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THE STABILITY OF PARAMETER ESTIMATION OF FUZZY VARIABLES

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Recently, the parameter estimations for normal fuzzy variables in the Nahmias' sense was studied by Cai [4]. These estimates were also studied for general T-related, but not necessarily normal fuzzy variables by Hong [10] In this paper, we report on some properties of estimators that would appear to be desirable, including unbiasedness. We also consider asymptotic or "large-sample" properties of a particular type of estimator.

Keywords: fuzzy variables, parameter estimation, consistency, MSE, stability of estimation

AMS Subject Classification: 28E10, 62L12

1. INTRODUCTION

Zadeh [23] and Nahmias [18] introduced the concept of a variable as a possible theoretical framework from which a rigorous theory could ultimately be constructed about fuzziness. Cai et al. [2, 3] established fuzzy variables as a basis for the probist reliability theory, but little attention has been paid to the parameter estimation issues of fuzzy variables except in the case of fuzzy software reliability modeling. Badard [1] considered the "minimum fuzziness estimator" for fuzzy models and studied the convergence of these estimators. Wang [22] presented various methods to estimate the membership function under the name of 'fuzzy statistics'. Dishkant [6] discussed the parameter estimation of fuzzy variables in the Zadeh's sense. Cai [4] studied parameter estimation methods for normal fuzzy variables in the Nahmias' sense. Recently, Hong [10] conducted additional investigations on the parameter estimations of general T-related, but not necessarily normal fuzzy variables using the results of Hong [11], Hong and Hwang [14], Mesiar [17], Marková [16] and Hong and Ro [13]. But as Cai [4] mentioned in his concluding remarks, the stability behavior of parameter estimates are still falling within the scope of open problems. The purpose of this paper is to introduce a way of studying stability properties of parameter estimations of T-related fuzzy variables in the Nahmias' sense, including unbiasedness, mean squared error and consistency.

2. PRELIMINARIES

First we recall some definitions and notations relevant to this subject.

Definition 2.1. For a base set Γ, suppose that $\mathcal{G}$ is the class of all subsets of Γ. Suppose a scale, σ, is defined on $\mathcal{G}$ and satisfies the following properties:

(i) $\sigma(\emptyset) = 0$ and $\sigma(\Gamma) = 1$.

(ii) For any collection of sets A_α of $\mathcal{G}$(finite, countable or uncountable),

$$\sigma\Big(\bigcup_\alpha A_\alpha\Big) = \sup_\alpha \sigma(A_\alpha).$$

Then σ is the scale measure and the triple $(\Gamma, \mathcal{G}, \sigma)$ is referred to as the pattern space.

Definition 2.2. A fuzzy variable X is a mapping from Γ to $\mathbb{R}$ (the real number line).

Definition 2.3. The membership function of a fuzzy variable X, denoted by μ_X, is a mapping from $\mathbb{R}$ to the unit interval $[0, 1]$ and is given by

$$\mu_X(x) = \sigma(\gamma : X(\gamma) = x)$$

for all $x \in \mathbb{R}$. Note that

$$\sup_x \mu_X(x) = \sigma\Big\{\bigcup_x (\gamma : X(\gamma) = x)\Big\} = \sigma(\Gamma) = 1.$$

In general we denote $X = x$ to be the subset $\{\gamma : X(\gamma) = x\}$ of $\mathcal{G}$.

It has been shown that the value of $\mu_X(x)$ at point x can be interpreted as the possibility that $X = x$ holds [2], though we are not asked to adopt Zadeh's definition of possibility measure [23]. Therefore the membership function of X can be viewed as the possibility of X and we arrive at the following definitions.

Definition 2.4. The possibility distribution function of a fuzzy variable X, denoted by π_X or μ_X, is a mapping from $\mathbb{R}$ to the unit interval $[0, 1]$ and is given by $\pi_X(x) = \mu_X(x) = \sigma(X = x)$ for all $x \in \mathbb{R}$.

A function $T : [0, 1] \times [0, 1] \to [0, 1]$ is said to be a t-norm [21] iff T is symmetric, associative, and non-decreasing in each argument, and furthermore, $T(x, 1) = x$ for all $x \in [0, 1]$. It is noted that $T(x, y) = \min(x, y)$ is the strongest t-norm.

