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ORBITAL SEMILINEAR COPULAS

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We introduce four families of semilinear copulas (i. e. copulas that are linear in at least one coordinate of any point of the unit square) of which the diagonal and opposite diagonal sections are given functions. For each of these families, we provide necessary and sufficient conditions under which given diagonal and opposite diagonal functions can be the diagonal and opposite diagonal sections of a semilinear copula belonging to that family. We focus particular attention on the family of orbital semilinear copulas, which are obtained by linear interpolation on segments connecting the diagonal and opposite diagonal of the unit square.

Keywords: copula, diagonal section, opposite diagonal section, orbital semilinear copula, semilinear copula

AMS Subject Classification: 62H10, 62H20

1. INTRODUCTION

Bivariate copulas (briefly copulas) are binary operations on the unit interval having 0 as absorbing element and 1 as neutral element and satisfying the condition of 2-monotonicity, i. e. a copula is a function $C : [0,1]^2 \to [0,1]$ satisfying the following conditions:

1. for all $x \in [0,1]$, it holds that

$$C(x,0) = C(0,x) = 0, \quad C(x,1) = C(1,x) = x\,;$$

2. for all $x, x', y, y' \in [0,1]$ such that $x \leq x'$ and $y \leq y'$, it holds that

$$V_C([x,x'] \times [y,y']) := C(x,y) + C(x',y') - C(x,y') - C(x',y) \geq 0\,.$$

$V_C([x,x'] \times [y,y'])$ is called the C-volume of the rectangle $[x,x'] \times [y,y']$. The copulas M and W, defined by $M(x,y) := \min(x,y)$ and $W(x,y) := \max(x+y-1,0)$, are called the Fréchet–Hoeffding upper and lower bounds: for any copula C it holds that $W \leq C \leq M$. A third important copula is the product copula Π defined by $\Pi(x,y) := xy$.

The diagonal section of a copula C is the function $\delta_C : [0,1] \to [0,1]$ defined by $\delta_C(x) = C(x,x)$. A diagonal function is a function $\delta : [0,1] \to [0,1]$ satisfying the following conditions:

(D1) $\delta(1) = 1$;
(D2) for all $x \in [0,1]$, it holds that $\delta(x) \leq x$;
(D3) δ is increasing and 2-Lipschitz continuous, i. e. for all $x, x' \in [0,1]$ such that $x \leq x'$, it holds that

$$0 \leq \delta(x') - \delta(x) \leq 2(x' - x) \,.$$

The diagonal section δ_C of a copula C is a diagonal function. Conversely, for any diagonal function δ, there exists at least one copula C with diagonal section $\delta_C = \delta$. For example, the copula K_δ defined by

$$K_\delta(x,y) = \min\left(x, y, \frac{\delta(x)+\delta(y)}{2}\right) \,, \tag{1}$$

is the greatest symmetric copula with diagonal section δ [13] (see also [5, 7]). Moreover, the Bertino copula B_δ defined by

$$B_\delta(x,y) = \min(x,y) + \min\{t - \delta(t) \mid t \in [\min(x,y), \max(x,y)]\} \,, \tag{2}$$

is the smallest copula with diagonal section δ. Note that B_δ is symmetric. In particular, $B_{\delta_M} = M$ and $B_{\delta_W} = W$.

The opposite diagonal section of a copula C is the function $\omega_C : [0,1] \to [0,1/2]$ defined by $\omega_C(x) = C(x, 1-x)$. An opposite diagonal function is a function $\omega : [0,1] \to [0,1/2]$ satisfying the following conditions:

(OD1) for all $x \in [0,1]$, it holds that $\omega(x) \leq \min(x, 1-x)$;

(OD2) ω is 1-Lipschitz continuous, i. e. for all $x, x' \in [0,1]$, it holds that

$$|\omega(x') - \omega(x)| \leq |x' - x| \,.$$

The opposite diagonal section ω_C of a copula C is an opposite diagonal function. Conversely, for any opposite diagonal function ω, there exists at least one copula C with opposite diagonal section $\omega_C = \omega$. For instance, the copula F_ω defined by

$$F_\omega(x,y) = \max(x+y-1, 0) + \min\left\{\omega(t) \mid t \in [\min(x, 1-y), \max(x, 1-y)]\right\} \,, \tag{3}$$

is the greatest copula with opposite diagonal section ω [2, 11]. Note that F_ω is opposite symmetric [2] (also called radially symmetric), i. e. $F_\omega(x,y) - F_\omega(1-y, 1-x) = x + y - 1$, for any $(x,y) \in [0,1]^2$. In particular, $F_{\omega_M} = M$ and $F_{\omega_W} = W$.

Although copulas were introduced in statistics already 50 years ago [15], during the past years there was a strong revival resulting in a panoply of new construction methods, and this largely inspired by the actual needs of statistical model building [10, 12]. For instance, several authors address the construction of copulas with a given diagonal section [4, 5, 7, 8, 9, 14], being related to the modelling of lower and upper tail dependence [16]. Modelling all four types of tail dependence is then related to constructing copulas with given diagonal and opposite diagonal sections [3].

The aim of this paper is to introduce new families of semilinear copulas. A copula C is called *semilinear* if for any $(x, y) \in [0, 1]^2$ there exists $\epsilon > 0$ such that C is linear on at least one of the segments $\{x\} \times [x - \epsilon, x]$, $\{x\} \times [x, x + \epsilon]$, $[y - \epsilon, y] \times \{y\}$ and $[y, y + \epsilon] \times \{y\}$. For instance, all piecewise linear copulas (in particular, M and W) are semilinear copulas since all their horizontal and vertical sections are piecewise linear. Also the product copula Π is semilinear, as all its horizontal and vertical sections are linear. Recently, Durante et al. [6] introduced two families of semilinear copulas with a given diagonal section, which they called lower and upper semilinear copulas. These copulas are obtained by linear interpolation on segments connecting the diagonal and one of the sides of the unit square. Lower and upper semilinear copulas are symmetric. In order to allow for non-symmetric semilinear copulas as well, De Baets et al. [1] have introduced two related families of semilinear copulas with a given diagonal section, called horizontal and vertical semilinear copulas.

