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Kybernetika, Vol. 45 (2009), No. 3, 475--490

Persistent URL: `http://dml.cz/dmlcz/140019`

KYBERNETIKA — VOLUME 45 (2009), NUMBER 3, PAGES 475 – 490

ASYMPTOTIC PROPERTIES AND OPTIMIZATION OF SOME NON–MARKOVIAN STOCHASTIC PROCESSES

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We study the limit behavior of certain classes of dependent random sequences (processes) which do not possess the Markov property. Assuming these processes depend on a control parameter we show that the optimization of the control can be reduced to a problem of nonlinear optimization. Under certain hypotheses we establish the stability of such optimization problems.

Keywords: nonmarkovian control sequence, average cost, attracting point, nonlinear optimitation, stability

AMS Subject Classification: 62M09, 93E20

1. INTRODUCTION

The theory of discrete-time Markov control processes with the long-run average cost is a well developed topic now. The basic methods here are dynamic programming and the related technique of optimality equations, which essentially exploit the Markov structure of the processes (see, e. g. [2, 5, 6, 11]). However there are applied stochastic control processes where the asymptotic behavior depends heavily on both, the control parameters and the whole past trajectory of the process. As far as we know there do not exist general methods of control optimization for such models.

In this paper we offer two examples of stochastic processes of the mentioned type and study the asymptotic behavior and optimization of a particular and new class of non-markovian discrete-time processes. We consider only the simplest "control": the choice of a parameter which minimizes the average cost per unite time over an infinite horizon.

We will show that the optimization problem can be reduced to the minimization of a certain nonlinear function that represents the limit value of the cost. In turn, to find this function we need to determinate strong attracting points of a random sequence associated with the processes under consideration. The methods to study the convergence to such points are well known (for example, stochastic approximation, martingale method, the method of differential equations; see the book [1]). Nevertheless, in our case an additional difficulty arises from the fact that the points of attraction depend on the control parameter.

In the last section of the paper we investigate the stability (robustness) of our parameter optimization problem. Under certain conditions we give an upper bound for "the stability index" introduced in [3]. We also give an example where the optimization problem is not stable.

2. EXAMPLES OF APPLIED CONTROL MODELS RESULTING NON–MARKOVIAN STOCHASTIC PROCESSES

Example 1. Non-markovian model of inventory control. In many well known models of discrete time optimal inventory control it is assumed that the successive demands on goods are independent and identically distributed (i.i.d.) random variables or i.i.d. random vectors, see e. g. [8]. Such assumptions lead to the Markov property of the corresponding control processes.

Let us consider a stock of only one commodity, for example certain product in a supermarket, and let $X_0,\ X_1,\ X_2,\ldots$ be the nonnegative random variables that represent the consecutive weekly demand for the commodity. We suppose that at the beginning of every week the owner of the supermarket orders a fixed amount $a \geq 0$ of the commodity. The control parameter a should be chosen to minimize the long-run average losses, which is in fact minus the average of the revenues:

$$W(a) := \limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathrm{E}\left\{c_0[a-X_t)^+] - r[\min(X_t,a)]\right\}, \tag{2.1}$$

here $r(y)$ is the one-stage revenue collected by selling y units of the commodity, and $c_0(y)$ is the one stage losses due to the excess y of the commodity. (We are considering some perishable goods, which can not be on sale more than a week.) We suppose c_0 and r are nonnegative and continuous.

Remark 1. Let us assume that $a \in A := [\theta_0, \theta_1]$, $0 \leq \theta_0 < \theta_1 < \infty$. So we can add a suitable constant to (2.1) in order to pass to an equivalent optimization criterion with a nonnegative cost function.

If the demands $X_0, X_1, X_2, \ldots$ are i.i.d. then the minimization of W is a simple problem. However we are going to suppose that for all $t = 1, 2, \ldots$ the conditional distribution function F_t (given a history up to $t-1$) of the random variable X_t depends as a parameter on the frequency of the past unsatisfied demands. That is: $F_t = F_{Z_t}$, where $Z_t := \frac{1}{t}\sum_{k=0}^{t-1} I_{\{X_k > a\}}$, and F_z, $z \in [0,1]$ is a given family of distribution functions (on $[0,\infty)$) such that if $z_2 > z_1$, then the random variable with distribution F_{z_2} is "stochastically less" than the random variable with distribution F_{z_1}.

Remark 2. The stochastic ordering can be understood, for example as the inequality:

$$F_{z_2}(x) \geq F_{z_1}(x), \quad \text{for} \quad z_2 > z_1 \quad \text{and all} \quad x \geq 0. \tag{2.2}$$

The above assumptions reflect the fact that a customer who has came across several lacks of a needed product can switch to other supermarket. In this situation finding a stock level a_* that minimizes $W(a)$ in (2.1) is not a trivial task. In Section 4 we will point out how to tackle this.