Recall that a continuous t-norm T is Archimedean iff $T(x, x) < x$ for all $x \in (0, 1)$. A well-known theorem (see [21]) asserts that for each continuous Archimedean t-norm there exists a continuous, decreasing function $f : [0, 1] \to [0, \infty]$ with $f(1) = 0$ such that

$$T(x, y) = f^{[-1]}(f(x) + f(y))$$

for all $x, y \in (0, 1)$. Here $f^{[-1]} : [0, \infty] \to [0, 1]$ is defined by

$$f^{[-1]}(y) = \begin{cases} f^{-1}(y) & \text{if } y \in [0, f(0)) \\ 0 & \text{if } y \in [f(0), \infty]. \end{cases}$$

The function f is called the additive generator of T and is unique up to positive multiplicative factors.

The concept and terminology of 'min-relatedness(unrelatedness)' [18] can be generalized to T-relatedness as follows.

Definition 2.5. Given a pattern space $(\Gamma, \mathcal{G}, \sigma)$ and a t-norm T, the sets $A_1, \ldots, A_n \subset \mathcal{G}$ are said to be mutually T-related if for any permutation denoted by $i_1, \ldots, i_n$ and any k between 1 and n,

$$\sigma\big(A_{i_1} \cap \cdots \cap A_{i_k}\big) = T\big(\sigma(A_{i_1}), \ldots, \sigma(A_{i_k})\big).$$

Definition 2.6. Given a pattern space $(\Gamma, \mathcal{G}, \sigma)$ and a t-norm T, the fuzzy variables $X_1, \ldots, X_n$ are said to be mutually T-related if the sets $\{X_1 = x_1\}, \ldots, \{X_n = x_n\}$ are mutually T-related for all $x_1, \ldots, x_n \in \mathbb{R}$.

Another useful notation for fuzzy variables is the modal value, as a modal value of fuzzy variable X may be imagined to be the 'expected' value of X. However we note that a fuzzy variable may not have a modal value [20].

Definition 2.7. (Rao and Rashed [20]) Let X be a fuzzy variable with membership function μ_X. A real number m is said to be a modal value of X, if $\mu_X(m) = 1$.

Definition 2.8. (Rao and Rashed [20]) A fuzzy variable X is said to be unimodal, if there exists a unique $a \in \mathbb{R}$ such that $\mu_X(a) = 1$. At this time we signify $\mathrm{E}X = a$.

A fuzzy variable X is of the *normal class* [20], denoted $N(a, b)$, if the membership function μ_X is given by

$$\mu_X(x) = \exp\left(-\left(\frac{x-a}{b}\right)^2\right) \quad \text{for} -\infty < x < \infty,$$

where a and b are constants.

Definition 2.9. A fuzzy variable X is of type H, denoted $H(a, b)$, if the membership function μ_X is given by

$$\mu_X(x) = H\left(\frac{x-a}{b}\right) \quad \text{for} -\infty < x < \infty,$$

where $H : \mathbb{R} \to [0, 1]$ is non-decreasing on $(-\infty, 0]$ and non-increasing on $[0, \infty)$ and a, $b \in \mathbb{R}$, $b > 0$.

It is noted that if $H(x) = e^{-x^2}$, then $N(a, b) = H(a, b)$.

We can describe the membership function of the fuzzy variable X of type H in the following manner:

$$\mu_X(x) = \begin{cases} H_-(x) & \text{for } x < c, \\ 1 & \text{for } c \le x \le d, \\ H_+(x) & \text{for } d < x. \end{cases}$$

We introduce the following function.

$$H_-^{-1}(\alpha) = \inf\{x : \mu_X(x) \geq \alpha\}$$
$$H_+^{-1}(\alpha) = \sup\{x : \mu_X(x) \geq \alpha\}$$

for $\alpha \in [0, 1]$

Heilpern [9] defined the expected interval of a fuzzy variable X of type H, which is denoted by $\mathrm{E}I(X)$, as follows.

