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A SOLUTION OF NONLINEAR DIFFUSION PROBLEMS BY SEMILINEAR REACTION–DIFFUSION SYSTEMS

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This paper deals with nonlinear diffusion problems involving degenerate parabolic problems, such as the Stefan problem and the porous medium equation, and cross-diffusion systems in population ecology. The degeneracy of the diffusion and the effect of cross-diffusion, that is, nonlinearities of the diffusion, complicate its analysis. In order to avoid the nonlinearities, we propose a reaction-diffusion system with solutions that approximate those of the nonlinear diffusion problems. The reaction-diffusion system includes only a simple reaction and linear diffusion. Resolving semilinear problems is typically easier than dealing with nonlinear diffusion problems. Therefore, our ideas are expected to reveal new and more effective approaches to the study of nonlinear problems.

Keywords: reaction-diffusion system approximation, degenerate parabolic problem, cross-diffusion system

AMS Subject Classification: 35K55, 35K57, 35K65, 76S05, 80A22

1. INTRODUCTION

We consider the following nonlinear diffusion problem: Find $\boldsymbol{z} = (z_1, \ldots, z_n) : \overline{\Omega} \times [0,T) \to \mathbb{R}^n$ $(n \in \mathbb{N})$ such that

$$\begin{cases} \dfrac{\partial z_i}{\partial t} = \Delta[a_i z_i + \phi_i(\boldsymbol{z})] + f_i(\boldsymbol{z}, \boldsymbol{\phi}(\boldsymbol{z})) & \text{in} \quad Q := \Omega \times (0,T),\ i = 1, \ldots, n, \\ a_i z_i + \phi_i(\boldsymbol{z}) = 0 & \text{on} \quad \partial\Omega \times (0,T),\ i = 1, \ldots, n, \\ \boldsymbol{z}(\cdot, 0) = \boldsymbol{z}_0 & \text{in} \quad \Omega. \end{cases} \tag{1}$$

Here, $\Omega \subset \mathbb{R}^N$ $(N \in \mathbb{N})$ is a bounded domain with smooth boundary $\partial\Omega$, T is a positive constant, a_i $(i = 1, \ldots, n)$ are non-negative constants, $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)$: $\mathbb{R}^n \to \mathbb{R}^n$, $\boldsymbol{f} = (f_1, \ldots, f_n)$: $\mathbb{R}^{2n} \to \mathbb{R}^n$ and $\boldsymbol{z}_0 = (z_{01}, \ldots, z_{0n})$: $\Omega \to \mathbb{R}^n$ are given functions.

Problem (1) includes numerous important problems. One of them is the two-phase Stefan problem, which is a typical model of the solid-liquid phase transition problem. In that problem, if $n = 1$, $a_1 = 0$ and

$$\phi(r) = d_1 \max(r - \lambda, 0) + d_2 \min(r, 0), \quad r \in \mathbb{R}, \tag{2}$$

where d_1, d_2 and λ are non-negative constants, then the functions z and $\phi(z)$ respectively represent physically the enthalpy, which is the sum of sensible and latent heats, and the temperature. The function f denotes an internal heating term. The equation can formally be rewritten as $\frac{\partial z}{\partial t} = \operatorname{div}(\phi'(z)\nabla z) + f(z, \phi(z))$. The diffusion $\phi'(z)$ vanishes in this problem, where z belongs to the interval $(0, \lambda)$. In this case, the problem has the degenerate character. Vanishing diffusion characterizes the presence of a free boundary. The solutions of this equation usually have jump discontinuities across the free boundary. Therefore, this problem is different from parabolic ones, and vanishing diffusion makes the problem difficult. Since the solution z can be discontinuous, the nonlinear function f should be independent of z directly, but it depends on $\phi(z)$.

The isentropic flow through a porous medium is also described by (1), the so-called porous medium equation, with $n = 1$, $a_1 = 0$ and

$$\phi(r) = |r|^{m-1} r, \quad r \in \mathbb{R},$$

where $m > 1$ is a constant. In this problem, the function z represents the fluid density. The equation is uniformly parabolic in any region where z is bounded away from zero; however, it is degenerate at points where $z = 0$. The most striking feature of this nonlinear degeneracy is that if the initial concentration has compact support, then the support of the solution is also compact even if $f = 0$. This is in stark contrast to the linear heat equation ($m = 1$) where there is an infinite speed of propagation. The nonlinear fluid-transfer process with an absorption $f(z) = -cz^l$ ($c, l > 0$) is more complicated. The interaction between nonlinear diffusion and absorption causes remarkable properties such as total extinction in finite time, support splitting and support re-splitting phenomena [9, 20]. Thus, the influence of the degenarate diffusion makes the dynamics complicated.