In the present paper, we first introduce four families of semilinear copulas with a given opposite diagonal section, called lower-upper, upper-lower, horizontal and vertical semilinear copulas. There is a great similarity between the case of a given opposite diagonal section and that of a given diagonal section (see also [2]), which can be explained by the existence of a linear transformation that maps copulas onto copulas in such a way that the diagonal is mapped onto the opposite diagonal and vice versa. In the second part of this paper, we consider the construction of semilinear copulas with given diagonal and opposite diagonal sections. Also here, four new families of semilinear copulas are introduced, called orbital, vertical, horizontal and radial semilinear copulas.

This paper is organized as follows. In Section 2, we recall some essential facts on semilinear copulas with a given diagonal section, while in Section 3, we introduce semilinear copulas with a given opposite diagonal section. In Section 4, we introduce the four families of semilinear copulas with given diagonal and opposite diagonal sections and provide for each family the conditions to be satisfied by a diagonal and opposite diagonal function such that they can be the diagonal and opposite diagonal sections of a semilinear copula belonging to that family. Finally, in Section 5, we derive some interesting properties of the family of orbital semilinear copulas.

2. SEMILINEAR COPULAS WITH A GIVEN DIAGONAL SECTION

Two different methods for constructing semilinear copulas with a given diagonal section have been presented recently. The first method is based on linear interpolation on segments connecting the diagonal with the left or lower side (resp. right or upper side) of the unit square; these symmetric copulas are called lower (resp. upper) semilinear copulas [6]. The second method is based on linear interpolation on segments connecting the diagonal with the lower or upper side (resp. left or right side) of the unit square; these in general non-symmetric copulas are called vertical (resp. horizontal) semilinear copulas [1]. The different interpolation schemes are depicted in Figure 1.

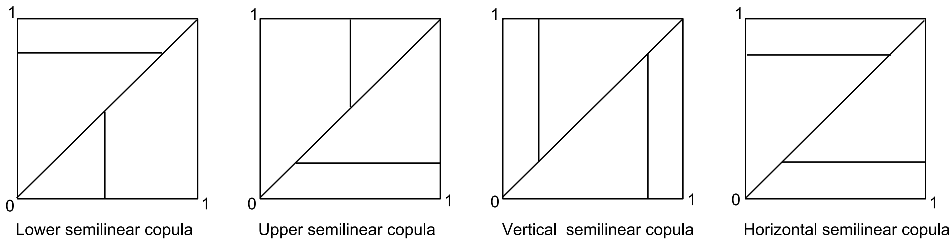

Fig. 1. Semilinear copulas with a given diagonal section.

We briefly recall the conditions on a diagonal function δ that guarantee the existence of a lower or vertical semilinear copula with δ as diagonal section.

Proposition 1. (Durante et al. [6]) Let δ be a diagonal function. The function $C^l_\delta : [0,1]^2 \to [0,1]$ defined by

$$C^l_\delta(x,y) = \begin{cases} y\dfrac{\delta(x)}{x}, & \text{if } y \le x\,, \\ x\dfrac{\delta(y)}{y}, & \text{otherwise}\,, \end{cases} \tag{4}$$

where the convention $\frac{0}{0} := 0$ is adopted, is a copula with diagonal section δ, called *lower semilinear copula with diagonal section* δ, if and only if

(i) the function $\lambda_\delta :]0,1] \to [0,1]$, defined by $\lambda_\delta(x) = \frac{\delta(x)}{x}$, is increasing;

(ii) the function $\rho_\delta :]0,1] \to [1,\infty[$, defined by $\rho_\delta(x) = \frac{\delta(x)}{x^2}$, is decreasing.

Proposition 2. (De Baets et al. [1]) Let δ be a diagonal function. The function $C^v_\delta : [0,1]^2 \to [0,1]$ defined by

$$C^v_\delta(x,y) = \begin{cases} y\dfrac{\delta(x)}{x}, & \text{if } y \le x\,, \\ \dfrac{x(y-x)}{1-x} + \dfrac{1-y}{1-x}\,\delta(x), & \text{otherwise}\,, \end{cases} \tag{5}$$

where the convention $\frac{0}{0} := 1$ is adopted, is a copula with diagonal section δ, called *vertical semilinear copula with diagonal section* δ, if and only if

(i) the function $\lambda_\delta :]0,1] \to [0,1]$, defined by $\lambda_\delta(x) = \frac{\delta(x)}{x}$, is increasing;

(ii) the function $\mu_\delta : [0,1[\to [0,1]$, defined by $\mu_\delta(x) = \frac{x-\delta(x)}{1-x}$, is increasing;

(iii) $\delta \ge \delta_\Pi$, i. e. for all $x \in [0,1]$, it holds that $\delta(x) \ge x^2$.

It can be easily proven that the upper semilinear copula with a given diagonal section can be regarded as a linear transform of a lower semilinear copula.

Proposition 3. Let δ be a diagonal function and $\hat{\delta}$ be the diagonal function defined by $\hat{\delta}(x) = 2x - 1 + \delta(1-x)$. The function $C^u_\delta : [0,1]^2 \to [0,1]$, defined by

$$C^u_\delta(x,y) = x + y - 1 + C^l_{\hat{\delta}}(1-x, 1-y)\,, \tag{6}$$

is a copula with diagonal section δ, called *upper semilinear copula with diagonal section* δ, if and only if

(i) the function $\mu_\delta : [0,1[\to [0,1]$, defined by $\mu_\delta(x) = \frac{x-\delta(x)}{1-x}$, is increasing;

(ii) the function $\sigma_\delta : [0,1[\to [1,\infty[$, defined by $\sigma_\delta(x) = \frac{\delta(x)-x^2}{(1-x)^2}$, is increasing.

Similarly, the horizontal semilinear copula with a given diagonal section is a linear transform of a vertical semilinear copula.

