrm{r}_{(xB)y}$.

Proof:

$$
\begin{aligned}
(A,a)\mathrm{r}_x(B,b)\mathrm{r}_y &= (A,a)(\mathrm{r}_x(B,b))\mathrm{r}_y \\
&= (A,a)((B,b)\mathrm{r}_{(xB)b}\mathrm{r}_b^{-1})\mathrm{r}_y \\
&= (A,a)(B,b)\mathrm{r}_{(xB)b}\mathrm{r}_b^{-1}(\mathrm{r}_{xB}^{-1}\mathrm{r}_{xB})\mathrm{r}_y(\mathrm{r}_{(xB)y}^{-1}\mathrm{r}_{(xB)y}) \\
&= (A,a)(B,b)(\mathrm{r}_{(xB)b}\mathrm{r}_b^{-1}\mathrm{r}_{xB}^{-1})(\mathrm{r}_{xB}\mathrm{r}_y\mathrm{r}_{(xB)y}^{-1})\mathrm{r}_{(xB)y} \\
&= (A,a)(B,b)\mathrm{r}_{xB,b}^{-1}\mathrm{r}_{xB,y}\mathrm{r}_{(xB)y}.
\end{aligned}
$$

□

Theorem 3.6. *For the Bol loop* $(Q,\circ)$ *the map*

$$\varphi\colon\ \{(A,a),x\} \mapsto (A,a)\mathrm{r}_x$$

gives an isomorphism of Mikheev's groupoid $(\mathrm{PsAut}(Q,\circ)\times Q,\star)$ *and the autotopism group* $\mathrm{Atop}(Q,\circ)$*. In particular* $(\mathrm{PsAut}(Q,\circ)\times Q,\star)$ *is a group.*

Proof: By Proposition 3.3 the map φ is a bijection of $\mathrm{PsAut}(Q,\circ)\times Q$ and $\mathrm{Atop}(Q,\circ)$. By Lemma 3.2 and Corollary 3.5

$$\varphi\left(\{(A,a),x\}\star\{(B,b),y\}\right) = \varphi\left(\{(A,a),x\}\right)\varphi\left(\{(B,b),y\}\right).$$

Thus $(\mathrm{PsAut}(Q,\circ)\times Q,\star)$ and $\mathrm{Atop}(Q,\circ)$ are isomorphic as groupoids. Furthermore since $\mathrm{Atop}(Q,\circ)$ is itself a group, so is $(\mathrm{PsAut}(Q,\circ)\times Q,\star)$. □

Theorem 1.2 is an immediate consequence.

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