Other examples of the problem (1) are cross-diffusion systems. Shigesada et al. [19] have proposed the following model to understand spatial and temporal behaviours of interacting species under the influence of the population pressure due to intra- and inter-specific interferences:

$$\begin{cases} \dfrac{\partial z_1}{\partial t} & = & \Delta\left[(a_1 + b_1 z_1 + c_1 z_2) z_1\right] + f_1(z_1, z_2), \\ \dfrac{\partial z_2}{\partial t} & = & \Delta\left[(a_2 + b_2 z_2 + c_2 z_1) z_2\right] + f_2(z_1, z_2), \end{cases} \tag{3}$$

where a_i, b_i, c_i ($i = 1, 2$) are non-negative constants and f_i ($i = 1, 2$) are given functions. In this problem, z_1 and z_2 represent the population densities of two competing species. The cross-diffusion parts imply that each species moves to avoid the congestion of the other species. The spatially segregating coexistence of two competing species occurs by the cross-diffusion effect, which is called cross-diffusion induced instability. Thus, the cross-diffusion effect makes the problem difficult and interesting.

There are a number of other exciting cross-diffusion models in population ecology (see, e. g., [8, 18]).

All of the above examples illustrate that the degeneracy of the diffusion and the effect of cross-diffusion, that is, the nonlinearities of the diffusion, often complicate its analysis. It might be useful for analysis of the problem to remove the nonlinearities of the diffusion. We would like to avoid the nonlinearities. A number of attempts have been made to remove the degeneracy of the diffusion. For example, continuous solutions can be treated for the Stefan problem by regularizing the enthalpy-temperature constitutive relation ϕ. Some regularized Stefan problems are also represented by (1).

In an attempt to avoid the nonlinearities of the diffusion, we consider semilinear reaction-diffusion systems with solutions that approximate those of problem (1). Numerous precedent studies have addressed such problems. But, each had different underlying motivations from ours. Knabner [10] has proposed a reaction-diffusion system with a reaction rate parameter as a macroscopic model of reactive solute transport in porous media. The singular limit as the reaction rate tends to infinity of the system has been studied and he has proved the limit equation can be described by the porous medium equation. Dancer et al. [3] have considered reaction-diffusion systems of Lotka–Volterra–Gause type to understand spatial patterns arising in population ecology. They have studied the case that the interspecific competition rates are very large. The limiting system can be described by the Stefan problem without latent heat, i. e., $\lambda = 0$ in (2). As a result, they have proved that the Stefan problem without latent heat can be approximated by a reaction-diffusion system. Motivated by their results, Hilhorst et al. [4] have tried to construct reaction-diffusion systems of which solutions approximate those of the Stefan problem with positive latent heat. Then, they have proposed a three components reaction-diffusion system and have shown that the limit equation takes the form of the Stefan problem [4, 5]. To avoid the lack of regularity, we have considered the combined use of the reaction-diffusion system approximation proposed by Hilhorst et al. with a regularization procedure [11, 13]. The convergence to the weak solutions of the Stefan problem and to that of a regularized Stefan problem have been investigated. Hilhorst et al. have proved only the convergence because they have used the compactness argument. We have considered the convergence rates for both approximations by Hilhorst et al. [4] and ours. Better rates are obtained for our reaction-diffusion system than for the one by Hilhorst et al. In addition, we have shown that the convergence rates provide information about convergence of approximate free boundaries by following Nochetto [17]. Iida et al. [7] have dealt with the cross-diffusion system (3) with $a_1, a_2, c_1 > 0$ and $b_1 = b_2 = c_2 = 0$. For a deeper understanding of the cross-diffusion mechanism, they have considered another way to avoid the congestion of the other species instead of cross-diffusion. As a result, they have proposed a reaction-diffusion system of which uniformly bounded solutions approximate a solution of the cross-diffusion system. They have also studied the relationship between the cross-diffusion induced instability and Turing's instability of the corresponding reaction-diffusion system. Hilhorst et al. have given a survey of these works recently [6].

Thus, all of the above references studied relations between nonlinear diffusion and reaction-diffusion, and achieved successful results in analysis of the nonlinear diffusion problems by means of reaction-diffusion systems. However, they considered individual problems. Noticing these problems can be represented in a unified form (1), we seek to address the general problem (1). We propose a reaction-diffusion system of which solution approximates that of (1) in the next section, and present our convergence results in Section 3.