Proposition 4. Let δ be a diagonal function. The function $C^u_\delta : [0,1]^2 \to [0,1]$, defined by

$$C^h_\delta(x,y) = C^v_\delta(y,x)\,, \tag{7}$$

is a copula with diagonal section δ, called *horizontal semilinear copula with diagonal section* δ, if and only if C^v_δ is a copula, i. e. under the conditions of Proposition 2.

Note that for any two lower (resp. upper, vertical, horizontal) semilinear copulas C_1 and C_2 it holds that $C_1 \le C_2$ if and only if $\delta_{C_1} \le \delta_{C_2}$. Since the function ρ_δ is decreasing, $\rho_\delta(x) \ge \rho_\delta(1) = 1$ for all $x \in]0,1]$. Therefore, $\delta(x) \ge x^2$, for any lower semilinear copula C. Similarly, since the function σ_δ is decreasing, $\delta(x) \ge x^2$, for any upper semilinear copula C. Note also that M and Π are examples of copulas that are at the same time lower, upper, vertical and horizontal semilinear copulas. Hence, Π is the smallest semilinear copula (of one of the above four types), i. e. every semilinear copula with a given diagonal section (of one of the above four types) is positive quadrant dependent (PQD).

3. SEMILINEAR COPULAS WITH A GIVEN OPPOSITE DIAGONAL SECTION

In analogy with the lower (resp. upper) and vertical (resp. horizontal) semilinear copulas with a given diagonal section δ, we introduce lower-upper (resp. upper-lower) and vertical (resp. horizontal) semilinear copulas with a given opposite diagonal section ω. For instance, the lower-upper semilinear copula is constructed based on linear interpolation on segments connecting the opposite diagonal with the left or upper side of the unit square. See also Figure 2 where the four different interpolation schemes are depicted.

Let C_ω be a copula with opposite diagonal section ω. The function C', defined by $C'(x,y) = x - C(x, 1-y)$, is again a copula whose diagonal section $\delta_{C'}$ is given by $\delta_{C'}(x) = x - \omega(x)$. This transformation permits to derive in a straightforward manner the conditions that guarantee the existence of a semilinear copula (of any of the above types) with a given opposite diagonal section.

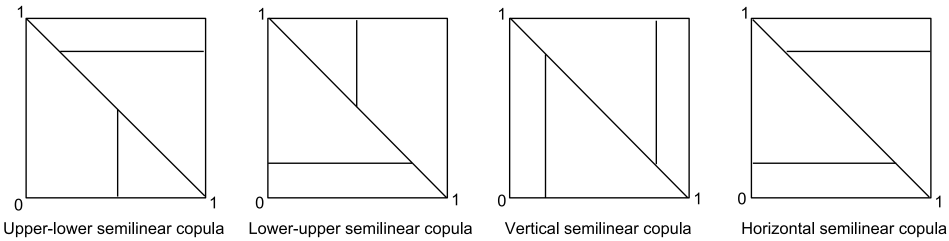

Fig. 2. Semilinear copulas with a given opposite diagonal section.

Proposition 5. Let ω be an opposite diagonal function. The function $C^{lu}_{\omega} : [0,1]^2 \to [0,1]$ defined by

$$C^{lu}_{\omega}(x,y) = \begin{cases} \dfrac{x}{1-y}\,\omega(1-y), & \text{if } x+y \le 1\,, \\ x+y-1+\dfrac{1-y}{x}\,\omega(x), & \text{otherwise}\,, \end{cases} \tag{8}$$

where the convention $\frac{0}{0} := 0$ is adopted, is a copula with opposite diagonal section ω, called *lower-upper semilinear copula with opposite diagonal section* ω, if and only if

(i) the function $\lambda_\omega :]0,1] \to [0,1]$, defined by $\lambda_\omega(x) = \frac{\omega(x)}{x}$, is decreasing;

(ii) the function $\eta_\omega :]0,1] \to [1,\infty[$, defined by $\eta_\omega(x) = \frac{x-\omega(x)}{x^2}$, is decreasing.

Proposition 6. Let ω be an opposite diagonal function. The function $C^{v}_{\omega} : [0,1]^2 \to [0,1]$, defined by

$$C^{v}_{\omega}(x,y) = \begin{cases} \dfrac{y}{1-x}\,\omega(x), & \text{if } x+y \le 1\,, \\ x+y-1+\dfrac{1-y}{x}\,\omega(x), & \text{otherwise}\,, \end{cases} \tag{9}$$

where the convention $\frac{0}{0} := 0$ is adopted, is a copula with opposite diagonal section ω, called *vertical semilinear copula with opposite diagonal section* ω, if and only if

(i) the function $\lambda_\omega :]0,1] \to [0,1]$, defined by $\lambda_\omega(x) = \frac{\omega(x)}{x}$, is decreasing;

(ii) the function $\nu_\omega : [0,1[\to [0,1]$, defined by $\nu_\omega(x) = \frac{\omega(x)}{1-x}$, is increasing;

(iii) $\omega \le \omega_\Pi$, i.e. for all $x \in [0,1]$, it holds that $\omega(x) \le x(1-x)$.

Proposition 7. Let ω be an opposite diagonal function and $\hat{\omega}$ be the opposite diagonal function defined by $\hat{\omega}(x) = \omega(1-x)$. The function $C^{ul}_{\delta} : [0,1]^2 \to [0,1]$, defined by

$$C^{ul}_{\omega}(x,y) = C^{lu}_{\hat{\omega}}(y,x)\,, \tag{10}$$

is a copula with opposite diagonal section ω, called *upper-lower semilinear copula with opposite diagonal section* ω, if and only if

(i) the function $\nu_\omega : [0,1[\to [0,1]$, defined by $\nu_\omega(x) = \frac{\omega(x)}{1-x}$, is increasing;

(ii) the function $\zeta_\omega : [0,1[\to [1,\infty[$, defined by $\zeta_\omega(x) = \frac{1-x-\omega(x)}{(1-x)^2}$, is increasing.

Similarly, the horizontal semilinear copula with a given opposite diagonal section is a linear transform of a vertical semilinear copula.

