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A NOTE ABOUT OPERATIONS LIKE T_W (THE WEAKEST t–NORM) BASED ADDITION ON FUZZY INTERVALS

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We investigate a relation about subadditivity of functions. Based on subadditivity of functions, we consider some conditions for continuous t-norms to act as the weakest t-norm T_W-based addition. This work extends some results of Marková-Stupňanová [15], Mesiar [18].

Keywords: fuzzy arithmetics, fuzzy intervals, triangular norms

AMS Subject Classification: 93E12, 62A10, 62F15

1. INTRODUCTION

Fuzzy arithmetic has grown in importance during recent years as a tool of advance in fuzzy optimization and control theory. The usual arithmetic operations of reals can be extended to the arithmetical operations on fuzzy intervals by means of Zadeh's extension principle [20] based on a triangular norm T. Fuzzy arithmetic based on the sup-(t-norm) convolution, with the controllability of the increase of fuzziness, enables us to construct more flexible and adaptable mathematical models in several intelligent technologies based on approximate reasoning and fuzzy logic. Hence a lot of effort is needed to find exact and good approximative computational formulas for fuzzy arithmetic operations. Some results on fuzzy arithmetic operations and their applications can be found in [1 – 18]. In this note, we are interested in some conditions of t-norm under which the addition of LR-fuzzy intervals act exactly as the T_W-based addition. Some results on the continuous t-norm based additions of fuzzy intervals which act exactly as the T_W-based addition can be found in [1, 9, 15, 18]. We investigate a relation of of subadditivity of functions. We consider some extensions of the result of Marková-Stupňanová [15], Mesiar [18] and Hong [9].

A function $T : [0,1] \times [0,1] \to [0,1]$ is said to be a t-norm [1] iff T is symmetric, associative, non-decreasing in each argument, and $T(x,1) = x$ for all $x \in [0,1]$. For arbitrary fuzzy quantities A_i, $i = 1,2,\dots,n$, $n \in N$, their T-sum is defined by means of the generalized extension principle [1].

$$A_1 \oplus_T \cdots \oplus_T A_n(z) = \sup_{\sum x_i = z} T(A_1(x_1), \dots, A_n(x_n)), \quad z \in \mathbb{R}, \tag{1}$$

where the usual extension of T to an n-ary operation is used.

If $T_1 \leq T_2$(the usual order of t-norm as two-place function), then for any fuzzy quantity A and B it is $A \oplus_{T_1} B \leq A \oplus_{T_2} B$. Let T_W denote the weakest t-norm defined by

$$T_W(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and let T_M denote the strongest t-norm defined by $T_M(x,y) = \min(x,y)$ for all $x, y \in [0,1]$. Consequently, for any t-norm T it is

$$A \oplus_W B \leq A \oplus_T B \leq A \oplus_M B.$$

A fuzzy quantity A is called fuzzy interval if it is continuous and for each $\alpha \in (0,1]$, the corresponding α-cut $A^\alpha = \{x \in \mathbb{R}; A(x) \geq \alpha\}$ is a non-empty convex closed subset of $\mathbb{R}$. If the support of A, $\operatorname{supp} A = \cup A^\alpha$, is bounded in $\mathbb{R}$, then the fuzzy interval A is so-called LR-fuzzy interval. If, for an LR-fuzzy interval A, $A^1 = [l_A, r_A]$ is a singleton (i. e., $l_A = r_A$) , then A is called an LR-fuzzy number.