2. A REACTION–DIFFUSION SYSTEM

We illustrate our ideas of construction of a reaction-diffusion system. The presence of nonlinearity of the diffusion renders it difficult to analyze. To remove the nonlinearity, let us define $u_i := \phi_i(\boldsymbol{z})$ and $v_i := z_i - \mu\phi_i(\boldsymbol{z})$, where μ is a positive constant. Thereby, we have

$$\mu\left(\frac{\partial u_i}{\partial t} - (a_i + 1/\mu)\Delta u_i\right) + f_i(\mu\boldsymbol{u} + \boldsymbol{v}, \boldsymbol{u}) = -\left(\frac{\partial v_i}{\partial t} - a_i\Delta v_i\right). \tag{4}$$

One can observe that z_i is divided into two types of state having high and low diffusivities. We suppose that it converts one type into the other as a very quick response to the environment, that is, we assume (4) equals to $F_i(\boldsymbol{u}, \boldsymbol{v})/\varepsilon$ with a sufficiently small parameter ε. Although many choices might exist for the conversion term F_i, we suggest the following one, which depends on the nonlinearity of the diffusion:

$$F_i(\boldsymbol{u}, \boldsymbol{v}) = \mu(\phi_i(\mu\boldsymbol{u} + \boldsymbol{v}) - u_i).$$

As explained below, this relation is chosen so that $\boldsymbol{u}$ and $\boldsymbol{\phi}(\mu\boldsymbol{u} + \boldsymbol{v})$ approach each other, where $(\boldsymbol{u}, \boldsymbol{v})$ is a solution of the ensuing reaction-diffusion system. Thus, the following $2n$ components reaction-diffusion system is proposed:

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = \left(a_i + \dfrac{1}{\mu}\right)\Delta u_i + \dfrac{1}{\mu} f_i(\mu\boldsymbol{u} + \boldsymbol{v}, \boldsymbol{u}) - \dfrac{1}{\varepsilon}(u_i - \phi_i(\mu\boldsymbol{u} + \boldsymbol{v})), & \\ \dfrac{\partial v_i}{\partial t} = a_i\Delta v_i + \dfrac{\mu}{\varepsilon}(u_i - \phi_i(\mu\boldsymbol{u} + \boldsymbol{v})) & \text{in } Q,\ i = 1, \dots, n, \\ \boldsymbol{u} = \boldsymbol{0},\ \boldsymbol{v} = \boldsymbol{0} & \text{on } \partial\Omega \times (0, T), \\ \boldsymbol{u}(\cdot, 0) = \boldsymbol{\phi}(\boldsymbol{z}_0), \quad \boldsymbol{v}(\cdot, 0) = \boldsymbol{z}_0 - \mu\boldsymbol{\phi}(\boldsymbol{z}_0) & \text{in } \Omega. \end{cases} \tag{5}$$

If $a_i = 0$ for some $i \in \{1, \dots, n\}$, i. e. degenerate case, then we regard the equations for u_i and v_i as the following equations consisting of a reaction-diffusion equation and an ordinary differential equation:

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = \dfrac{1}{\mu}\Delta u_i + \dfrac{1}{\mu} f_i(\mu\boldsymbol{u} + \boldsymbol{v}, \boldsymbol{u}) - \dfrac{1}{\varepsilon}(u_i - \phi_i(\mu\boldsymbol{u} + \boldsymbol{v})), & \\ \dfrac{\partial v_i}{\partial t} = \dfrac{\mu}{\varepsilon}(u_i - \phi_i(\mu\boldsymbol{u} + \boldsymbol{v})) & \text{in } Q, \\ u_i = 0 & \text{on } \partial\Omega \times (0, T), \\ u_i(\cdot, 0) = \phi_i(\boldsymbol{z}_0), \quad v_i(\cdot, 0) = z_{0i} - \mu\phi_i(\boldsymbol{z}_0) & \text{in } \Omega. \end{cases} \tag{6}$$

Let $(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon)$ be a solution of (5). We expect that $\boldsymbol{z}^\varepsilon := \mu \boldsymbol{u}^\varepsilon + \boldsymbol{v}^\varepsilon$ and $\boldsymbol{u}^\varepsilon$ approximate the solution $\boldsymbol{z}$ of the original problem (1) and $\boldsymbol{\phi}(\boldsymbol{z})$, respectively. Now, we examine the reaction part of (5). If the value of u_i^ε is greater than that of $\phi_i(\boldsymbol{z}^\varepsilon)$, then that of u_i^ε decreases. Conversely, if the value of u_i^ε is less than that of $\phi_i(\boldsymbol{z}^\varepsilon)$, then that of u^ε increases. Consequently, the value of u_i^ε approaches that of $\phi_i(\boldsymbol{z}^\varepsilon)$:

$$u_i^\varepsilon \approx \phi_i(\boldsymbol{z}^\varepsilon). \tag{7}$$

On the other hand, it follows from (5) that

$$\frac{\partial z_i^\varepsilon}{\partial t} = a_i \Delta z_i^\varepsilon + \Delta u_i^\varepsilon + f_i(\boldsymbol{z}^\varepsilon, \boldsymbol{u}^\varepsilon).$$