Proposition 8. Let ω be an opposite diagonal function and $\hat{\omega}$ be the opposite diagonal function defined by $\hat{\omega}(x) = \omega(1-x)$. The function $C^h_\omega : [0,1]^2 \to [0,1]$, defined by

$$C^h_\omega(x,y) = C^v_{\hat{\omega}}(y,x)\,, \tag{11}$$

is a copula with opposite diagonal section ω, called *horizontal semilinear copula with opposite diagonal section* ω, if and only if $C^v_{\hat{\omega}}$ is a copula, i. e. under the conditions of Proposition 6.

Note that for any two lower-upper (resp. vertical, upper-lower, horizontal) semilinear copulas C_1 and C_2 it holds that $C_1 \le C_2$ if and only if $\omega_{C_1} \le \omega_{C_2}$. From Propositions 5 and 6, it follows that $\omega(x) \le x(1-x)$ for any lower-upper, upper-lower, horizontal or vertical semilinear copula. Note also that W and Π are examples of copulas that are at the same time lower-upper, vertical, upper-lower and horizontal semilinear copulas. Hence, Π is the greatest semilinear copula (of one of the above four types), i. e. every semilinear copula with a given opposite diagonal section (of one of the above four types) is negative quadrant dependent (NQD).

4. SEMILINEAR COPULAS WITH GIVEN DIAGONAL AND OPPOSITE DIAGONAL SECTIONS

In this section we introduce four new families of semilinear copulas. Their construction is based on linear interpolation on segments connecting the diagonal and opposite diagonal or connecting the diagonal or opposite diagonal and one of the sides of the unit square. Since in any of the four triangular sectors of the unit square delimited by the diagonal and opposite diagonal, we can either interpolate between a point on the diagonal and a point on the opposite diagonal, or between a point on the sides of the unit square and a point on the diagonal or opposite diagonal, there are 16 possible combinations of interpolation schemes. Based on symmetry considerations, we will consider only the four combinations depicted in Figure 3.

Clearly, in general, given a diagonal function δ and an opposite diagonal function ω, there need not exist a copula that has δ as diagonal section and ω as opposite diagonal section. For instance, the diagonal function $\delta(x) = x^2$ and the opposite diagonal function $\omega(x) = \min(x, 1-x)$ cannot be the diagonal and opposite diagonal sections of a copula since $\delta(1/2) = 1/4 \ne 1/2 = \omega(1/2)$.

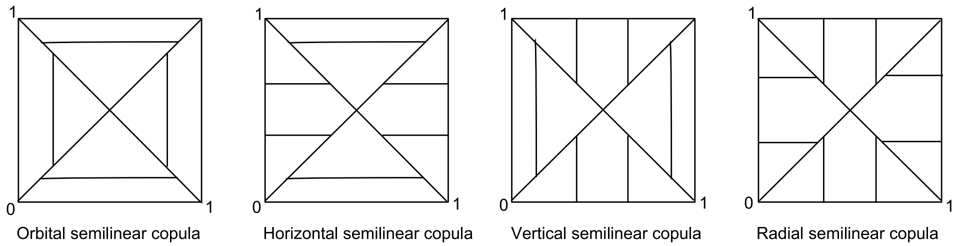

Fig. 3. Semilinear copulas with given diagonal and opposite diagonal sections.

Proposition 9. Let δ and ω be diagonal and opposite diagonal functions such that $\delta(1/2) = \omega(1/2)$. The function $C^o_{\delta,\omega} : [0,1]^2 \to [0,1]$, defined by $C^o_{\delta,\omega}(x,y) =$

$$\begin{cases} \dfrac{x+y-1}{2y-1}\,\delta(y) + \dfrac{y-x}{2y-1}\,\omega(1-y), & \text{if } (x \le y \wedge x+y > 1) \vee (x > y \wedge x+y \le 1)\,, \\[2ex] \dfrac{x+y-1}{2x-1}\,\delta(x) + \dfrac{x-y}{2x-1}\,\omega(x), & \text{otherwise}\,, \end{cases} \tag{12}$$

where the convention $\frac{0}{0} := \delta(1/2)$ is adopted, is a copula with diagonal section δ and opposite diagonal section ω, called *orbital semilinear copula with diagonal section δ and opposite diagonal section ω*, if and only if:

(i) the functions $\vartheta_{\delta,\omega}, \psi_{\delta,\omega} : [0,1/2[\, \cup\,]1/2,1] \to [0,1]$, defined by

$$\vartheta_{\delta,\omega}(x) = \frac{\omega(x) - \delta(x)}{1-2x}\,, \quad \psi_{\delta,\omega}(x) = \frac{\omega(1-x) - \delta(x)}{1-2x}\,,$$

are increasing on the interval $[0,1/2[$ and on the interval $]1/2,1]$;

(ii) for all $x, x' \in [0,1/2[$, such that $x < x'$, it holds that

$$\omega(x) + \omega(1-x) \le \frac{\delta(x')(1-2x) - \delta(x)(1-2x')}{x'-x}\,,$$

$$\delta(x) + \delta(1-x) \ge \frac{\omega(x')(1-2x) - \omega(x)(1-2x')}{x'-x}\,;$$

(iii) for all $x, x' \in]1/2,1]$, such that $x < x'$, it holds that

$$\omega(x') + \omega(1-x') \le \frac{\delta(x')(1-2x) - \delta(x)(1-2x')}{x'-x}\,,$$

$$\delta(x') + \delta(1-x') \ge \frac{\omega(x')(1-2x) - \omega(x)(1-2x')}{x'-x}\,.$$

Proof. The function $C^{o}_{\delta,\omega}$ defined by (12) clearly satisfies the boundary conditions of a copula. Therefore, it suffices to prove that conditions (i)−(iii) are equivalent with the property of 2-monotonicity.