Example 2. Message repeating in a damaging communication network. Taking into account the high capacity of modern communication channels, the following example refers to a hypothetical situation of an armed conflict between rather developed countries, where a significant part of the communication means such as satellites could be destroyed or suppressed electronically. It is known that in military communication networks, some important messages (commands) must be confirmed by the receiver. If a sender has no confirmation during a prescribed period $a \geq 0$, then he sends again the message. The reason for such duplication is that in a poor (damaged) packets switching network with a fixed routing a message can be lost or delayed an inadmissible time. We consider the simplified model where at discrete instants $t = 0, 1, 2, \ldots$ a sender emits messages and receives confirmations of its delivery after random delays denoted correspondingly by $X_0, X_1, X_2, \ldots$. For a chosen control parameter $a \geq 0$, if the sender has no confirmation on the tth message before the time $t + a$, he transmits a copy of it at the time $t + a$. The confirmation of the above copy delays Y_t units of time, where $Y_0, Y_1, Y_2, \ldots$ is a sequence of random variables having the same distribution as those of $X_0, X_1, X_2, \ldots$

We set the problem of optimal "control" as finding a parameter a_* that minimizes the following performance index:

$$W(a) := \limsup_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T-1} \mathrm{E}\big[\min\{X_t, a + Y_t\}\big]. \tag{2.3}$$

Let us note that $\min\{X_t, a + Y_t\}$ is the resulting delay of the confirmation on the tth message.

As in the Example 1, the assumption that X_0, X_1, $X_2, \ldots$ are independent and identically distributed makes the problem trivial with solution $a_* = 0$. This is not the case when we take into account that a copy is sent only if $X_t > a$, and we admit that a considerable number of users which share the channels of the network apply the same procedure of copy sending. One can see that in this case the increasing of the frequency of the message duplication raises the loading of the network. So it is reasonable to expect that in a damaged network this leads to an increase of message delays (for example, due to the growth of queues in pocket switching nodes, see [10]). We suppose that for each $t = 1, 2, \ldots$ the random variables X_t, Y_t are conditionally independent given the "history" X_0, $X_1, \ldots$, X_{t-1} and have the same conditional distribution function F_{Z_t}. The latter depends as a parameter on the value of the random variable

$$Z_t := \frac{1}{t} \sum_{k=0}^{t-1} I_{\{X_k > a\}},$$

the frequency of message repeating. Here F_z, $z \in [0, 1]$ is a given family of distribution functions defined on the interval $[0, \infty)$, such that if $z_2 > z_1$, then the

F_{z_2}-distributed random variable is (in some sense) stochastically greater than the F_{z_1}-distributed random variable. In Section 4, we consider the problem of minimization of $W(a)$ in (2.3) in the above outlined setting.

3. DEFINITION OF THE CLASS OF STOCHASTIC PROCESSES UNDER CONSIDERATION AND THE CORRESPONDING OPTIMIZATION PROBLEM

Let us assume that $A := [\theta_0, \theta_1]$, $0 \le \theta_0 < \theta_1 < \infty$ is a given set of admissible values of a "control parameter" a. We also suppose that F_z, $z \in [0,1]$ is a family of distribution functions of nonnegative random variables, and $n \ge 1$ is a fixed nonnegative integer, and we interpret $t = 0, 1, 2, \ldots$ as the (discrete) time.

For an arbitrary but fixed $a \in A$ we define the n-dimensional "controlled" process

$$\overline{X}_t = \left(X_t^{(1)}, X_t^{(2)}, \ldots, X_t^{(n)} \right), \quad t = 0, 1, 2, \ldots$$

by means of the following recursive method.

1. Let $z_0 \in [0,1]$ be fixed and let $\left\{ X_0^{(1)}, \ldots, X_0^{(n)} \right\}$ be i.i.d. random variables with distribution function F_{z_0}.

2. For a $t \ge 1$, if we have observed the "part of the trajectory" $X_0^{(1)}, X_1^{(1)}, \ldots, X_{t-1}^{(1)}$, we calculate

$$Z_t := \frac{1}{t} \sum_{k=0}^{t-1} I_{\{X_k^{(1)} > a\}}, \tag{3.1}$$

and let the random variables $X_t^{(1)}, \ldots, X_t^{(n)}$ be conditionally (given Z_t) i.i.d. with the conditional distribution function F_{Z_t}.

That is, for $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$,

$$\begin{aligned} &\Pr\left(X_t^{(1)} \le x_1, X_t^{(2)} \le x_2, \ldots, X_t^{(n)} \le x_n \,/\, \overline{X}_0, \overline{X}_1, \ldots, \overline{X}_{t-1} \right) \\ &= \Pr\left(X_t^{(1)} \le x_1, X_t^{(2)} \le x_2, \ldots, X_t^{(n)} \le x_n \,/\, Z_t \right) \\ &= F_{Z_t}(x_1) F_{Z_t}(x_2) \ldots F_{Z_t}(x_n). \end{aligned} \tag{3.2}$$

Remark 3. As we have explained in the Examples 1 and 2 (considering the cases with $n = 1$ and $n = 2$), the definition in 2 means that the conditional distribution of the random vector $\overline{X}_t$, given a history of the process up to time t, is determined by the family of distribution functions F_z, $z \in [0,1]$ and by the frequency of the "past events" pointed out in (3.1).