Therefore, (7) implies that

$$\frac{\partial z_i^\varepsilon}{\partial t} \approx \Delta[a_i z_i^\varepsilon + \phi_i(\boldsymbol{z}^\varepsilon)] + f_i(\boldsymbol{z}^\varepsilon, \boldsymbol{\phi}(\boldsymbol{z}^\varepsilon)).$$

Thus, it can be expected formally that $\boldsymbol{z}^\varepsilon$ approximates $\boldsymbol{z}$. Moreover, from (7), we can expect that $\boldsymbol{u}^\varepsilon$ is close to $\boldsymbol{\phi}(\boldsymbol{z})$. These are our ideas of construction of the reaction-diffusion system as an approximation to the nonlinear diffusion problems.

One can observe that the diffusion coefficient for v_i is the lower bound of the diffusion and the parameter μ seems to control the upper bound of the diffusion. Indeed, we impose such condition for μ in the degenerate case (see $(\mathrm{H5})_\mathrm{D}$ below).

We note that the nonlinearity of the diffusion $\boldsymbol{\phi}$ moves to reaction parts of the reaction-diffusion system (5). The reaction-diffusion system includes only simple reactions and linear diffusions. Dealing with semilinear problems is generally easier than with nonlinear problems such as problems (1). Therefore, our ideas are expected to be useful for analysis of nonlinear problems and for development of good numerical methods. In fact, we have constructed and analyzed a numerical method for the nonlinear degenerate parabolic problem by means of the reaction-diffusion system (6) [14, 15]. Numerical experiments have been carried out to demonstrate the effectiveness and efficiency of the method.

3. MATHEMATICAL RESULTS

This section is devoted to state our theoretical results. Since the analyses of both the degenerate diffusion problems and the cross-diffusion mechanisms are difficult, we have analyzed in each case independently, i. e. in the cases of degenerate parabolic systems without cross-diffusion [12, 14] and non-degenerate cross-diffusion systems [16]. In this paper, we also mention a degenerate cross-diffusion system.

First, we introduce our results for the degenerate parabolic systems. Before stating our results, we describe assumptions and definitions of weak solutions of (1) and (5). The following assumptions are imposed on the data:

$(\mathrm{H1})_\mathrm{D}$ $a_i = 0$ for $i = 1, \ldots, n$.

$(\mathrm{H2})_{\mathrm{D}}$ For each $i = 1, 2, \cdots, n$, the function ϕ_i depends only on the ith variable; we denote $\phi_i(\boldsymbol{\eta}) = \phi_i(\eta_i)$ for $\boldsymbol{\eta} \in \mathbb{R}^n$. Moreover, ϕ_i is nondecreasing and Lipschitz continuous with $\phi_i(0) = 0$. Also, ϕ_i grows at least linearly at infinity; more precisely, there exist positive constants L_ϕ, C_1 and C_2 such that

$$0 \le \phi_i'(\eta) \le L_\phi,$$
$$|\phi_i(\eta)| \ge C_1|\eta| - C_2 \quad \text{for a.e. } \eta \in \mathbb{R}.$$

$(\mathrm{H3})_{\mathrm{D}}$ $\boldsymbol{f}$ is Lipschitz continuous and independent of the first n variables.

$(\mathrm{H4})_{\mathrm{D}}$ $\boldsymbol{z}_0 \in L^2(\Omega)^n$.

$(\mathrm{H5})_{\mathrm{D}}$ μ satisfies $0 < \mu < L_\phi^{-1}$.

We introduce a weak solution of (1).

Definition 1. A function $\boldsymbol{z} \in (L^2(Q) \cap H^1(0,T;H^{-1}(\Omega)))^n$ is said to be a weak solution of (1) if it satisfies

$$a_i z_i + \phi_i(\boldsymbol{z}) \in L^2(0,T;H_0^1(\Omega)),$$
$$\int_0^T \left\langle \frac{\partial z_i}{\partial t}, \varphi_i \right\rangle + \int_0^T \langle \nabla[a_i z_i + \phi_i(\boldsymbol{z})], \nabla\varphi_i\rangle = \int_0^T \langle f_i(\boldsymbol{z}, \boldsymbol{\phi}(\boldsymbol{z})), \varphi_i\rangle,$$
$$z_i(\cdot, 0) = z_0 \quad \text{a.e. in } \Omega$$

for all functions $\varphi_i \in L^2(0,T;H_0^1(\Omega))$ and $i = 1, 2, \cdots, n$. Here, $\langle\cdot,\cdot\rangle$ denotes both the inner product of $L^2(\Omega)$ and the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

The unique existence of the weak solution of (1) under the conditions $(\mathrm{H1})_{\mathrm{D}}$–$(\mathrm{H4})_{\mathrm{D}}$ is known. We define a weak solution of (5).