For any rectangle $[x,x']\times[y,y']$ such that $y'<x$ and $x+y>1$ or $y>x'$ and $x'+y'<1$ (otherwise stated, any rectangle entirely situated inside either the triangular sector that contains the right side of the unit square or the triangular sector that contains the left side of the unit square), the volume is given by

$$V_{C^{o}_{\delta,\omega}}([x,x']\times[y,y'])=(\vartheta_{\delta,\omega}(x')-\vartheta_{\delta,\omega}(x))(y'-y)\,.$$

Positivity of these volumes is clearly equivalent with the increasingness of the function $\vartheta_{\delta,\omega}$ on the intervals $[0,1/2[$ and $]1/2,1]$.

Similarly, for any rectangle $[x,x']\times[y,y']$ such that $y>x'$ and $x+y>1$ or $y'<x$ and $x'+y'<1$ (otherwise stated, any rectangle entirely situated inside either the triangular sector that contains the upper side of the unit square or the triangular sector that contains the lower side of the unit square), the volume is given by

$$V_{C^{o}_{\delta,\omega}}([x,x']\times[y,y'])=(\psi_{\delta,\omega}(y')-\psi_{\delta,\omega}(y))(x'-x)\,.$$

Positivity of these volumes is clearly equivalent with the increasingness of the function $\psi_{\delta,\omega}$ on the intervals $[0,1/2[$ and $]1/2,1]$.

Finally, expressing that the volume of any rectangle of the type $[x,x']\times[x,x']$ or of the type $[x,x']\times[1-x',1-x]$ (otherwise stated, rectangles whose diagonal or opposite diagonal is situated on the diagonal or opposite diagonal of the unit square) is positive is equivalent with conditions (ii) and (iii). □

Note that conditions (ii) and (iii) can be reformulated by means of the functions $\vartheta_{\delta,\omega}$ and $\psi_{\delta,\omega}$ in the following way:

(ii') for all $x,x'\in[0,1/2[$ such that $x<x'$, it holds that

$$\vartheta_{\delta,\omega}(x)+\psi_{\delta,\omega}(x)\le\frac{\delta(x')-\delta(x)}{x'-x}\,,\quad \psi_{\delta,\omega}(1-x)-\vartheta_{\delta,\omega}(x)\ge\frac{\omega(x')-\omega(x)}{x'-x}\,;$$

(iii') for all $x,x'\in]1/2,1]$ such that $x<x'$, it holds that

$$\vartheta_{\delta,\omega}(x')+\psi_{\delta,\omega}(x')\ge\frac{\delta(x')-\delta(x)}{x'-x}\,,\quad \psi_{\delta,\omega}(1-x')-\vartheta_{\delta,\omega}(x')\le\frac{\omega(x')-\omega(x)}{x'-x}\,.$$

In case δ and ω are differentiable functions, the latter two conditions are equivalent with

(ii”) for all $x<1/2$, it holds that

$$\delta'(x)\ge\vartheta_{\delta,\omega}(x)+\psi_{\delta,\omega}(x)\,,\quad \omega'(x)\le\psi_{\delta,\omega}(1-x)-\vartheta_{\delta,\omega}(x)\,;$$

(iii”) for all $x>1/2$, it holds that

$$\delta'(x)\le\vartheta_{\delta,\omega}(x)+\psi_{\delta,\omega}(x)\,,\quad \omega'(x)\ge\psi_{\delta,\omega}(1-x)-\vartheta_{\delta,\omega}(x)\,.$$

Example 1. Consider the diagonal function $\delta(x) = x^2$ and the opposite diagonal function $\omega(x) = (1/2)\min(x, 1-x)$. Note that $\delta(1/2) = \omega(1/2) = 1/4$. Clearly, the functions $\vartheta_{\delta,\omega}$ and $\psi_{\delta,\omega}$ are increasing on the intervals $[0, 1/2[$ and $]1/2, 1]$. For all $0 \le x < x' \le 1/2$, it holds that

$$\vartheta_{\delta,\omega}(x) + \psi_{\delta,\omega}(x) = x < x + x' = \frac{\delta(x') - \delta(x)}{x' - x}$$

and

$$\psi_{\delta,\omega}(1-x) - \vartheta_{\delta,\omega}(x) = 1 - x > \frac{1}{2} = \frac{\omega(x') - \omega(x)}{x' - x}\,,$$

whence condition (ii') is satisfied. Similarly, condition (iii') is satisfied, and therefore the function $C^o_{\delta,\omega}$ defined by (12) is the orbital semilinear copula with diagonal section δ and opposite diagonal section ω.

Next, we consider the construction of the horizontal (resp. vertical) semilinear copula with given diagonal and opposite diagonal sections. It is constructed by interpolating in the x-direction (resp. y-direction). As it is again possible to connect the two types by means of a linear transformation, we will make explicit the conditions to be fulfilled by δ and ω for just one type.

Proposition 10. Let δ and ω be diagonal and opposite diagonal functions such that $\delta(1/2) = \omega(1/2)$. The function $C^h_{\delta,\omega} : [0,1]^2 \to [0,1]$, defined by $C^h_{\delta,\omega}(x, y) =$

$$\begin{cases} \dfrac{x+y-1}{2y-1}\,\delta(y) + \dfrac{y-x}{2y-1}\,\omega(1-y), & \text{if } (x \le y \wedge x+y > 1) \vee (x > y \wedge x+y \le 1)\,, \\ \dfrac{x}{1-y}\,\omega(1-y), & \text{if } x+y \le 1 \wedge y \ge 1/2\,, \\ \dfrac{x}{y}\,\delta(y), & \text{if } x \le y < 1/2\,, \\ \dfrac{y(x-y)}{1-y} + \dfrac{1-x}{1-y}\,\delta(y), & \text{if } 1/2 \le y \le x\,, \\ x+y-1 + \dfrac{1-x}{y}\,\omega(1-y), & \text{if } x+y \ge 1 \wedge y \le 1/2\,, \end{cases} \tag{13}$$

where the convention $\frac{0}{0} := 0$ is adopted, is a copula with diagonal section δ and opposite diagonal section ω, called *horizontal semilinear copula with diagonal section δ and opposite diagonal section ω*, if and only if:

(i) the function $\psi_{\delta,\omega}$ is increasing on the interval $[0, 1/2[$ and on the interval $]1/2, 1]$;