Now we suppose that it is given a measurable one-stage cost function:

$$c : A \times [0, \infty)^n \to [0, \infty),$$

and introduce the performance index:

$$W(a) := \limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathrm{E}c\left(a, X_t^{(1)}, \ldots, X_t^{(n)}\right), \tag{3.3}$$

which is the average cost per unit of time over infinite horizon.

Under the assumptions that will be given on the next section we have that for all $a \in A$, $W(a) < \infty$. It makes sense to set the following optimization problems:

(a) To find

$$W_* := \inf_{a\in A} W(a); \tag{3.4}$$

(b) To find $a_* \in A$ such that

$$W(a_*) = W_*,$$

provided that such an "optimal control" a_* exists.

4. ASYMPTOTIC DISTRIBUTIONS OF $\overline{X}_T$ AND CALCULATION OF $W(A)$

We can perform our analysis for $n = 2$ without any loss of generality. For a moment we fix $a \in A$ and denote:

$$\varphi_a(z) := 1 - F_z(a), \quad z \in [0,1]. \tag{4.1}$$

Assumption 1.

(a) The second derivative $\varphi_a''(z)$ exists and is bounded on $z \in [0,1]$.

(b) For $z \in [0,1]$, $\varphi_a'(z) < 1$, and also for each $x \in [0,\infty)$ the function $z \to F_z(x)$ is continuous on $[0,1]$.

Assumption 1 yields that the equation $\varphi_a(z) = z$ has in $[0,1]$ a unique root, which will be denoted in what follows as $z_* = z_*(a)$.

Remark 4.

(a) In Example 1 of Section 2, supposing that the applications $z \to F_z(x)$ are smooth enough, condition (2.2) provides that $\varphi_a'(z) \le 0$.

(b) In Example 2 of Section 2 we have supposed something as decreasing of values of $F_z(x)$ as z increases. Thus under smoothness condition, to meet Assumption 1 (b), this decreasing should be "not too fast" (see the below Figures 1 and 2).

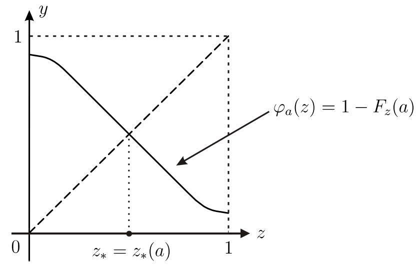

Fig. 1. A possible behaviour of $\varphi_a(z)$ in Example 1.

(c) As an example of the family of distribution functions $F_z(x)$, $z \in [0,1]$ that satisfies Assumption 1, as well as Assumptions 3 and 4, one can consider

$$F_z(x) = \begin{cases} 1 - e^{-\lambda(z)x}, & x > 0, \\ 0, & x \le 0. \end{cases}$$

Here $\lambda \in C^2[0,1]$ is any positive function such that $\mid \lambda'(z) \mid \le \alpha$, $z \in [0,1]$ for some $\alpha < 1$.

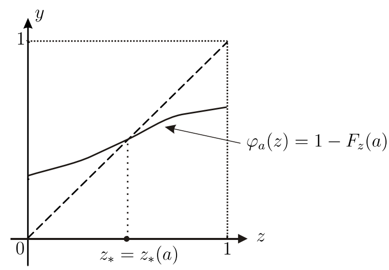

Fig. 2. A possible behaviour of $\varphi_a(z)$ in Example 2.

Proposition 1. Suppose that Assumption 1 holds. Then

(1)
$$Z_t \xrightarrow{\text{a.s.}} z_* \text{ as } t \to \infty \tag{4.2}$$

(2)
$$\overline{X}_t \xrightarrow{\text{weakly}} \overline{X}_* \text{ as } t \to \infty, \tag{4.3}$$

where $\overline{X}_* = \left(X_*^{(1)}, X_*^{(2)}\right)$, and $X_*^{(1)}, X_*^{(2)}$ are i.i.d. random variables with the distribution function F_{z_*}.

Assumption 2.

(a) The function $c(a,x_1,x_2)$ is continuous in $(x_1,x_2)\in[0,\infty)\times[0,\infty)$.

(b) There exist finite constants K_1,K_2 such that

$$\sup_{a\in A} c(a,x_1,x_2)\le K_1+K_2(x_1+x_2),\ x_1,x_2\in[0,\infty). \tag{4.4}$$

(c) There exist $p>1$, $K<\infty$ such that

$$\sup_{z\in[0,1]}\int_0^\infty x^p \mathrm{d}F_z(x)\le K. \tag{4.5}$$

Proposition 2. Suppose that Assumptions 1 and 2 hold. Then

$$W(a)=\int_0^\infty\int_0^\infty c(a,x_1,x_2)\,\mathrm{d}F_{z_*(a)}(x_1)\,\mathrm{d}F_{z_*(a)}(x_2)<\infty. \tag{4.6}$$

Corollary 1. Let Assumption 1 and 2 hold for every $a\in A$. Then:

$$W_*=\inf_{a\in A}\int_0^\infty\int_0^\infty c(a,x_1,x_2)\,\mathrm{d}F_{z_*(a)}(x_1)\,\mathrm{d}F_{z_*(a)}(x_2).$$

The existence of an "optimal control" a_* such that $W(a_*)=W_*$ would follow, for example, from the continuity of the function $W(a)$, $a\in A$ in (4.6).