Let $\mathcal{J}$ be the class of all fuzzy quantities defined on $[0,\infty]$ which are continuous non-increasing with the strict maximum 1 in the point 0 and its members be called shapes. Any fuzzy interval A can be written as a quadruple

$$A = (l_A, \gamma_A, A_*, A^*)$$

where A_* and A^* are defined by $A^*(x) = A(x + r_A)$ and $A_*(x) = A(l_A - x)$, (if $l_A = -\infty$, then $A_* = \phi$; similarly if $r_A = \infty$ then $A^* = \phi$; otherwise A_*, $A^* \in \mathcal{J}$),

$$A(x) = \begin{cases} 1 & \text{if } x \in [l_A, r_A], \\ A^*(x - r_A) & \text{if } x > r_A, \\ A_*(l_A - x) & \text{if } x < l_A. \end{cases}$$

It is known [3] that the T-sum of fuzzy intervals is defined by

$$\begin{aligned} A \oplus_T B &= (l_A, r_A, A_*, A^*) \oplus_T (l_B, r_B, B_*, B^*) \\ &= (l_A + l_B, r_A + r_B, A_* \oplus_T B_*, A^* \oplus_T B^*), \end{aligned}$$

where, by the convention, $\infty + x = \infty$ and $-\infty + x = -\infty$ for all $x \in \mathbb{R}$, and for any element S from $\mathcal{J} \cup \{\phi\}$, $S \oplus_T \phi = S$.

Following that above argument, it suffices to study only about T-sums of shapes instead of fuzzy intervals.

2. T_W BASED ADDITION ON FUZZY INTERVALS

For the weakest of t-norm T_W, we have the following result directly from the extension principle.

Theorem 1. Let $S_i \in \mathcal{J}$, $i = 1, \ldots, n$, $n \in N$. Then

$$S_1 \oplus_W \cdots \oplus_W S_n = \max(S_1, \ldots, S_n).$$

Corollary 1. Let $A_i = (l_i, r_i, S_i, R_i)$, $i = 1, \dots, n$, $n \in N$ be fuzzy intervals. Then

$$A_1 \oplus_W \cdots \oplus_W A_n = \left(\sum l_i, \sum r_i, \max S_i, \max R_i\right).$$

Recall that a continuous t-norm T is Archimedean iff $T(x,x) < x$ for all $x \in (0,1)$. Every Archimedean t-norm T is representable by a continuous and decreasing function $f : [0,1] \to [0,\infty]$ with $f(1) = 0$ and $T(x,y) = f^{[-1]}(f(x) + f(y))$, where $f^{[-1]}$ is the pseudo-inverse of f, defined by

$$f^{[-1]}(y) =: \begin{cases} f^{-1}(y) & \text{if } y \in [0, f(0)], \\ 0 & \text{if } y \in [f(0), \infty]. \end{cases}$$

The function f is the additive generator of T. If f is bounded, then it can be chosen uniquely so that $f(0) = 1$ and the corresponding t-norm T is called a nilpotent t-norm. If f is unbounded, the corresponding t-norm T is called a strict t-norm. For two fuzzy quantities A and B, their T-sum defined by (1) can be written into

$$A \oplus_T B(z) = \sup_{x+y=z} f^{[-1]}(f(A(x)) + f(B(y))). \tag{2}$$

The following theorem is due to Marková-Stupňanová [15] and Mesiar [18, Theorem 4] and gives some sufficient conditions for a t-norm T to act as the T_W-based addition.

Theorem 2. (Marková-Stupňanová [15], Mesiar [18]) Let T be a continuous Archimedean t-norm (strict or nilpotent)with additive generator f. Let $S_i \in \mathcal{J}$, $i = 1, \dots, n$, $n \in N$, be shapes such that all composites $f \circ S_i$ are concave functions. Then

$$S_1 \oplus_T \cdots \oplus_T S_n = \max(S_1, \dots, S_n).$$

A function h is subadditive if for all $s, t \geq 0$

$$h(s+t) \leq h(s) + h(t).$$

We say that $S \in \mathcal{J}$ is a $\oplus_T-$idempotent if $S \oplus_T S = S$.

The following result is due to Marková-Stupňanová [15, Theorems 5, 6] and Hong [9, Theorem 3].

Theorem 3. (Hong [9], Marková-Stupňanová [15]) Let $S \in \mathcal{J}$ and let T be a continuous Archimedean t-norm (strict or nilpotent) with additive generator f. Then $f \circ S$ is subadditive if and only if S is a $\oplus_T$- idempotent.