(ii) the function λ_δ is increasing on the interval $]0, 1/2]$ and the function $\lambda_\omega$is decreasing on the interval $[1/2, 1]$;

(iii) the function μ_δ is increasing on the interval $[1/2, 1[$ and the function ν_ω is increasing on the interval $[0, 1/2]$;

(iv) for all $x \in [0, 1/2]$, it holds that

$$\min[(1-x)\delta(x) - x\omega(1-x)\,, x\delta(1-x) - (1-x)\omega(x)] \geq 0\,;$$

(v) for all $x \in [1/2, 1]$, it holds that

$$\min[x(1-2x) - (1-x)\omega(1-x) + x\delta(x)\,, (\delta(1-x) + 2x - 1)(1-x) - x\omega(x)] \geq 0\,.$$

Proof. The proof is similar to that of Proposition 9. □

Example 2. Consider the diagonal function $\delta(x) = \frac{x}{2-x}$ and the opposite diagonal function $\omega(x) = \frac{2}{3}\min(x, 1-x)$. Note that $\delta(1/2) = \omega(1/2) = 1/3$. The first three conditions of Proposition 10 are trivially fulfilled. Moreover, for all $x \in [0, 1/2]$, it holds that

$$(1-x)\delta(x) - x\omega(1-x) = \frac{x(3-x)(1-2x)}{3(2-x)} \geq 0$$

and

$$x\delta(1-x) - (1-x)\omega(x) = \frac{x(1-x)(1-2x)}{3(1+x)} \geq 0\,,$$

which implies condition (iv). Condition (v) holds similarly. Therefore the function $C^h_{\delta,\omega}$ is the horizontal semilinear copula with diagonal section δ and opposite diagonal section ω.

The vertical semilinear copula $C^v_{\delta,\omega}$ with diagonal section δ and opposite diagonal section ω is defined by

$$C^v_{\delta,\omega}(x, y) = C^h_{\delta,\hat{\omega}}(y, x)\,, \tag{14}$$

with $\hat{\omega}$ the opposite diagonal function defined by $\hat{\omega}(x) = \omega(1-x)$ and $C^h_{\delta,\hat{\omega}}$ the horizontal semilinear copula with diagonal section δ and opposite diagonal section $\hat{\omega}$, provided the latter is properly defined. In fact, the conditions on δ and ω are exactly conditions (i) – (v) of Proposition 10.

Finally, we consider the case where the interpolation is done on segments connecting points on the diagonal or opposite diagonal and points on the sides of the unit square. We call a semilinear copula that results from this combination of interpolation schemes a radial semilinear copula.

Proposition 11. Let δ and ω be diagonal and opposite diagonal functions such that $\delta(1/2) = \omega(1/2)$. The function $C^r_{\delta,\omega} : [0,1]^2 \to [0,1]$, defined by $C^r_{\delta,\omega}(x,y) =$

$$\begin{cases} \dfrac{x}{y}\,\delta(y), & \text{if } x \le y \le 1/2\,, \\ \dfrac{y}{x}\,\delta(x), & \text{if } y \le x \le 1/2\,, \\ \dfrac{x}{1-y}\,\omega(1-y), & \text{if } x+y \le 1, y \ge 1/2\,, \\ \dfrac{y}{1-x}\,\omega(x), & \text{if } x+y \le 1, x \ge 1/2\,, \\ x+y-1+\dfrac{1-x}{y}\,\omega(1-y), & \text{if } x+y \ge 1, y \le 1/2\,, \\ x+y-1+\dfrac{1-y}{x}\,\omega(x) & \text{if } x+y \ge 1, x \le 1/2\,, \\ \dfrac{y(x-y)}{1-y} + \dfrac{1-x}{1-y}\,\delta(y), & \text{if } 1/2 \le y \le x\,, \\ \dfrac{x(y-x)}{1-x} + \dfrac{1-y}{1-x}\,\delta(x), & \text{if } 1/2 \le x \le y\,, \end{cases} \tag{15}$$

where the convention $\frac{0}{0} := 0$ is adopted, is a copula with diagonal section δ and opposite diagonal section ω, called *radial semilinear copula with diagonal section* δ *and opposite diagonal section* ω, if and only if

(i) the function λ_δ is increasing on $]0,1/2]$, the function ρ_δ is decreasing on $]0,1/2]$, the function λ_ω is decreasing on $]0,1/2]$ and the function η_ω is decreasing on $]0,1/2]$;

(ii) the function μ_δ is increasing on $[1/2,1[$, the function σ_δ is increasing on $[1/2,1[$, the function ν_ω is increasing on $[1/2,1[$ and the function ζ_ω is increasing on $[1/2,1[$.

Note that M, Π and W are examples of copulas that are at the same time orbital, vertical, horizontal and radial semilinear copulas with given diagonal and opposite diagonal sections.

5. PROPERTIES OF ORBITAL SEMILINEAR COPULAS WITH A GIVEN DIAGONAL OR OPPOSITE DIAGONAL SECTION

In this section we will further study the family of orbital semilinear copulas. It is the only family for which the interpolation in all the triangular sectors occurs between points on the diagonal and points on the opposite diagonal, and therefore it has no counterpart at all in the families of semilinear copulas which were constructed before by giving either a diagonal or an opposite diagonal section.

It is well known that M (resp. W) is the only copula with diagonal sectionδ_M (resp. opposite diagonal section ω_W). As these copulas are orbital semilinear copulas, they are obviously the only such copulas with that given behaviour. In general, however, M (resp. W) is not the only copula with opposite diagonal section ω_M

(resp. diagonal section δ_W). For instance, the copula K_{δ_W} defined in (1) differs from W.

In the context of orbital semilinear copulas, however, M and W take up a unique role again.

Proposition 12.

(i) M is the only orbital semilinear copula which has ω_M as opposite diagonal section;

(ii) W is the only orbital semilinear copula which has δ_W as diagonal section.