Definition 2. A pair of functions $(\boldsymbol{u}, \boldsymbol{v}) \in L^2(Q)^{2n}$ is a weak solution of (5) if it satisfies

$$u_i, v_i \in L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)),$$
$$\int_0^T \left\langle \frac{\partial u_i}{\partial t}, \varphi_i \right\rangle + \left(a_i + \frac{1}{\mu}\right) \int_0^T \langle \nabla u_i, \nabla\varphi_i\rangle$$
$$= \frac{1}{\mu}\int_0^T \langle f_i(\mu\boldsymbol{u} + \boldsymbol{v}, \boldsymbol{u}), \varphi_i\rangle - \frac{1}{\varepsilon}\int_0^T \langle u_i - \phi_i(\mu\boldsymbol{u} + \boldsymbol{v}), \varphi_i\rangle, \tag{8}$$
$$\int_0^T \left\langle \frac{\partial v_i}{\partial t}, \varphi_i \right\rangle + a_i \int_0^T \langle \nabla v_i, \nabla\varphi_i\rangle = \frac{\mu}{\varepsilon}\int_0^T \langle u_i - \phi_i(\mu\boldsymbol{u} + \boldsymbol{v}), \varphi_i\rangle,$$
$$u_i(\cdot, 0) = \phi_i(\boldsymbol{z}_0), \quad v_i(\cdot, 0) = \boldsymbol{z}_0 - \mu\phi_i(\boldsymbol{z}_0) \quad \text{a.e. in } \Omega \tag{9}$$

for all functions $\varphi_i \in L^2(0,T;H_0^1(\Omega))$ and $i \in \{i \mid a_i > 0\}$, and

$$u_i \in L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)),\ v_i \in H^1(0,T;L^2(\Omega)),$$
$$\frac{\partial v_i}{\partial t} = \frac{\mu_i}{\varepsilon}(u_i - \phi_i(\mu\boldsymbol{u} + \boldsymbol{v})) \quad \text{a.e. in } Q$$

and (8), (9) for all functions $\varphi_i \in L^2(0,T;H_0^1(\Omega))$ and $i \in \{i \mid a_i = 0\}$.

Unique solvability of the problem under the assumptions $(\mathrm{H1})_{\mathrm{D}}-(\mathrm{H4})_{\mathrm{D}}$ and $\mu > 0$ can be easily verified. We can prove the following convergence theorem (see [12, 14, 15]).

Theorem 1. Suppose $(\mathrm{H1})_{\mathrm{D}}-(\mathrm{H5})_{\mathrm{D}}$ are satisfied. Let $\boldsymbol{z}$ be a weak solution of (1) and $(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon)$ be a weak solution of (6). Then,

$$\begin{aligned} \boldsymbol{u}^\varepsilon \to \boldsymbol{\phi}(\boldsymbol{z}) \quad & \text{strongly in } L^2(Q)^n,\ \text{a.e. in } Q, \\ & \text{and weakly in } L^2(0,T;H^1(\Omega))^n, \\ \mu\boldsymbol{u}^\varepsilon + \boldsymbol{v}^\varepsilon \rightharpoonup \boldsymbol{z} \quad & \text{weakly in } L^2(Q)^n \cap H^1(0,T;H^{-1}(\Omega))^n \end{aligned}$$

as ε tends to zero.

Next, we consider non-degenerate cross-diffusion systems. The general cross-diffusion systems are quite difficult to deal with. Even for the problem (3), only partial results are available on the existence of solutions (see [1, 2] and references therein). We restrict the nonlinear diffusivity to the Lipschitz continuous function. This restriction facilitates the analysis. The following assumptions are made:

$(\mathrm{H1})_{\mathrm{C}}$ $a_i > 0$ for all $i = 1, \ldots, n$.

$(\mathrm{H2})_{\mathrm{C}}$ $\boldsymbol{\phi}$ is a Lipschitz continuous function satisfying $\boldsymbol{\phi}(\boldsymbol{0}) = \boldsymbol{0}$.

$(\mathrm{H3})'_{\mathrm{C}}$ $\boldsymbol{\phi}$ satisfies

$$\sum_{i=1}^n \sum_{j=1}^n (\phi_i)_j(\boldsymbol{\eta})\xi_i\xi_j \ge 0$$

for almost all $\boldsymbol{\eta}, \boldsymbol{\xi} \in \mathbb{R}^n$.

Here, $(\phi_i)_j$ denotes the derivative of the ith component of $\boldsymbol{\phi}$ with respect to the jth variable.

$(\mathrm{H4})_{\mathrm{C}}$ $\boldsymbol{f}$ is a Lipschitz continuous function and dependent only on the first n variables.