The following theorem is due to Hong [9] which generalize Theorems 2, 3 and 4 of Marková-Stupňanová [15] and Theorem 4 and 5 of Mesiar [18].

Theorem 4. (Hong [9]) Let T be a continuous Archimedean t-norm with additive generator f. Let $S_i \in \mathcal{J}$, $i = 1, \dots, n$, $n \in N$, be shapes such that $f \circ \max(S_i)$ is subadditive. Then

$$S_1 \oplus_T \cdots \oplus_T S_n = \max(S_1, \dots, S_n).$$

3. MAIN RESULTS

We first consider the relation between subadditivity of all composites $f \circ S_i$ and subadditivity of $f \circ \max(S_i)$. The condition that $f \circ S_i$, $i = 1, \dots, n$, $n \in N$ are concave implies that $f \circ \max(S_i)$ is concave. Likewise, can we prove that the condition that $f \circ S_i$, $i = 1, \dots, n$, $n \in N$ are subadditive implies that $f \circ \max(S_i)$ is subadditive? Indeed, both conditions have nothing to do with each other. The following example shows the subadditivity of $f \circ S_i$, $i = 1, \dots, n$, $n \in N$ does not imply the subadditivity of $f \circ \max(S_i)$.

Example 1. Łukasiewicz t-norm T_L, $T_L(x, y) = \max(x + y - 1, 0)$ with additive generator $f(x) = 1 - x$. Let

$$1 - S_1(x) = \begin{cases} \frac{3}{2}x & x \in [0, 1/4], \\ \frac{1}{2}x + \frac{1}{4} & x \in [1/4, 3/4], \\ \frac{3}{2}x - \frac{1}{2} & x \in [3/4, 1], \\ 1 & \text{otherwise}, \end{cases}$$

and let $1 - S_2(x) = x$, $x \in [0, 1]$, and 1, otherwise. Then clearly $f \circ S_2$ is subadditive. For $f \circ S_1$, there are many cases to check. For example, we consider the case that $s \in [0, 1/4]$, $t \in [3/4, 1]$. Then

$$\begin{aligned} & f \circ S_1(s) + f \circ S_1(t) - f \circ S_1(s + t) \\ = \; & \frac{3}{2}s + \frac{3}{2}t - \frac{1}{2} - \min\left\{\frac{3}{2}(s + t) - \frac{1}{2},\ 1\right\} \\ \geq \; & \frac{3}{2}s + \frac{3}{2}t - \frac{1}{2} - \left(\frac{3}{2}(s + t) - \frac{1}{2}\right) \\ = \; & 0 \end{aligned}$$

Similarly we can easily check for all other cases to show that $f \circ S_1$ is subadditive. But from the fact that $f \circ \max(S_1, S_2)\,(1) = 1$, $f \circ \max(S_1, S_2)\,(1/4) = (1/4)$ and $f \circ \max(S_1, S_2)\,(3/4) = (5/8)$ we see that $f \circ \max(S_1, S_2)$ is not subadditive.

Note 1. In general, nondecreasing concave function is subadditive. But the converse is not true. $f \circ S_1$ in Example 1 is a counter example for this.

The following example shows the subadditivity of $f \circ \max(S_i)$ does not imply the subadditivity of $f \circ S_i$, $i = 1, \dots, n$, $n \in N$.

Example 2. Łukasiewicz t-norm T_L, $T_L(x, y) = \max(x + y - 1, 0)$ with additive generator $f(x) = 1 - x$. Let

$$1 - S_1(x) = \begin{cases} x & x \in [0, 1/4], \\ \frac{3}{2}x - \frac{1}{8} & x \in [1/4, 3/8], \\ \frac{1}{2}x + \frac{1}{4} & x \in [3/8, 1/2], \\ x & x \in [1/2, 1], \\ 1 & \text{otherwise}, \end{cases}$$

$$1 - S_2(x) = \begin{cases} x & x \in [0, 1/2], \\ \frac{3}{2}x - \frac{1}{4} & x \in [1/2, 3/4], \\ \frac{1}{2}x + \frac{1}{2} & x \in [3/4, 1], \\ 1 & \text{otherwise.} \end{cases}$$