P r o o f. We will prove (ii), the proof of (i) being similar. Suppose that $C^o_{\delta_W,\omega}$ is the orbital semilinear copula with diagonal section δ_W and a not yet specified opposite diagonal section ω. Note that $\omega(1/2) = \delta_W(1/2) = 0$. From condition (ii) of Proposition 9, it follows that for all $x \in [0, 1/2[$, it must hold that $\omega(x)+\omega(1-x) = 0$, which implies that $\omega = \omega_W$. Hence, $C^o_{\delta_W,\omega} = C^o_{\delta_W,\omega_W} = W$. □

We now investigate the situation where either the given diagonal function is the diagonal section δ_Π of the product copula Π, or the given opposite diagonal function ω is the opposite diagonal section ω_Π of Π. In fact, we will prove a more general statement by introducing parametrized families of diagonal (resp. opposite diagonal) functions that contain δ_Π (resp. ω_Π).

Proposition 13. Let δ and ω be diagonal and opposite diagonal functions such that $\delta(1/2) = \omega(1/2)$.

(i) If $\delta(x) = \alpha x^2 + (1-\alpha)x$ for arbitrary $\alpha \in]0, 1]$, then the function $C^o_{\delta,\omega}$ given by (12) is an orbital semilinear copula if and only if the function $\phi_\omega : [0, 1/2[\cup]1/2, 1] \to [-1, 1]$, defined by

$$\phi_\omega(x) = \frac{2\omega(x) + \alpha(2x^2 - x + 1/2) - 1}{2(1-2x)} ,$$

is decreasing and α-Lipschitz continuous on both intervals $[0, 1/2[$ and $]1/2, 1]$.

(ii) If $\omega(x) = \alpha x(1-x)$ for arbitrary $\alpha \in]0, 1]$, then the function $C^o_{\delta,\omega}$ given by (12) is an orbital semilinear copula if and only if the function $\phi_\delta : [0, 1/2[\cup]1/2, 1] \to [-1, 1]$, defined by

$$\phi_\delta(x) = \frac{\delta(x) - \alpha x^2}{1-2x} ,$$

is increasing and α-Lipschitz continuous on both intervals $[0, 1/2[$ and $]1/2, 1]$.

P r o o f. We will prove (ii). The proof of (i) is analogous. Let $C^o_{\delta,\omega}$ be the function defined by (12) with $\omega(x) = \alpha x(1-x)$. Note that the functions $\vartheta_{\delta,\omega}$ and $\psi_{\delta,\omega}$ coincide. This function is an orbital semilinear copula if and only if conditions (i)–(iii) of Proposition 9 are satisfied. For the sake of simplicity, we use the equivalent conditions (ii') and (iii'), although the differentiability of δ is not explicitly assumed. In the case $x \in [0,1/2[$, it should therefore hold that $\delta'(x) \geq \vartheta_{\delta,\omega}(x) + \psi_{\delta,\omega}(x)$, or

$$\delta'(x) \geq \frac{\omega(x) + \omega(1-x) - 2\delta(x)}{1-2x},$$

which is equivalent to the condition that the function ϕ_δ is increasing on $[0,1/2[$, as can be readily verified by computing the derivative of ϕ_δ. Furthermore, it is required that $\vartheta_{\delta,\omega}$ is increasing on $[0,1/2[$, or, equivalently, $\alpha - \phi'_\delta(x) \geq 0$. It follows that ϕ_δ must be α-Lipschitz continuous on $[0,1/2[$. Since $\phi_\delta(0) = 0$, the increasingness of ϕ_δ implies that $\delta(x) \geq \alpha x^2$ for any $x \in [0,1/2[$, whence

$$\delta(x) + \delta(1-x) \geq \alpha(x^2 + (1-x)^2) = \omega'(x)(1-2x) + 2\omega(x),$$

which is equivalent to $\omega'(x) \leq \psi_{\delta,\omega}(1-x) - \vartheta_{\delta,\omega}(x)$ for all $x \in [0,1/2[$. Hence, on the subinterval $[0,1/2[$, all conditions of Proposition 9 are satisfied. In case $x \in]1/2,1]$, the proof is similar. □

Example 3.

(i) Let $\delta(x) = \alpha x^2 + (1-\alpha)x$, with $\alpha \in]0,1]$ and consider the opposite diagonal function

$$\omega(x) = \left(1 - \frac{\alpha}{2}\right) \min(x, 1-x).$$

The function ϕ_ω is decreasing and α-Lipschitz continuous on $[0,1/2[$ and $]1/2,1]$, whence for any $\alpha \in]0,1]$ the functions δ and ω are the diagonal and opposite diagonal sections of an orbital semilinear copula.

(ii) Let $\omega(x) = \alpha x(1-x)$, with $\alpha \in]0,1]$, and consider the diagonal function

$$\delta(x) = \left(1 - \frac{\alpha}{2}\right) \max(2x-1, 0) + \frac{\alpha}{2}x.$$

The function ϕ_δ is increasing and α-Lipschitz continuous on $[0,1/2[$ and $]1/2,1]$, whence for any $\alpha \in]0,1]$ the functions δ and ω are the diagonal and opposite diagonal sections of an orbital semilinear copula.

To conclude this section, we lay bare the necessary and sufficient conditions on a diagonal function δ guaranteeing that the corresponding Bertino copula is an orbital semilinear copula and we investigate also the symmetry and opposite symmetry properties of orbital semilinear copulas.