Definition 2.10. (Hilpern [9]) Assume $\int_{-\infty}^{c} x \,\mathrm{d}H_-(x) < \infty$ and $\int_{d}^{\infty} x \,\mathrm{d}H_+(x) < \infty$. Then

$$\mathrm{E}I(X) = \left[\int_{-\infty}^{c} x \,\mathrm{d}H_-(x), \int_{d}^{\infty} x \,\mathrm{d}H_+(x)\right] = [\mathrm{E}H_-, \mathrm{E}H_+].$$

Definition 2.11. (Hilpern [9]) The center of the expected interval of a fuzzy variable of type H is called the expected value of this variable. It is denoted by $\mathrm{E}V(X)$, i. e., $\mathrm{E}V(X) = \frac{1}{2}(\mathrm{E}H_- + \mathrm{E}H_+)$.

The following two lemmas give us simple formulas for calculating the expected interval and expected value of a fuzzy variable.

Lemma 2.1. (Hilpern [9]) Let X be a fuzzy variable of type H with $\mathrm{E}H_- < \infty$, $\mathrm{E}H_+ < \infty$. Then

$$\mathrm{E}H_- = c - \int_{-\infty}^{c} H_-(x) \,\mathrm{d}x,$$
$$\mathrm{E}H_+ = d + \int_{d}^{\infty} H_+(x) \,\mathrm{d}x.$$

Lemma 2.2. (Hilpern [9]) Let H_- and H_+ be continuous with $\mathrm{E}H_- < \infty, \mathrm{E}H_+ < \infty$. Then

$$\mathrm{E}H_- = \int_0^1 H_-^{-1}(t) \,\mathrm{d}t \quad \text{and} \quad \mathrm{E}H_+ = \int_0^1 H_+^{-1}(t) \,\mathrm{d}t.$$

A similar definition was defined by Chanas and Nowakowski in [5].

Suppose that $X_1, \ldots, X_n$ are fuzzy variables defined on $(\Gamma, \mathcal{G}, \sigma)$. Then we have the following result from Nahmias [18].

Theorem 2.1. (Nahmias [18]) Let $X_1, \ldots, X_n$, be mutually T-related and $Z = X_1 + \cdots + X_n$, then $\mu_Z(z) = \sup_{x_1 + \cdots + x_n = z} T(\mu_{X_1}(x_1), \mu_{X_2}(x_2), \ldots, \mu_{X_n}(x_n))$.

The following results are due to Hong and Hwang [14], Mesiar [17] and Hong and Ro [13].

Theorem 2.2. Let T be an continuous Archimedean t-norm with additive generator f and let $X_1, \ldots, X_n$ be mutually T-related fuzzy variables with $\mu_{X_i}(x) = H(x - a_i)$ $i = 1, \ldots, n$. If $f \circ H$ is a convex function on $\mathbb{R}$, then the membership function of $S_n = X_1 + \cdots + X_n$ is

$$\mu_{S_n}(z) = f^{[-1]}\left(nf\left(H\left(\frac{z - A_n}{n}\right)\right)\right),$$

where $A_n = a_1 + \cdots + a_n$.

The following results are generalizations of Theorem 5.2 [15] due to Dubois and Prade [7] and Hong and Kim [12].

Theorem 2.3. Let X and Y be two T-related fuzzy variables with modal values a and b respectively. Then $a + b$ is a modal value for the fuzzy variables $Z = X + Y$.

Theorem 2.4. Let $X_1, \ldots, X_n$ be mutually min-related fuzzy variables $H(a_1, b_1)$, $\ldots, H(a_n, b_n)$, and $\alpha_1, \ldots, \alpha_n$ non-zero scalars. Then $Z = \sum_{i=1}^n \alpha_i X_i$ become $H(\sum_{i=1}^n \alpha_i a_i, \sum_{i=1}^n |\alpha_i| b_i)$.