Then $1 - S_i(x) = f \circ S_i,\ i = 1, 2$, and $f \circ S_1(1/4) + f \circ S_1(1/8) = 1/4 + 1/8 < 7/16 = f \circ S_1(3/8)$ and hence $f \circ S_1$ is not subadditive. Similarly, since $f \circ S_2(1/4) + f \circ S_1(1/2) = 1/4 + 1/2 < 7/8 = f \circ S_2(3/4)$, $f \circ S_2$ is not subadditive. But $f \circ \max(S_i)\,(x) = \min(f \circ S_i)\,(x) = x,\ x \in [0, 1]$, and 1 otherwise, is subadditive.

We consider another sufficient condition for a t-norm T to act as the T_W-based addition. This result also generalize Theorem 2, 3 and 4 of Marková-Stupňanová [15] and Theorem 4 and 5 of Mesiar [18].

Theorem 5. Let T be a continuous Archimedean t-norm with additive generator f. Let $S_i \in \mathcal{J}$, $i = 1, \dots, n$, $n \in N$, be shapes such that all composites $f \circ S_i$ are subadditive. Then

$$S_1 \oplus_T \cdots \oplus_T S_n = \max(S_i) \oplus_T \cdots \oplus_T \max(S_i).$$

P r o o f. We prove for $k = 2$ and the case for $k = n \in \mathbf{N}$ is similar. Since $\max(S_1, S_2) \leq S_1 \oplus_T S_2$, and S_1 and S_2 are $\oplus_T$-idempotent, we have

$$\begin{aligned} S_1 \oplus_T S_2 &\leq \max(S_1, S_2) \oplus_T \max(S_1, S_2) \\ &\leq (S_1 \oplus_T S_2) \oplus_T (S_1 \oplus_T S_2) \\ &= (S_1 \oplus_T S_1) \oplus_T (S_2 \oplus_T S_2) \\ &= S_1 \oplus_T S_2, \end{aligned}$$

where the first equality comes from commutative law based on $\oplus_T$ addition operation. This implies $S_1 \oplus_T S_2 = \max(S_1, S_2) \oplus_T \max(S_1, S_2)$ and completes the proof. □

Note 2. In Theorem 2 if $f \circ S_i$, $i = 1, \dots, n$, $n \in N$ are non-decreasing concave, then $f \circ \max(S_i) = \min(f \circ S_i)$ is non-decreasing concave and hence is subadditive. Then $\max(S_i)$ is $\oplus_T$-idempotent by Theorem 3. Therefore Theorem 5 generalizes Theorem 2.

The following theorem gives a relation between subadditivity of all composites $f \circ S_i$ and subadditivity of $f \circ \max(S_i)$.

Corollary 2. Let T be a continuous Archimedean t-norm with additive generator f. Let $S_i \in \mathcal{J}$, $i = 1, \dots, n$, $n \in N$. If $f \circ S_i$ is subadditive for $i = 1, \dots, n$ and $S_1 \oplus_T \cdots \oplus_T S_n = \max(S_i)$, then $f \circ \max(S_i)$ is subadditive.

P r o o f. Suppose that $S_1 \oplus_T S_2 = \max(S_1, S_2)$. Since $S_1,\ S_2$ are $\oplus_T$- idempotent,

$$\begin{aligned} \max(S_1, S_2) &= S_1 \oplus_T S_2 \\ &= (S_1 \oplus_T S_1) \oplus_T (S_2 \oplus_T S_2) \\ &= (S_1 \oplus_T S_2) \oplus_T (S_1 \oplus_T S_2) \\ &= \max(S_1, S_2) \oplus_T \max(S_1, S_2). \end{aligned}$$

In general, we have $\max(S_i) = \max(S_i) \oplus_T \max(S_i)$ by the mathematical induction. Therefore $f \circ \max(S_i)$ is subadditive by Theorem 3, which completes the proof. □

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