Proposition 14. Let δ be a diagonal function, then the corresponding Bertino copula B_δ defined in (2) is an orbital semilinear copula if and only if for any $x \in [0,1/2]$ and any $t \in [x, 1-x]$, it holds that

$$t - \delta(t) \geq \min(x - \delta(x), 1 - x - \delta(1-x)). \tag{16}$$

Proof. Let B_δ be the Bertino copula defined in (2) and suppose that B_δ is an orbital semilinear copula. The opposite diagonal section ω_B of this Bertino copula is given by

$$\omega_B(x) = \min(x, 1-x) - \min\{t - \delta(t) \mid t \in [\min(x, 1-x), \max(x, 1-x)]\}\,.$$

Let C_{δ,ω_B} be the orbital semilinear copula with diagonal section δ and opposite diagonal section ω_B, then it obviously coincides with B_δ. In the triangular sector described by $x \le y \le 1-x$, it holds that

$$\frac{x+y-1}{2x-1}\,\delta(x) + \frac{x-y}{2x-1}\,\omega_B(x) = x - \min\{t - \delta(t) \mid t \in [x, y]\}\,.$$

Now, consider the function f defined by $f(t) = t - \delta(t)$, then

$$(x+y-1)\delta(x) + (x-y)\left(x - \min_{t\in[x,1-x]} f(t)\right) = (2x-1)\left(x - \min_{t\in[x,y]} f(t)\right),$$

or equivalently,

$$(1-2x)\min_{t\in[x,y]} f(t) = (1-x-y)\min_{t\in[x,x]} f(t) + (y-x)\min_{t\in[x,1-x]} f(t)\,.$$

Let $t^* \in [x, 1-x]$ be such that $\min_{t\in[x,1-x]} f(t) = f(t^*)$ (note that t^* is not necessarily unique). The above equality, with $y = t^*$, reduces to

$$(1-2x)f(t^*) = (1-x-t^*)f(x) + (t^*-x)f(t^*)\,,$$

which implies that either $t^* = 1-x$ or $f(t^*) = f(x)$. This means that on the interval $[x, 1-x]$ the minimal value of f is attained in at least one of the points $t = x$ or $t = 1-x$, whence condition (16) follows. The three other triangular sectors lead to the same condition. The above reasoning can obviously be traversed in the converse direction. □

Corollary 1. Let δ be a diagonal function. If

(i) δ is 1-Lipschitz continuous on $[0, 1/2]$,

(ii) for every $x \in [0, 1/2]$, it holds that $\delta(1-x) = \delta(x) + 1 - 2x$,

then the corresponding Bertino copula B_δ is an orbital semilinear copula.

Proof. Condition (ii) expresses that the function $f(t) = t - \delta(t)$ is symmetric w.r.t. 1/2 and it therefore suffices to show that

$$t - \delta(t) \ge \min(x - \delta(x), 1 - x - \delta(1-x)) = x - \delta(x)\,,$$

for any $x \in [0, 1/2]$ and any $t \in [x, 1/2]$. As condition (i) simply states that the function $f(t) = t - \delta(t)$ is increasing on $[0, 1/2]$, the latter is trivially fulfilled. □

Proposition 15. Let ω be an opposite diagonal function, then the corresponding copula F_ω defined in (3) is an orbital semilinear copula if and only if for any $x \in [0,1/2]$ and any $t \in [x, 1-x]$, it holds that

$$\omega(t) \geq \min(\omega(x), \omega(1-x))\,. \tag{17}$$

Corollary 2. Let ω be an opposite diagonal function. If

(i) ω is increasing on $[0,1/2]$,

(ii) for every $x \in [0,1/2]$ it holds that $\omega(1-x) = \omega(x)$,

then the corresponding copula F_ω is an orbital semilinear copula.

Conditions (16) and (17) respectively express that the function $f(t) = t - \delta(t)$ and the function ω satisfy a restricted form of convexity by considering only intervals symmetric w.r.t. the point $1/2$. Note that δ_W satisfies the conditions of Corollary 1, while ω_M satisfies the conditions of Corollary 2. This confirms that $B_{\delta_W} = W$ and $F_{\omega_M} = M$ are orbital semilinear copulas.

Proposition 16. Let δ and ω be diagonal and opposite diagonal functions such that $\delta(1/2) = \omega(1/2)$.

(i) Let $\delta(x) = \alpha x^2 + (1-\alpha)x$ for arbitrary $\alpha \in]0,1]$ (see Proposition 13), then the smallest orbital semilinear copula with diagonal section δ is the Bertino copula B_δ defined in (2).

(ii) Let $\omega(x) = \alpha x(1-x)$ for arbitrary $\alpha \in]0,1]$ (see Proposition 13), then the greatest orbital semilinear copula with opposite diagonal section ω is the copula F_ω defined in (3).

Proof. One easily verifies that the given diagonal function satisfies the sufficient conditions of Corollary 1. Hence, the Bertino copula is an orbital semilinear copula. As it is the smallest copula with diagonal section δ, it is obviously also the smallest orbital semilinear copula with this diagonal section. The same reasoning applies to F_ω. $\square$

The following proposition is a matter of direct verification.

Proposition 17. An orbital (resp. radial) semilinear copula $C^o_{\delta,\omega}$ (resp. $C^r_{\delta,\omega}$) is

(i) opposite symmetric if and only if the function $f(x) = x - \delta(x)$ is symmetric w.r.t. $x = 1/2$, i. e. $\delta(1-x) = \delta(x) + 1 - 2x$ for all $x \in [0,1/2]$;

(ii) symmetric if and only if ω is symmetric w.r.t. $x = 1/2$, i. e. $\omega(x) = \omega(1-x)$ for all $x \in [0,1/2]$.

Under the conditions of Corollary 1, the Bertino copula B_δ is an orbital semilinear copula that is both symmetric and opposite symmetric. Similarly, under the conditions of Corollary 2, the copula F_ω is also an orbital semilinear copula that is both symmetric and opposite symmetric.

6. CONCLUSIONS

We have introduced four new types of semilinear copulas and have derived necessary and sufficient conditions on given diagonal and opposite diagonal functions such that a copula of one of the considered types exists that has these functions as diagonal and opposite diagonal sections. The most interesting new copulas are the so-called orbital semilinear copulas which are obtained based on linear interpolation on segments connecting points on the diagonal and opposite diagonal of the unit square solely. The extreme copulas M and W are both orbital semilinear copulas, as well as the product copula Π. Moreover, the smallest copula whose diagonal section coincides with the diagonal section of the product copula and also the greatest copula whose opposite diagonal section coincides with the opposite diagonal section of the product copula turn out to be orbital semilinear copulas different from Π, as follows from Proposition 16.

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