Assumption 3.

(a) The function c is continuous on $A\times[0,\infty)\times[0,\infty)$.

(b) $|\varphi_a'(z)|<1$ for each $z\in[0,1]$, $a\in A$.

(c) For each $z\in[0,1]$, F_z has a density f_z such that

$$\sup_{z\in[0,1]}\sup_{x\in A} f_z(x)=:M_0<\infty. \tag{4.7}$$

Proposition 3. Under Assumptions $1,2$ and 3 the function W in (4.6) is continuous on A.

5. ESTIMATING THE STABILITY IN THE OPTIMIZATION PROBLEM

We proceed from the setting suggested in [3] and developed in the case of the average cost, for instance, in [4, 7] to deal with the mentioned "stability". We suppose that the family of distribution functions F_z, $z\in[0,1]$ is known only approximately, i.e. in place of $\{F_z\}$ we dispose an approximation $\widetilde{F}_z$, $z\in[0,1]$.

Now we assume that the last family satisfies Assumptions 1 and 2 for every $a \in A$ and that there is a parameter $\widetilde{a}_*$ minimizing the cost:

$$\widetilde{W}(a) := \int_0^\infty \int_0^\infty c(a, x_1, x_2)\, \mathrm{d}\widetilde{F}_{\widetilde{z}_*(a)}(x_1)\, \mathrm{d}\widetilde{F}_{\widetilde{z}_*(a)}(x_2),\ a \in A \tag{5.1}$$

in the "approximating problem" determined by $\widetilde{F}_z, \quad z \in [0,1]$.

In (5.1) $\widetilde{z}_* = \widetilde{z}(a)$ is the unique root of the equation $1 - \widetilde{F}_z(a) = z$.

Remark 5. To guarantee the existence of $\widetilde{a}_*$ it suffices to impose on c and $\widetilde{F}_z$, $z \in [0,1]$ the conditions of Proposition 3.

We analyze now the case when the parameter $\widetilde{a}_*$ can be found (at least theoretically), and it is applied to the "original" process determined by the family $\{F_z\}$ (see (3.1), (3.2)). If we use the index

$$\Delta := W(\widetilde{a}_*) - W_* \geq 0, \tag{5.2}$$

as a measure of "stability", we have the following two questions:

- Can Δ be bounded by the disturbance term

$$\delta := \sup_{z\in[0,1]} \sup_{x \geq 0} |F_z(x) - \widetilde{F}_z(x)|\ ? \tag{5.3}$$

- Is it possible that Δ does not approach to zero despite of $\delta \to 0$?

Depending on the conditions imposed the answer to each above questions can be positive.

In order to fix ideas and simplify computations we only consider the particular case when $\overline{X}_0, \overline{X}_1, \overline{X}_2, \dots$ are one dimensional, i.e. $n = 1$ in (3.2).

Assumption 4.

(a) There exists a constant $L < \infty$ such that for all $x, y \in [0, \infty),\ a \in A$

$$|c(a,x) - c(a,y)| \leq L|x - y|.$$

(b) There exists a constant $\alpha < 1$ such that for all $z \in [0,1]$, and $x \geq 0$

$$\left| \frac{\partial}{\partial z} F_z(x) \right| \leq \alpha$$

Remark 6. Assumption 4 (b) strengthens both Assumption 1 (b) and Assumption 3 (b).

Proposition 4. Suppose that the Assumptions 1, 2 and 4 hold with (x_1, x_2) being replaced by x, and that for the approximating family $\{\widetilde{F}_z\}$ the Assumptions 1 and 2 are satisfied and there exists the above defined optimal $\widetilde{a}_*$. Then

$$\Delta \le M(p)\delta^{1-\frac{1}{p}}, \tag{5.4}$$

where δ was defined in (5.3) and

$$M(p) = 2L(1-\alpha)^{1/p-1}\left(\frac{2K}{p}\right)^{1/p}, \tag{5.5}$$

where the constants $p > 1$ and $K < \infty$ were defined in (4.5).

The next example shows that in Proposition 4, Assumption 4 (b) is essential.

Example 3. An unstable optimization problem for $n = 1$. Let us assume that $\gamma : [0,1] \to [1,\infty)$, and $\beta : [0,1] \to (0,\infty)$ are $C^2[0,1]-$functions, and we define the functions

$$b(z) := \frac{\gamma(z)-1}{\beta(z)}, \tag{5.6}$$

$$d(z) := \frac{\gamma(z)}{\beta(z)}, \tag{5.7}$$

We specify particular functions γ and β in such a way that the graphs of $\varphi_a(z)$, $z \in [0,1]$ defined below in (5.8) have the qualitative behavior as it is shown on Figure 3 (with $\varphi_a'(z) < 1$ for $a \neq 2$).