3. LARGE–SAMPLE PROPERTIES

Recently, Cai [4] and Hong [10] discussed parameter estimation issues for fuzzy variables. In this section, we discuss some properties of estimators that would appear to be desirable, including unbiaseness. Throughout this section, let X be a unimodal fuzzy variable defined on the pattern space $(\Gamma, \mathcal{G}, \sigma)$ and

$$\mu_X(x) = \sigma(X = x) = H\left(\frac{x - a}{b}\right).$$

X can be imagined as some quantitative representation of an object. To estimate a and b, we conduct experiments on the object n times with X_i being the corresponding quantitative representation for the ith experiment. We assume that $X_1, \ldots, X_n$ are mutually T-related and $x_1, \ldots, x_n$ are their realizations.

Definition 3.1. An estimator D is said to be an unbiased estimator of $\tau(\theta)$ if

$$\mathrm{E}V(D) = \tau(\theta).$$

Otherwise, we say that D is a biased estimator of $\tau(\theta)$.

Example 3.1. Consider a fuzzy variable from a membership function $N(\theta)$ with $\theta = (a, b)$. Suppose $T =$ " min " and a is unknown, but b is known. Cai [4] introduced two estimators in estimating a,

$$\hat{a_1} = \frac{X_1 + \cdots + X_n}{n},$$
$$\hat{a_2} = \frac{1}{2}\left(\max_{1 \le i \le n} X_i + \min_{1 \le i \le n} X_i\right).$$

Since $\mathrm{E}V\hat{a_1} = \mathrm{E}V\hat{a_2} = a$, both estimators are unbiased for a.

A general idea is to select the estimator that tends to be closest or "most concentrated" around the true value of the parameter. It might be reasonable to say that D_2 is more concentrated than D_1 about $\tau(\theta)$ if

$$\{\mu_{D_1}(x) \geq \alpha\} \supset \{\mu_{D_2}(x) \geq \alpha\}$$

for all $\alpha \geq 0$.

Example 3.2. Let us reconsider Example 3.1. Suppose T is the classical product, i. e., $T(x, y) = xy$ and $b = 1$. Then by Theorem 2.2,

$$\mu_{\hat{a_1}}(x) = e^{-n(x-a)^2} \quad \text{and} \quad \mu_{\hat{a_2}}(x) = e^{-2(x-a)^2}$$

noting that $\max_{1\leq i\leq n} X_i = \min_{1\leq i\leq n} X_i = X$. So for $n \geq 2$, $\hat{a_1}$ is more concentrated than $\hat{a_2}$ about a.

Definition 3.2. Let X be a fuzzy variable of type H. We denote $|\mathrm{E}V|(X) = \frac{1}{2}(|\mathrm{E}H_-| + |\mathrm{E}H_+|)$. We call $|\mathrm{E}V|(X - \mathrm{E}V(X))^2$ the variance of X and denote it by $\mathrm{Var}(X)$.

If D is an unbiased estimator of $\tau(\theta)$, one with a smaller expected variance will tend to be more concentrated and thus may be preferable. In Example 3.2 we can easily find that for $n \geq 2$

$$\mathrm{Var}(\hat{a_1}) \leq \mathrm{Var}(\hat{a_2}).$$

Definition 3.3. If D is an estimator of $\tau(\theta)$, the bias is given by

$$b(D) = \mathrm{E}V(D) - \tau(\theta),$$

and the mean squared error (MSE) of D is given by

$$\mathrm{MSE} = |\mathrm{E}V|[D - \tau(\theta)]^2.$$

So far we have discussed properties of estimators. Estimators are defined for any fixed sample size n. These are examples of "small-sample" properties. It is also useful to consider asymptotic or "large-sample" properties of a particular type of estimator. An estimator may have undesirable properties for small n, but still be a reasonable estimator in certain applications if it has good asymptotic properties as the sample size increases. Also it is quite often possible to evaluate the asymptotic properties of an estimator when small-sample properties are difficult to determine.

For a fuzzy variable X and any subset C of the real numbers, the quantity

$$\mathrm{Nes}(X|C) = 1 - \sup_{x\notin C} \mu_X(x)$$

is considered to be a measure of the necessity of X belonging to C [23].