$(\mathrm{H5})_{\mathrm{C}}$ $\boldsymbol{z}_0 \in H_0^1(\Omega)^n$.

The conditions $(\mathrm{H1})_{\mathrm{C}}$, $(\mathrm{H2})_{\mathrm{C}}$ and $(\mathrm{H3})'_{\mathrm{C}}$ imply that the systems are uniformly parabolic. Indeed, since for almost all $\boldsymbol{\xi} \in \mathbb{R}^n$

$$\boldsymbol{\xi}^T \boldsymbol{A}\boldsymbol{\xi} = \sum_{i=1}^n a_i\xi_i^2 + \sum_{i=1}^n \sum_{j=1}^n (\phi_i)_j(\boldsymbol{z})\xi_i\xi_j,$$

there exists a positive constant C such that

$$\min_{0\le i\le n}\{a_i\}|\boldsymbol{\xi}|^2 \le \boldsymbol{\xi}^T \boldsymbol{A}\boldsymbol{\xi} \le C|\boldsymbol{\xi}|^2.$$

Here, $\boldsymbol{A}$ denotes the diffusion matrix, that is, the entries A_{ij} are $a_i + (\phi_i)_i(\boldsymbol{z})$ if $i = j$ and are $(\phi_i)_j(\boldsymbol{z})$ if $i \neq j$. The cross-diffusion system (3) with non-negative bounded solutions is uniformly parabolic if $a_i > 0$ and

$$8b_1c_2 > c_1^2, \ 8b_2c_1 > c_2^2. \tag{10}$$

We consider more general problems which possess block triangular diffusion matrices those diagonal blocks have the same structure as the above. We use the following notations: $\mathbb{R}^n = \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_l}$, $\boldsymbol{z} = (z_{11}, \ldots, z_{1m_1}, z_{21}, \ldots, z_{2m_2}, \ldots, z_{l1}, \ldots, z_{lm_l})$, $\boldsymbol{\phi} = (\phi_{11}, \ldots, \phi_{1m_1}, \phi_{21}, \ldots, \phi_{2m_2}, \ldots, \phi_{l1}, \ldots, \phi_{lm_l})$ and so on. The following conditions are imposed on $\boldsymbol{\phi}$ instead of $(\mathrm{H3})'_{\mathrm{C}}$ and on μ:

$(\mathrm{H3})_{\mathrm{C}}$ For each $i = 1, \ldots, l-1$ and $j = 1, \ldots, m_i$, the function ϕ_{ij} is independent of the rsth $(r > i, s = 1, \ldots, m_r)$ variables. Moreover, $\boldsymbol{\phi}$ satisfies

$$\sum_{j=1}^{m_i} \sum_{s=1}^{m_i} (\phi_{ij})_{is}(\boldsymbol{\eta}) \xi_{ij} \xi_{is} \geq 0$$

for all $i = 1, \ldots, l$ and almost all $\boldsymbol{\eta}, \boldsymbol{\xi} \in \mathbb{R}^n$.

$(\mathrm{H6})_{\mathrm{C}}$ μ satisfies

$$0 < \mu < 4 \min_{1 \leq i \leq l} \min_{1 \leq s \leq m_i} \left\{ a_{is} \Big/ \left(m_i \sum_{j=1}^{m_i} \operatorname*{ess\,sup}_{\boldsymbol{\eta} \in \mathbb{R}^n} (\phi_{ij})_{is}(\boldsymbol{\eta})^2 \right) \right\}.$$

The cross-diffusion system (3) with $a_1, a_2 > 0$, $b_1, b_2, c_1 \geq 0$ and $c_2 = 0$ can be treated as long as the solution is non-negative and bounded. The condition (10) is no longer needed. Our problem contains the situation considered by Iida et al. [7].

The unique existence of the weak solution of (5) can be established easily provided that $(\mathrm{H1})_{\mathrm{C}}$, $(\mathrm{H2})_{\mathrm{C}}$, $(\mathrm{H4})_{\mathrm{C}}$, $(\mathrm{H4})_{\mathrm{D}}$ and $\mu > 0$ are satisfied. We can also prove the existence of the weak solution of the cross-diffusion system (1) by means of the reaction-diffusion system (5). We have prepared sufficiently to state our results for the non-degenerate cross-diffusion systems.

Theorem 2. (Murakawa [16]) Assume that $(\mathrm{H1})_{\mathrm{C}} - (\mathrm{H6})_{\mathrm{C}}$ are satisfied. Let $(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon)$ be the weak solution of (5).