The family $\{\varphi_a(z)\}$ of functions indexed by $a \in [1.9, 2.1]$ is given by the following equality:

$$\varphi_a(z) := \gamma(z) - a\beta(z), \quad z \in [0,1]. \tag{5.8}$$

For any $z \in [0,1]$ let F_z be the uniform distribution on the interval $[b(z), d(z)] \subset [0,\infty)$.

Also we choose a one-stage cost function (in (3.3)) as

$$c(a,x) := a^3 + x, \;\; a \in [1.9, 2.1], \;\; x \ge 0. \tag{5.9}$$

From (5.6) – (5.8) it is easy to see that for each

$$a \in [1.9, 2.1], \; z \in [0,1], \;\; \text{we have that } \; 1 - F_z(a) = \varphi_a(z), \tag{5.10}$$

and this corresponds to the definition of $\varphi_a(z)$ in (4.1).

For every $a \neq 2$ it is not hard to check the fulfilment of Assumption 1, Assumption 2, Assumption 3 (a), (c) and Assumption 4 (a). However in this example Assumption 3 (b) and Assumption 4 (b) are not satisfied. We show that the latter leads to the discontinuity of the function $W(a)$, $a \in [1.9, 2.1]$ in the one dimensional analog to (4.6) and to the lack of stability in the corresponding optimization problem over $a \in [1.9, 2.1]$.

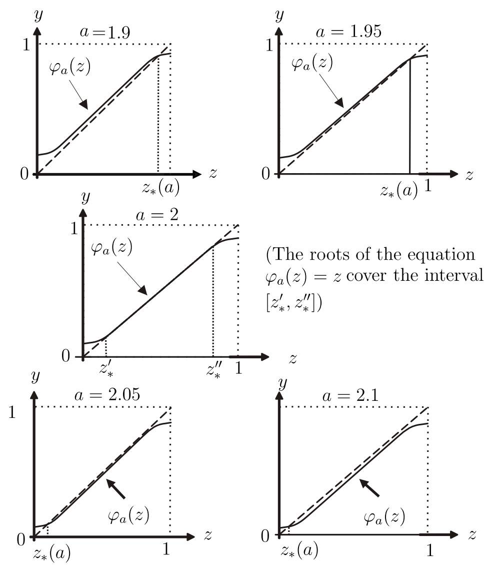

Fig. 3. The graphs of the function $\varphi_a(z)$, $z \in [0, 1]$ in Example 3.

By Proposition 2 and (5.9) we obtain for $a \neq 2$ that

$$W(a) = a^3 + \frac{1}{2}\left[b(z_*(a)) + d(z_*(a))\right],$$

where $z_*(a)$ is a root of the equation $\varphi_a(z) = z$.

Calculating $z_*(a)$ and $W(a)$ numerically we get the following graph (see also the above graphs of $\varphi_a(z)$):

Thus $W_* = \inf_{a\in[1.9,2.1]} W(a) = \lim_{a\to 2+} W(a)$.

Now, since for small enough $\varepsilon > 0$ we have that (5.8) defines properly $\varphi_a(z)$ for $a \in [1.9, 2.1 + \varepsilon]$, we let

$$\widetilde{\varphi}_a(z) := \varphi_{a+\varepsilon}(z) = \gamma(z) - (a + \varepsilon)\beta(z),\ z \in [0, 1],\ a \in [1.9, 2.1],$$

and similarly to (5.6), (5.7) we define:

$$\widetilde{b}(z) = \frac{\gamma(z) - 1}{\beta(z)} - \varepsilon,$$

$$\widetilde{d}(z) = \frac{\gamma(z)}{\beta(z)} - \varepsilon.$$

Now let the approximation family $\widetilde{F}_z$, $z \in [0, 1]$ be the uniform distribution on $[\widetilde{b}(z), \widetilde{d}(z)]$. For this family $1 - \widetilde{F}_z(a) = \widetilde{\varphi}_a(z)$, and the conditions of Proposition 2 hold for $a \neq 2 - \varepsilon$. (For $a = 2 - \varepsilon$ the equation $\widetilde{\varphi}_a(z) = z$ has a continuum of roots.)

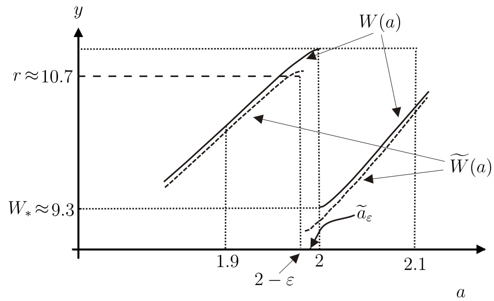

Fig. 4. The graphs of the average costs $W(a)$ and $\widetilde{W}(a)$, $a \in [1.9, 2.1]$.

Thus we can use Proposition 2 and the one dimensional analog of (5.1) to calculate $\widetilde{W}(a)$, $a \in [1.9, 2.1]$ shown as a dotted line in Figure 4. The infimum of W is obtained as $a \to (2-\varepsilon)+$.