Definition 3.4. A sequence of functions $\{D_n(X_1,\ldots,X_n)\}$ is called a consistent sequence of estimators for $\tau(\theta)$ if for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \text{Nes}(D_n|(\tau(\theta)-\varepsilon,\tau(\theta)+\varepsilon)) = 1.$$

Often one takes some liberty with the terminology and considering D_n as a representative of $\{D_n\}$ refers to D_n as a consistent estimator for $\tau(\theta)$.

Definition 3.5. An estimator D_n for $\tau(\theta)$ is said to be mean squared consistent if

$$\lim_{n\to\infty} |\text{E}V|[D_n - \tau(\theta)]^2 = 0.$$

Definition 3.6. An estimator D_n for $\tau(\theta)$ is said to be asymptotically unbiased if

$$\lim_{n\to\infty} \text{E}V(D_n) = \tau(\theta).$$

Example 3.3. Let $H(0,1) = 1+x$ on $[-1,0]$, $1-\frac{1}{2}x$ on $[0,2]$ and 0, otherwise and let $X_i = H(a,b),\ i=1,2,\ldots,n$ with $a>0,\ b>2$. In [10], Hong suggested that if $T =$ "min", for given possibility α, the estimate of b is given by

$$\hat{b} = \frac{\max_{1\le i\le n} X_i - \min_{1\le i\le n} X_i}{H_+^{-1}(\alpha) - H_-^{-1}(\alpha)}$$

and the estimate of a is given by

$$\hat{a} = \frac{1}{2}\left(\max_{1\le i\le n} X_i + \min_{1\le i\le n} X_i) + \frac{\hat{b}}{2}(H_-^{-1}(\alpha) + H_+^{-1}(\alpha)\right)$$

If $\alpha = \frac{1}{2}$, then it is not hard to show that

$$\mu_{\hat{b}}(x) = \begin{cases} 1-\frac{1}{2}x & \text{if } 0\le x\le 2,\\ 1+\frac{1}{2}x & \text{if } -2\le x\le 0,\\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu_{\hat{a}}(x) = \begin{cases} 1+\frac{2(x-a)}{4a+3b} & \text{if } -a-\frac{3}{2}b\le x\le a,\\ 1-\frac{2(x-a)}{4a+5b} & \text{if } a\le x\le 3a+\frac{5}{2}b,\\ 0 & \text{otherwise}, \end{cases}$$

noting that

$$\mu_{\max X_i - \min X_i}(x) = \begin{cases} 1-\frac{1}{3b}x & \text{if } 0\le x\le 3b,\\ 1+\frac{1}{3b}x & \text{if } -3b\le x\le 0,\\ 0 & \text{otherwise}, \end{cases}$$

and

$$\mu_{\max X_i + \min X_i}(x) = \begin{cases} 1 - \frac{x-2a}{4b} & \text{if } 2a \le x \le 2a+4b, \\ 1 + \frac{x-2a}{2b} & \text{if } 2a-2b \le x \le 2a, \\ 0 & \text{otherwise.} \end{cases}$$

Then $EV(\hat{b}) = 0$ and $EV(\hat{a}) = a + \frac{1}{4}b$, hence $\hat{a}$, $\hat{b}$ are biased estimators of a, b respectively. Now, by Nguyen's Theorem,

$$\left\{\mu_{(\hat{b}-b)^2}(x) \ge \alpha\right\} = \left[(-2\alpha+2-b)^2, (-2\alpha+2+b)^2\right]$$

and hence

$$\begin{aligned} \mathrm{MSE}(\hat{b}) &= |EV|[\hat{b}-b]^2 \\ &= \frac{1}{2}\int_0^1 \left((-2\alpha+2-b)^2 + (-2\alpha+2+b)^2\right) \mathrm{d}\alpha \\ &= b^2 + \frac{4}{3}. \end{aligned}$$

Now, if $b = 4$, then

$$\mu_{\hat{a}-a}(x) = \begin{cases} 1 + \frac{x}{2a+6} & \text{if } -2a-6 \le x \le 0, \\ 1 - \frac{x}{2a+10} & \text{if } 0 \le x \le 2a+10 \end{cases}$$

and

$$\{\mu_{(\hat{a}-a)^2}(x) \ge \alpha\} = [-(2a+6)(2a+10)(1-\alpha)^2, (2a+10)^2(1-\alpha)^2].$$