Then, there exist a weak solution $\boldsymbol{z} \in (L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)))^n$ of (1) and subsequences $\{\boldsymbol{u}^{\varepsilon_k}\}$, $\{\boldsymbol{v}^{\varepsilon_k}\}$ of $\{\boldsymbol{u}^\varepsilon\}$, $\{\boldsymbol{v}^\varepsilon\}$ such that

$$\begin{aligned} \mu \boldsymbol{u}^{\varepsilon_k} + \boldsymbol{v}^{\varepsilon_k} \to \boldsymbol{z} \quad & \text{strongly in } L^2(Q)^n, \text{ a.e. in } Q, \\ & \text{and weakly in } L^2(0,T;H^1(\Omega))^n \cap H^1(0,T;H^{-1}(\Omega))^n, \\ \boldsymbol{u}^{\varepsilon_k} \rightharpoonup \boldsymbol{\phi}(\boldsymbol{z}) \quad & \text{weakly in } L^2(0,T;H^1(\Omega))^n \end{aligned}$$

as ε_k tends to zero.

Our results show that the weak solutions of the degenerate parabolic problems and of the cross-diffusion systems can be approximated by those of the reaction-diffusion system (5).

Finally, we touch on degenerate cross-diffusion systems. Because it is very difficult to deal with these general problems, we consider problems of the following type:

$$\begin{cases} \dfrac{\partial z_1}{\partial t} = \Delta[a_1 z_1 + \phi_1(z_1, \phi_2(z_2))] + f_1(z_1, \phi_2(z_2)) & \text{in } Q, \\ \dfrac{\partial z_2}{\partial t} = \Delta\phi_2(z_2) + f_2(z_1, \phi_2(z_2)) & \text{in } Q, \\ a_1 z_1 + \phi_1(z_1, \phi_2(z_2)) = 0, \quad \phi_2(z_2) = 0 & \text{on } \partial\Omega \times (0, T), \\ \boldsymbol{z}(\cdot, 0) = \boldsymbol{z}_0 & \text{in } \Omega, \end{cases} \tag{11}$$

where $a_1 > 0$, ϕ_1 is a Lipschitz continuous function so that $\phi_1(\mathbf{0}) = 0$ and $(\phi_1)_1(\boldsymbol{\eta}) \geq 0$ for almost all $\boldsymbol{\eta} \in \mathbb{R}^2$, ϕ_2 is a function satisfying the condition $(\mathrm{H2})_{\mathrm{D}}$, $\boldsymbol{f}$ is Lipschitz continuous, and the initial datum $\boldsymbol{z}_0$ belongs to $H_0^1(\Omega) \times L^2(\Omega)$. Analogous to the above discussion, the following reaction-diffusion system is proposed as an approximation to (11):

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \left(a_1 + \dfrac{1}{\mu}\right) \Delta u_1 + \dfrac{1}{\mu} f_1(\mu u_1 + v_1, u_2) - \dfrac{1}{\varepsilon}(u_1 - \phi_1(\mu u_1 + v_1, u_2)), \\ \dfrac{\partial v_1}{\partial t} = a_1 \Delta v_1 + \dfrac{\mu}{\varepsilon}(u_1 - \phi_1(\mu u_1 + v_1, u_2)), \\ \dfrac{\partial u_2}{\partial t} = \dfrac{1}{\mu}\Delta u_2 + \dfrac{1}{\mu} f_2(\mu u_1 + v_1, u_2) - \dfrac{1}{\varepsilon}(u_2 - \phi_2(\mu u_2 + v_2)), \\ \dfrac{\partial v_2}{\partial t} = \dfrac{\mu}{\varepsilon}(u_2 - \phi_2(\mu u_2 + v_2)) & \text{in } Q, \\ \boldsymbol{u} = \mathbf{0}, \ v_1 = 0 & \text{on } \partial\Omega \times (0, T), \\ \boldsymbol{u}(\cdot, 0) = \boldsymbol{\phi}(\boldsymbol{z}_0), \quad \boldsymbol{v}(\cdot, 0) = \boldsymbol{z}_0 - \mu\boldsymbol{\phi}(\boldsymbol{z}_0) \text{ in } \Omega. \end{cases} \tag{12}$$

Here, the parameter μ satisfies $0 < \mu < \min\{4a_1 / \operatorname{ess\,sup}_{\boldsymbol{\eta} \in \mathbb{R}^n} (\phi_1)_1(\boldsymbol{\eta})^2, L_\phi^{-1}\}$. The problem is uniquely solvable. Combining proofs of Theorem 1 and of Theorem 2, we obtain the following result.