Replacing in (5.2) $\widetilde{a}_*$ by any $\widetilde{a}_\varepsilon$ such that

$$\widetilde{W}(\widetilde{a}_\varepsilon) - \inf_{a \in [1.9,2.1]} \widetilde{W}(a) < \varepsilon,$$

$a_\varepsilon < 2$, we conclude that for each (small enough) $\varepsilon > 0$,

$$\Delta_\varepsilon := W(\widetilde{a}_\varepsilon) - W_* = r - W_* \geq 1.2$$

On the other hand, it is almost evident that for the current example in (5.3) $\delta \to 0$ as $\varepsilon \to 0$.

Remark 7. To obtain the equation (4.6) we use the almost surely convergence in (4.2). The computer simulation made to evaluate the rate of convergence have shown that if the values of $\varphi'_a(z)$ are "significantly less" than 1, then $Z_t,\ t = 1, 2, \ldots$ defined in (3.1) approach the root $z_*(a)$ "fast enough" (with t being several hundred Z_t is "very close" to $z_*(a)$).

On the other hand, when in some neighborhood of $z_*(a)$ we have $\varphi'_a(z) < 1$, but $\varphi'_a(z) \approx 1$ (as in the case $a = 1.95$ in Figure 3), then Z_t converges to $z_*(a)$ "very slowly" (even values of t of order 30 000 do not provide a good approximation).

6. PROOFS

The proof of the first part of Proposition 1 follows from Theorem 9.2.11 in [1], the simplified version of which we cite for reference convenience.

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\{\mathcal{F}_t, t = 0, 1, 2, \ldots\} \subset \mathcal{F}$, and let $\{\xi_t, t = 1, 2, \ldots\}$, $\{\gamma_t, t = 0, 1, \ldots\}$ be sequences of random variables adapted to $\{\mathcal{F}_t\}$, such that, $\gamma_t \geq 0$, $\sum_{t=0}^{\infty} \gamma_t = \infty$, $\sum_{t=0}^{\infty} \gamma_t^2 < \infty$ almost surely.

Let also $\Gamma \subset \mathbb{R}$ be a bounded closed interval and $h : \Gamma \to \mathbb{R}$ be a continuous function such that for any $z_0 \in \Gamma$,

$$Z_{t+1} = Z_t + \gamma_t h(Z_t) + \gamma_t \xi_{t+1} \in \Gamma \tag{6.1}$$

for $t = 1, 2, \ldots$.

Theorem. (Duflo [1]) Suppose that there exists a function $V \in C^1(G)$, where $G \supset \Gamma$ is an open interval, such that:

(a) $[h(z)]^2 \leq \text{const}[V(z)+1],\ z \in \Gamma;$

(b) $|V'(z) - V'(y)| \leq \text{const}|z-y|,\ z, y \in \Gamma;$

(c) $\mathrm{E}[\xi_{t+1}|\mathcal{F}_t] = 0$, $\mathrm{E}[\xi_{t+1}^2|\mathcal{F}_t] = O[V(Z_t)+1]$; for $t = 0, 1, 2, \ldots$ almost surely;

(d) For some $z_* \in \Gamma$, $h(z_*) = 0$, $V(z_*) = 0$ and $V'(z)h(z) < 0$, $V(z) > 0$ for $z \in \Gamma \backslash \{z_*\}$.

Then

$$Z_t \xrightarrow{\text{a.s.}} z_* \quad \text{as} \quad t \to \infty.$$

To prove Proposition 1 we first rewrite Z_{t+1} in (3.1) as follows:

$$\begin{aligned}
Z_{t+1} &= \frac{1}{t+1}\sum_{k=0}^{t} I_{\{X_k^{(1)}>a\}} = \frac{t}{t+1}Z_t + \frac{1}{t+1}I_{\{X_t^{(1)}>a\}} \\
&= Z_t - \frac{1}{t+1}Z_t + \frac{1}{t+1}I_{\{X_t^{(1)}>a\}} - \frac{1}{t+1}\Pr(X_t^{(1)} > a \,/\, Z_t) \\
&\quad + \frac{1}{t+1}\Pr(X_t^{(1)} > a \,/\, Z_t) \\
&= Z_t + \tfrac{1}{t+1}[\varphi_a(Z_t) - Z_t] + \frac{1}{t+1}\left[I_{\{X_t^{(1)}>a\}} - \varphi_a(Z_t)\right].
\end{aligned} \tag{6.2}$$

We set the following elements to apply the above theorem:

- $\mathcal{F}_t := \sigma(X_0^{(1)}, X_1^{(1)}, \ldots, X_{t-1}^{(1)});$
- $\Gamma = [0,1]$ and $G \supset [0,1]$ an open interval;
- $h(z) := \varphi_a(z) - z, \qquad z \in [0,1];$ (6.3)
- $\gamma_t := \frac{1}{t+1};$
- $\xi_{t+1} := I_{\{X_t^{(1)}>a\}} - \varphi_a(Z_t);$ (6.4)
- $V(z) := [\varphi_a(z) - z]^2,$ (6.5)

(where V is extended to G in any smooth way).