Hence

$$\begin{aligned} \mathrm{MSE}(\hat{a}) &= |EV|[\hat{a}-a]^2 \\ &= \frac{1}{2}\left[\left((2a+6)(2a+10) + (2a+10)^2\right)\int_0^1 (1-\alpha)^2 \,\mathrm{d}\alpha\right] \\ &= \frac{4}{3}a^2 + 12a + \frac{80}{3}. \end{aligned}$$

Theorem 3.1. If D is an estimator of $\tau(\theta)$ and $T = \min$, then

$$\mathrm{MSE}(D) \le \mathrm{Var}(D)(1 + 2|b(D)|) + |b(D)|(2 + |b(D)|)$$

Proof.

$$\begin{aligned} \mathrm{MSE}(D) &= |EV|[D-\tau(\theta)]^2 \\ &= |EV|[D - EV(D) + EV(D) - \tau(\theta)]^2 \\ &= |EV|\big[(D-EV(D))^2 + 2(EV(D)-\tau(\theta))(D-EV(D)) \\ &\qquad + (EV(D)-\tau(\theta))^2\big] \end{aligned}$$

Now, by the linearity of the expected interval [Theorem 2. [9]], and the inequality $|\mathrm{E}V|(D-\mathrm{E}V(D)) \le 1+|\mathrm{E}V|(D-\mathrm{E}V(D))^2 = 1+\mathrm{Var}(D)$,

$$\begin{aligned} \mathrm{MSE}(D) &\le |\mathrm{E}V|(D-\mathrm{E}V(D))^2 + 2|\mathrm{E}V(D)-\tau(\theta)||\mathrm{E}V|(D-\mathrm{E}(D)) \\ &\quad +|\mathrm{E}V|\big(\mathrm{E}V(D)-\tau(\theta)\big)^2 \\ &\le \mathrm{Var}(D) + 2|\mathrm{E}V(D)-\tau(\theta)|(\mathrm{Var}(D)+1) + (\mathrm{E}V(D)-\tau(\theta))^2 \\ &= \mathrm{Var}(D)(1+2|b(D)|) + |b(D)|(2+|b(D)|). \end{aligned}$$ □

The following result follows immediately from Theorem 3.1.

Theorem 3.2. Let $T = \min$. An estimator D_n for $\tau(\theta)$ is mean squared consistent if it is asymptotically unbiased and $\lim_{n\to\infty} \mathrm{Var}(D_n) = 0$.

The following theorem is a fuzzy version of Markov inequality. We denote $\mathrm{Pos}(X|D) = \sup_{x\notin D} \mu_X(x)$.

Theorem 3.3. Let X be a fuzzy variable with $\mathrm{E}(X) = 0$. Then for any $\varepsilon > 0$,

$$\mathrm{Pos}(X|(-\varepsilon,\varepsilon)) \le \frac{2|\mathrm{E}V|(X^2)}{\varepsilon^2}.$$

Proof. Let $\{\mu_X(x) \ge \alpha\} = [X_-(\alpha), X_+(\alpha)]$ for $0 \le \alpha \le 1$. Then by Nguyen's Theorem,

$$\big\{\mu_{X^2}(x) \ge \alpha\big\} \supset \big[X_-(\alpha)X_+(\alpha), \max\{X_-(\alpha)^2, X_+(\alpha)^2\}\big].$$

Now,

$$\begin{aligned} \varepsilon^2 \sup_{x\notin(-\varepsilon,\varepsilon)} \mu_X(x) &= \varepsilon^2 \max\{\mu_X(-\varepsilon), \mu_X(\varepsilon)\} \\ &\le \int_0^1 \max\{X_-(\alpha)^2, X_+(\alpha)^2\}\,\mathrm{d}\alpha \\ &\le \int_0^1 \max\{X_-(\alpha)^2, X_+(\alpha)^2\}\,\mathrm{d}\alpha + \left|\int_0^1 X_-(\alpha)X_+(\alpha)\,\mathrm{d}\alpha\right| \\ &\le 2|\mathrm{E}V|(X^2), \end{aligned}$$

where the first inequality above comes from the fact that for $\alpha \le \mu_X(-\varepsilon)$, $\varepsilon^2\mu_X(-\varepsilon) \le X_-(\alpha)^2$ and for $\alpha \le \mu_X(\varepsilon)$, $\varepsilon^2\mu_X(\varepsilon) \le X_+(\alpha)^2$. Therefore, we have

$$\mathrm{Pos}(X|(-\varepsilon,\varepsilon)) = \sup_{x\notin(-\varepsilon,\varepsilon)} \mu_X(x) \le \frac{2|\mathrm{E}V|(X^2)}{\varepsilon^2}.$$