Corollary 1. Let $(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon)$ be the weak solution of (12). Then, there exist a weak solution $\boldsymbol{z}$ of (11) and subsequences $\{\boldsymbol{u}^{\varepsilon_k}\}$, $\{\boldsymbol{v}^{\varepsilon_k}\}$ of $\{\boldsymbol{u}^\varepsilon\}$, $\{\boldsymbol{v}^\varepsilon\}$ such that

$$\begin{array}{ll} \mu u_1^{\varepsilon_k} + v_1^{\varepsilon_k} \to z_1 & \text{strongly in } L^2(Q), \text{ a.e. in } Q, \\ & \text{and weakly in } L^2(0, T; H^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega)), \\ u_1^{\varepsilon_k} \rightharpoonup \phi_1(z_1, \phi_2(z_2)) & \text{weakly in } L^2(0, T; H^1(\Omega)) \\ \mu u_2^\varepsilon + v_2^\varepsilon \rightharpoonup z_2 & \text{weakly in } L^2(Q) \cap H^1(0, T; H^{-1}(\Omega)), \\ u_2^\varepsilon \to \phi_2(z_2) & \text{strongly in } L^2(Q), \text{ a.e. in } Q, \\ & \text{and weakly in } L^2(0, T; H^1(\Omega)) \end{array}$$

as ε_k tends to zero.

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REFERENCES

[1] L. Chen and A. Jüngel: Analysis of a multidimensional parabolic population model with strong cross-diffusion. SIAM J. Math. Anal. *36* (2006), 301–322.

[2] L. Chen and A. Jüngel: Analysis of a parabolic cross-diffusion population model without self-diffusion. J. Differential Equations *224* (2006), 39–59.

[3] E. N. Dancer, D. Hilhorst, M. Mimura, and L. A. Peletier: Spatial segregation limit of a competition-diffusion system. European J. Appl. Math. *10* (1999), 97–115.

[4] D. Hilhorst, M. Iida, M. Mimura, and H. Ninomiya: A competition-diffusion system approximation to the classical two-phase Stefan problem. Japan J. Indust. Appl. Math. *18* (2001), 161–180.

[5] D. Hilhorst, M. Iida, M. Mimura, and H. Ninomiya: Relative compactness in L^p of solutions of some $2m$ components competition-diffusion systems. Discrete Contin. Dyn. Syst. *21* (2008), 1, 233–244.

[6] D. Hilhorst, M. Mimura, and H. Ninomiya: Fast reaction limit of competition-diffusion systems. In: Handbook of Differential Equations: Evolutionary Equations Vol. 5 (C. M. Dafermos and M. Pokorny, eds.). Elsevier/North Holland, Amsterdam 2009, pp. 105–168.

[7] M. Iida, M. Mimura, and H. Ninomiya: Diffusion, cross-diffusion and competitive interaction. J. Math. Biol. *53* (2006), 617–641.

[8] T. Kadota and K. Kuto: Positive steady states for a prey-predator model with some nonlinear diffusion terms. J. Math. Anal. Appl. *323* (2006), 1387–1401.

[9] R. Kersner: Nonlinear heat conduction with absorption: space localization and extinction in finite time. SIAM J. Appl. Math. *43* (1983), 1274–1285.

[10] P. Knabner: Mathematische Modelle für Transport und Sorption gelöster Stoffe in porösen Medien. Verlag Peter Lang, Frankfurt 1991.

[11] H. Murakawa: On reaction-diffusion system approximations to the classical Stefan problems. In: Proc. Czech–Japanese Seminar in Applied Mathematics 2005 (M. Beneš, M. Kimura and T. Nakaki, eds.), COE Lecture Note Vol. 3, Faculty of Mathematics, Kyushu University 2006, pp. 117–125.

[12] H. Murakawa: Reaction-diffusion system approximation to degenerate parabolic systems. Nonlinearity *20* (2007), 2319–2332.

[13] H. Murakawa: A regularization of a reaction-diffusion system approximation to the two-phase Stefan problem. Nonlinear Anal. *69* (2008), 3512–3524.

[14] H. Murakawa: A Solution of Nonlinear Degenerate Parabolic Problems by Semilinear Reaction-Diffusion Systems. Ph.D. Thesis, Graduate School of Mathematics, Kyushu University, 2008.

[15] H. Murakawa: Discrete-time approximation to nonlinear degenerate parabolic problems using a semilinear reaction-diffusion system. Preprint.

[16] H. Murakawa: A relation between cross-diffusion and reaction-diffusion. Preprint.

[17] R. H. Nochetto: A note on the approximation of free boundaries by finite element methods. RAIRO Modél. Math. Anal. Numér. *20* (1986), 355–368.

[18] A. Okubo and S. A. Levin: Diffusion and Ecological Problems: Modern Perspectives. Second edition. (Interdisciplinary Applied Mathematics 14.) Springer–Verlag, New York 2001.

[19] N. Shigesada, K. Kawasaki, and E. Teramoto: Spatial segregation of interacting species. J. Theor. Biol. *79* (1979), 83–99.

[20] K. Tomoeda: Support re-splitting phenomena caused by an interaction between diffusion and absorption. Proc. Equadiff–11 2005, pp. 499–506.

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