The conditions (a) and (b) of the Theorem hold because of (6.5) and Assumption 1 (a). Regarding the condition (c), we have (see (6.2), (6.4) and (3.2)):

$$\mathrm{E}[\xi_{t+1}/\mathcal{F}_t] = \mathrm{E}[(I_{\{X_t^{(1)}>a\}} - \varphi_a(Z_t))\,/\,\mathcal{F}_t] = \varphi_a(Z_t) - \varphi_a(Z_t) = 0.$$

Also $0 \le \xi_{t+1} \le 1$, $t = 0, 1, 2, \ldots$. Finally, let z_* be a root of the equation $\varphi_a(z) = z$ which is unique by Assumption 1.

Then from (6.3) and (6.5) we get that $h(z_*) = 0$, $V(z_*) = 0$, $V(z) > 0$, $z \ne z_*$ and $V'(z)h(z) = 2[\varphi_a(z) - z]^2(\varphi_a'(z) - 1) < 0$ for $z \ne z_*$ by the Assumption 1 (b). Now (4.2) in Proposition 1 follows from the above Theorem.

To prove (4.3) it suffices to observe that for any $x_1, x_2 \in [0, \infty)$ by (3.2), (4.2), Assumption 1 (b) and the bounded convergence theorem we have:

$$\Pr(X_t^{(1)} \le x_1, X_t^{(2)} \le x_2) = \mathrm{E}\{F_{Z_t}(x_1)F_{Z_t}(x_2)\} \to F_{z_*}(x_1)F_{z_*}(x_2). \qquad \square$$

Proof of Proposition 2. Taking into account (3.3) it is enough to show that as $t \to \infty$,

$$\mathrm{E}c(a, X_t^{(1)}, X_t^{(2)}) \to \mathrm{E}c(a, X_*^{(1)}, X_*^{(2)}), \tag{6.7}$$

where the i.i.d. random variables $X_*^{(1)}$ and $X_*^{(2)}$ have distribution function F_{z_*}.

From (4.4) we have:

$$\big[c(a, x_1, x_2)\big]^p \le B(p)\big[1 + (x_1^p + x_2^p)\big],$$

where $B(p) = B(p, K_1, K_2)$ is a finite constant.

Also in view of (4.5), for $i = 1, 2$,

$$\begin{aligned} \mathrm{E}[X_t^{(i)}]^p &= \mathrm{E}\Big[\mathrm{E}\Big\{[X_t^{(i)}]^p\,/\,Z_t\Big\}\Big] = \mathrm{E}\int_0^\infty x^p \mathrm{d}F_{Z_t}(x) \\ &\le \sup_{z\in[0,1]} \int_0^\infty x^p \mathrm{d}F_z(x) \le K. \end{aligned}$$

Therefore $\sup_{t\ge 0} \mathrm{E}\big[c(a, X_t^{(1)}, X_t^{(2)})\big]^p < \infty$.

Now from (4.3), Assumption 2 (a) and the well-known sufficient condition of convergence of expectations, we obtain (6.7). $\square$

Proof of Proposition 3. For an arbitrary but fixed $a \in A$, we choose a sequence $\{a_n\}_{n\ge 1} \in A$ such that $a_n \to a$ as $n \to \infty$. As previously, let $z_* = z_*(a)$ and $z_n = z_n(a)$ be the roots of, respectively, the equations $\varphi_a(z) = z$ and $\varphi_{a_n}(z) = z$. We remark that the uniqueness of the roots follows from Assumption 1. Thus

$$\begin{aligned} |z_n - z| &= |\varphi_{a_n}(z_n) - \varphi_a(z)| \le |\varphi_a(z) - \varphi_a(z_n)| \\ &\quad + |\varphi_a(z_n) - \varphi_{a_n}(z_n)| \le \alpha|z - z_n| + |a - a_n|M_0, \end{aligned} \tag{6.8}$$

where M_0 is the constant from (4.7), and $\alpha = \alpha(a) := \sup_{z\in[0,1]} |\varphi_a'(z)| < 1$ by Assumption 3 (b) and Assumption 1 (a).

From (6.8) we get:

$$|z_n - z| \leq \frac{M_0}{1-\alpha} \mid a - a_n|. \tag{6.9}$$

Let us denote by $X^{(1)}_{*,n}, X^{(2)}_{*,n}$ two i.i.d. random variables with the distribution function F_{z_n}, and as previously, $X^{(1)}_*, X^{(2)}_*$ be i.i.d. random variables with the distribution function F_{z_*}. Using (6.9), Assumption 1 (b) we see that

$$(X^{(1)}_{*,n}, X^{(2)}_{*,n}) \overset{\text{weakly}}{\longrightarrow} (X^{(1)}_*, X^{(2)}_*) \qquad (\text{as } n \to \infty). \tag{1}$$

Moreover, interpreting a_n, a as degenerative random variables and making use of Assumption 3 (a) we see that $c(a_n, X^{(1)}_{*,n}, X^{(2)}_{*,n}) \overset{\text{weakly}}{\longrightarrow} c(a, X^{(1)}_*, X^{(2)}_*)$ (as $n \to \infty$). Here we do again the procedure of the previous proof and use the Assumption 2 (b) and (c) to get:

$$\sup_{n\geq 1} \mathrm{E}\big[c(a_n, X^{(1)}_{*,n}, X^{(2)}_{*,n})\big]^p < \infty.$$