From the above inequality, we have

$$\mathrm{Nes}(X|(-\varepsilon,\varepsilon)) \ge 1 - \frac{2|\mathrm{E}V|(X^2)}{\varepsilon^2}.$$ □

Using this inequality, we have the following result.

Theorem 3.4. If an estimator D_n for $\tau(\theta)$ is mean squared consistent and $\lim_{n\to\infty} \mathrm{E}(D_n - \tau(\theta)) = 0$, then it is simply consistent.

Proof. By the assumption, we can easily have that

$$\lim_{n\to\infty} |\mathrm{E}V|[D_n - \tau(\theta) - \mathrm{E}(D_n - \tau(\theta))]^2 = 0.$$

We have for large n

$$\begin{aligned} & \mathrm{Nes}(D_n - \tau(\theta)|(-\varepsilon, \varepsilon)) \\ \geq\ & \mathrm{Nes}(D_n - \tau(\theta)|(\mathrm{E}(D_n - \tau(\theta)) - \varepsilon/2, \mathrm{E}(D_n - \tau(\theta)) + \varepsilon/2)) \\ =\ & \mathrm{Nes}(D_n - \tau(\theta) - \mathrm{E}(D_n - \tau(\theta))|(-\varepsilon/2, \varepsilon/2)) \\ \geq\ & 1 - \frac{8|\mathrm{E}V|[D_n - \tau(\theta) - \mathrm{E}(D_n - \tau(\theta))]^2}{\varepsilon^2}, \end{aligned}$$

where the last inequality comes from Theorem 3.3 and hence we complete the proof. □

By Theorem 6 [13] and Theorem 3 [15], we also have the following result about simply consistent.

Theorem 3.5. Let T be a continuous Archimedean t-norm with additive generator f and $f \circ H$ is either a convex function on $\mathbb{R}$ or a convex function on $[A, B]$ and $H = 0$, otherwise for some constants A, B such that $A < 0 < B$. Let $X_i = H(a, 1)$, $i = 1, 2, \ldots, n$. Then $(X_1 + \cdots + X_n)/n$ is a consistent estimator for a.

Note. If $T = \min$ in Theorem 3.5, then $(X_1 + \cdots + X_n)/n$ is not a consistent estimator for a, since for any n, $(X_1 + \cdots + X_n)/n = H(a, 1)$.

Example 3.4. Let T be product t-norm with the additive generator $f(x) = -\log x$ and $H(x) = e^{-|x|}$. Then $f \circ H(x) = |x|$ which is convex. Let $X_i = H(a, 1)$, $i = 1, 2, \ldots, n$, then by Theorem 3.5, $(X_1 + \cdots + X_n)/n$ is a consistent estimator for a. By a result from Badard [1] or Theorem 2.2, we have that

$$\mu_{(X_1+\cdots+X_n)/n}(x) = e^{-n|x-a|}.$$

Since $\int_a^\infty e^{-n|x-a|}\,\mathrm{d}x = \infty$ for all $n \geq 1$, $|\mathrm{E}V|[(X_1 + \cdots + X_n)/n - a]^2 = \infty$ for all $n \geq 1$. Therefore $(X_1 + \cdots + X_n)/n$ is not a mean squared consistent estimator for a.

4. CONCLUSION

In this paper, we have discussed properties of estimators of T-related fuzzy variables such as unbiasedness and MSE. We considered asymptotic or large-sample properties of a particular type of estimator. We also have evaluated the asymptotic properties of an estimator when the small-sample properties are difficult to determine.

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