Therefore

$$\mathrm{E}c(a_n, X^{(1)}_{*,n}, X^{(2)}_{*,n}) \longrightarrow \mathrm{E}c(a, X^{(1)}_*, X^{(2)}_*),$$

or in view of (4.6) $W(a_n) \to W(a)$. □

P r o o f o f P r o p o s i t i o n 4. From the definitions in (5.2) we have:

$$\begin{aligned} \Delta = W(\widetilde{a}_*) - \inf_{a\in A} W(a) &\leq |W(\widetilde{a}_*) - \widetilde{W}(\widetilde{a}_*)| + |\inf_{a\in A} \widetilde{W}(a) - \inf_{a\in A} W(a)| \\ &\leq 2\sup_{a\in A} |W(a) - \widetilde{W}(a)|, \end{aligned} \tag{6.10}$$

where analogously to (4.6) and (5.1) we have

$$W(a) = \mathrm{E}c(a, X_*),\ \widetilde{W}(a) = \mathrm{E}c(a, \widetilde{X}_*),$$

and the random variables X_*, $\widetilde{X}_*$ have, respectively the distribution functions F_{z_*} and $\widetilde{F}_{\widetilde{z}_*}$.

Let $a \in A$ be arbitrary but fixed. Using Assumption 4 (a) and the two equivalent definitions of the Kantorovich distance

$$\ell(F, G) := \int_{-\infty}^{\infty} |F(x) - G(x)|\, \mathrm{d}x$$

between the distribution functions F and G, we can see (consult [9], Chapt. 8, for details), that

$$\big|\mathrm{E}c(a, X_*) - \mathrm{E}c(a, \widetilde{X}_*)\big| \leq L \int_0^{\infty} \big|F_{z_*}(x) - \widetilde{F}_{\widetilde{z}_*}(x)\big|\, \mathrm{d}x \tag{6.11}$$

For arbitrary $b \geq 0$,

$$
\begin{aligned}
I \quad &:= \quad \int_0^\infty |F_{\widetilde{z}_*}(x) - \widetilde{F}_{\widetilde{z}_*}(x)|\,\mathrm{d}x \leq \int_0^b |F_{z_*}(x) - \widetilde{F}_{\widetilde{z}_*}(x)|\,\mathrm{d}x \\
&\quad + \frac{1}{b^{p-1}} \int_b^\infty x^{p-1} \big|(1 - F_{z_*}(x)) - (1 - \widetilde{F}_{\widetilde{z}_*}(x))\big|\,\mathrm{d}x \\
&\leq \quad \int_0^b |F_{z_*}(x) - \widetilde{F}_{\widetilde{z}_*}(x)|\,\mathrm{d}x + \frac{1}{b^{p-1}} \\
&\quad \times \left[\int_0^\infty x^{p-1}(1 - F_{z_*}(x))\,\mathrm{d}x + \int_0^\infty x^{p-1}(1 - \widetilde{F}_{\widetilde{z}_*}(x))\,\mathrm{d}x \right].
\end{aligned}
\tag{6.12}
$$

We integrate by parts and use (4.5) to find that the second term on the right-hand side of (6.12) is less than $\dfrac{2}{pb^{p-1}}K$.

The next task is to bound the first summand on the right-hand side of (6.12). We observe that by Assumption 4 (b)

$$
\begin{aligned}
&|F_{z_*}(x) - \widetilde{F}_{\widetilde{z}_*}(x)| \leq |F_{z_*}(x) - F_{\widetilde{z}_*}(x)| + |F_{\widetilde{z}_*}(x) - \widetilde{F}_{\widetilde{z}_*}(x)| \\
&\leq \alpha |z_* - \widetilde{z}_*| + \delta = \alpha |F_{z_*}(a) - \widetilde{F}_{\widetilde{z}_*}(a)| + \delta,
\end{aligned}
\tag{6.13}
$$

where δ was defined in (5.3).

Letting $\varepsilon := \sup_{x \geq 0} |F_{z_*}(x) - \widetilde{F}_{\widetilde{z}_*}(x)|$, we see from (6.13) that

$$
\varepsilon \leq \alpha\varepsilon + \delta, \quad \text{or} \quad \varepsilon \leq \frac{\delta}{1-\alpha}.
$$

Therefore we can strengthen inequality (6.12) as follows:

$$
I \leq \frac{b}{1-\alpha}\delta + \frac{2}{pb^{p-1}}K. \tag{6.14}
$$

We choose $b = \left[\frac{2(1-\alpha)K}{\delta p}\right]^{1/p}$ to make both summands in the last inequality equal. Finally, comparing (6.10), (6.11) and (6.14) we obtain the desired inequality (5.4). $\square$

ACKNOWLEDGEMENT

The authors would like to thank the referees for their valuable suggestions to improve the text of the paper. The research of the first author was supported by SEP-CONACyT, under grant CB-2005-C01-49854-F and PIFI 2007, UAM-I-CA-56. The second author was partially supported by PIFI 2007, UAM-I-CA-55. The third author was partially supported by PIFI 2007, UAM-I-CA-56.

(Received December 11, 2007.)

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